The cut separator problem

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Abstract

Given $G = (V, E)$ an undirected graph and two specified nonadjacent nodes a and b of V, a cut separator is a subset $F = \delta(C) \subseteq E$ such that $a, b \in V \setminus C$ and a and b belong to different connected components of the graph induced by $V \setminus C$. Given a nonnegative cost vector $c \in \mathbb{R}_+^{|E|}$, the optimal cut separator problem is to find a cut separator of minimum cost. This new problem is closely related to the vertex separator problem. In this paper, we give a polynomial time algorithm for this problem. We also present four equivalent linear formulations, and we show their tightness. Using these results we obtain an explicit short polyhedral description of the dominant of the cut separator polytope.

Keywords: *Combinatorial Optimization, Polyhedra, Polynomial Time Algorithms, Separator Problem*

1   Introduction

Let $G = (V, E)$ be an undirected graph and a and b be two specified nonadjacent nodes of V. Let $\{A, B, C\}$ be a partition of V such that there is no edge between A and B, $a \in A$ and $b \in B$. The subset $F = \delta(C)$ will be called a *cut separator* where $\delta(C)$ is the set of edges having exactly one end-node in C. The *optimal cut separator problem* is, given G, a, b, and a cost vector $c \in \mathbb{R}_+^{|E|}$, to find a cut separator of minimum cost.

This new problem is closely related to the well known vertex separator problem (see, e.g., [1, 4]) where we still look for a partition $\{A, B, C\}$ such that there is no edge between A and B, but we would like to minimize the cost of the vertices of C (here a cost is associated with each vertex). Some restrictions on the size of A and B are generally introduced to take into account practical constraints. The vertex separator problem has many applications in different areas (connectivity problems, linear algebra, finite element problems, etc.) (see [1, 4] and the references therein).

If $G = (V, E)$ is a graph and $F \subseteq E$, the $0 - 1$ vector $z^F \in \mathbb{R}^{|E|}$ with $z^F(e) = 1$ if $e \in E$ and $z^F(e) = 0$ if not is called the *incidence vector* of F.

Define the polyhedron

$$Q_{ab}(G) = \operatorname{conv}\{z^F : F \text{ is a cut separator}\}.$$

$Q_{ab}(G)$ is called the cut separator polyhedron. This polyhedron is difficult to describe unless $P = NP$. This comes from the fact that when the edge weights are negative, the minimum

cut separator problem can be reduced to a classical maximum cut problem. Let $D_{ab}(G)$ be the dominant of $Q_{ab}(G)$, *i.e.*,

$$D_{ab}(G) = \text{conv}\{x^F \mid F \text{ is a cut separator}\} + \mathbb{R}_+^{|E|}.$$

If $c \geq 0$ then the cut separator problem is equivalent to solving the linear program

$$\min \left\{ \sum_{e \in E} c_e z_e, \ z \in D_{ab}(G) \right\}.$$

Given $w \in \mathbb{R}^E$ and $F \subseteq E$, $w(F)$ will denote $\sum_{e \in F} w(e)$. If u and v are two distinct nodes of G, we denote by $\mathcal{P}_{uv}$ the set of paths of G from u to v.

It is easy to see that any $z \in D_{ab}(G)$ must satisfy the following inequalities:

$$z(P) \geq 2 \qquad \forall P \in \mathcal{P}_{ab}, \tag{1.1}$$
$$z(P \setminus \{e\}) \geq 1 \quad \forall P \in \mathcal{P}_{ab}, \forall e \in P. \tag{1.2}$$

Define the polyhedron

$$\overline{D}_{ab}(G) = \{z \geq 0; z \text{ satisfies (1.1) and (1.2)}\}.$$

Obviously, $D_{ab}(G) \subseteq \overline{D}_{ab}(G)$. We will prove later that $D_{ab}(G) = \overline{D}_{ab}(G)$ and hence we obtain a complete description of $D_{ab}(G)$. A simple polynomial time combinatorial algorithm will also be provided for the minimum cut separator problem.

