A magnetic algorithm for a maximum stable set in a graph

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Abstract

We give a characterization in terms of forbidden subgraphs of a class of graphs for which the stability number (maximum size of a stable set) can be computed in polynomial time by a sequence of reductions. Each reduction uses a special pair of vertices of the graph called a magnet and transforms the graph into another one having one vertex less and having the same stability number as the previous graph.

Keywords: *stable set, graph transformation.*

1 Introduction

In a graph G a magnet is a pair u, v of adjacent vertices such that the neighbours of u which are not neighbours of v are completely linked to the neighbours of v which are not neighbours of u. It has been shown (see [2]) that one can reduce the graph G by removing the two vertices u, v and introducing a new vertex linked only to all common neighbours of u and v in G. The new graph has one vertex less and it has the same stability number as the initial graph, i.e., the maximum size of a stable set has not changed.

Let $N(u)$ be the set of neighbours of vertex u; then in a magnet the sets $N(u) - N(v) - \{v\}$ and $N(v) - N(u) - \{u\}$ are completely linked. Notice that the definition allows either set (or both) to be empty. In such cases we get known reductions. We characterize in terms of forbidden subgraphs the family of graphs such that for any subgraph the repeated use of reductions based on magnets will produce a reduced graph which is simply a stable set (whose size is precisely the stability number of the initial graph). This provides a combinatorial algorithm for finding in polynomial time the stability number of graphs belonging to this class; these graphs are perfect (so they can be recognised in polynomial time). The complete proofs can be found in [3] with some additional results.

2 Preliminary results

We use the result of Chvatal and Rusu [1] stating that a graph containing no induced cycle of length at least 5 is either a stable set or it contains a magnet.

Following [2] we call *demagnetisation* of G a graph F which contains no magnet and for which there is a sequence of graphs $G = G_0, \cdots, G_k = F$ such that each G_i is obtained from G_{i-1} by application of the magnet reduction (MR). We define two graphs H_1 and H_2 on 6 and 7 vertices respectively (see Figure 1). These are the ones to be excluded if one wants to avoid having cycles of length at least 5 in the transformed graph obtained by a MR. We can then derive the following :

Lemma 1 *Let G be a $\{H_1, H_2, C_k (k \geq 5)\}$-free graph and let $