In the rest of the paper, LP_1 will denote the following linear program:

$$\min \left\{ \sum_{e \in E} c_e z_e, \ z \in \overline{D}_{ab}(G) \right\}.$$

We will use the following notation. If e is an edge with end-nodes u and v, then we write $e = (uv)$.

The paper is organized as follows. Next Section is devoted to a polynomial time algorithm for the minimum cut separator problem. In Section 3, we present two Mixed Integer Programming formulations, and we prove the equivalence between their linear relaxations and LP_1. We also show that all linear relaxations are tight which implies that $\overline{D}_{ab}(G) = D_{ab}(G)$. Finally, Section 4 concludes with avenues for further research.

Notice that some results are given without proofs due to space limitation. All proofs can be found in [2].

2   A polynomial time algorithm for the minimum cut separator problem

A simple polynomial time algorithm is described below to solve the minimum cut separator problem.

We build a weighted directed graph $G' = (V', E')$ as follows. Each vertex $i \in V$ is replaced by two new vertices u_i and v_i. The new graph contains a directed arc $(v_i u_i)$ with an infinite weight for each $u \in V$. Given an undirected edge $e = (ij) \in E$, we define 6 directed arcs as follows: $(v_i u_j)$ and $(v_j u_i)$ with an infinite weight, $(v_i v_j)$, $(v_j v_i)$, $(u_i u_j)$ and $(u_j u_i)$ with a weight equal to c_e.

Let us now consider a minimum weight cut $\delta^+(D)$ separating v_a and u_b by computing a maximum flow from v_a to u_b. We assume here that $D \subset V'$ contains v_a and $\delta^+(D)$ is the set of arcs going from D to $V' \setminus D$. Now let $A = \{i \in V, u_i \in D \text{ and } v_i \in D\}$,

$B = \{i \in V, u_i \notin D \text{ and } v_i \notin D\}$ and $C = V \setminus \{A \cup B\}$. If $\delta^+(D)$ is a finite weight cut, then all vertices of C will satisfy $u_i \in D$ and $v_i \notin D$. We can also observe that there are no edges with one end in A and one end in B. We also have $a \in A$ and $b \in B$. The weight of $\delta^+(D)$ is clearly equal to the weight of $\delta(C)$. Hence, by computing a minimum weight cut $\delta^+(D)$ separating v_a and u_b we get a minimum weight cut separator.

Since we have a polynomial time algorithm to solve the minimum cut separator problem, there is a chance to provide a full description of $D_{ab}(G)$.

Observe that the graph G' considered above is generally not bipartite. One may also remark that one can slightly modify the algorithm to impose that a given vertex c belongs to C. It is also easy to impose that c belongs to $A \cup C$ (resp. $B \cup C$).

3 Linear formulations

In this section, we present two mixed integer formulations for the cut separator problem. Then we show that the linear relaxations of these formulations are equivalent to LP_1.

Consider a variable $y_u \in \{-1, 0, 1\}$ associated with each $u \in V$ such that $y_u = 1$ if $u \in A$, $y_u = -1$ if $u \in B$ and $y_u = 0$ if $u \in C$. Then the cut separator problem can obviously be formulated through this program:

$$(IP_2) \begin{cases} \min \sum_{e \in E} c_e z_e & \\ z_{(uv)} \geq |y_u - y_v| & \forall (uv) \in E \\ 1 \geq |y_u| & \forall u \in V \\ y_a = 1, y_b = -1 & \\ z_{(uv)} \leq 1 & \forall (uv) \in E \\ y_u \in Z & \forall u \in V \end{cases}$$

We use LP_2 to denote the linear relaxation of IP_2.

Notice that the constraints involving the absolute value can obviously be replaced by linear constraints.

Proposition 3.1 *Each feasible solution of LP_2 satisfies the constraints of LP_1.*

proof: Let (y, z) be a solution of LP_2. Let P be any path in $\mathcal{P}_{ab}$. Then we have $z(P) \geq \sum_{(uv) \in P} |y_u - y_v| \geq |y_a - y_b| = 2$. Since $z(e) \leq 1$ for any $e \in E$, inequality $z(P) \geq 2$ leads to $z(P \setminus e) \geq 1$ for any $e \in P$. The positivity of z ends the proof. ∎

Let $G = (V, E)$ be a graph and $d : V \times V \to \mathbb{R}^+$ be a function such that $d(u, v) = d(v, u)$ for all $u, v \in V$; $d(u, u) = 0$ for all $u \in V$; and $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in V$. Such function is called semimetric. The cut separator problem can be formulated as the following mixed integer program.

$$(IP_3) \begin{cases} \min \sum_{(uv) \in E} c_{(uv)} d(u, v) & \\ d \text{ is a semimetric} & \\ d(a, b) = 1 & \\ d(a, u) + d(b, v) \geq 1 & \forall (uv) \in E \\ d(a, v) + d(b, u) \geq 1 & \forall (uv) \in E \\ |d(a, u) + d(b, v) - d(a, v) - d(b, u)| \leq d(u, v) & \forall (uv) \in E \\ d(u, v) \leq 1 & \forall u, v \in V \\ d(a, u) \in \{0, 1\} & \forall u \in V \\ d(b, u) \in \{0, 1\} & \forall u \in V \end{cases}$$

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Notice that the variables $d(u, v)$ are not restricted to be integer when $\{u, v\} \cap \{a, b\} = \emptyset$. This comes from the fact that if d^* is a an optimal solution of (IP_3) then $d^*(u, v) \in \{0, 1\}$ for all $(uv) \in E$.

This formulation is inspired by the formulation of [3] for the multiway cut problem. We denote the linear relaxation of IP_3 by LP_3.

Proposition 3.2 LP_2 and LP_3 are equivalent.

Proposition 3.3 Every extremal optimal solution of LP_1 can be transformed into a feasible solution of LP_3 having the same cost.

Combination of the previous Propositions clearly implies that the linear problems LP_1, LP_2 and LP_3 have the same optimal objective value.

Proposition 3.4 The optimal objective value of LP_1, LP_2 and LP_3 is equal to the minimum cost of a cut separator.

Theorem 3.5 $D_{ab}(G) = \overline{D}_{ab}(G)$.

Proof: The previous Proposition implies that minimizing any positive objective function over $\overline{D}_{ab}(G)$ (This is LP_1) is equivalent to minimizing the same objective function over $D_{ab}(G)$. If the objective function is negative, we get an unbounded problem in both cases. This implies that $D_{ab}(G)$ and $\overline{D}_{ab}(G)$ are defined by the same set of valid inequalities. In other words, we have $D_{ab}(G) = \overline{D}_{ab}(G)$. ∎

Theorem (3.5) implies that all the extreme points of $\overline{D}_{ab}(G)$ are $0 - 1$ vectors. Consequently, we can impose that $z_e \leq 1$ for any edge e without loosing the optimality property. Moreover, constraints $z(P \setminus \{e\}) \geq 1$ become redundant when constraints $z_e \leq 1$ are considered. Therefore the relaxation LP_4 defined below is also tight. In other words, the polytope considered in LP_4 is exactly the intersection of $D_{ab}(G)$ and $[0, 1]^{|E|}$.

$$(LP_4) \begin{cases} \min \sum_{e \in E} c_e z_e \\ z(P) \geq 2 & \forall P \in \mathcal{P}_{ab} \\ 0 \leq z_e \leq 1 & \forall e \in E \end{cases}$$

4 Conclusion

We gave a full description of the dominant of $Q_{ab}(G)$. However, describing $Q_{ab}(G)$ is much harder since the maximum cut separator problem has the same difficulty as the classical maximum cut problem. Semidefinite relaxations and linear relaxations may be proposed for the maximum cut separator problem.

Other variations of the cut separator problem can be studied. It may be useful to integrate some size constraints related the the sets A, B and C defining the cut.

In fact, the results provided in this paper and in [2] lead to a large new class of graph partitioning problems that can be solved in polynomial time.

References

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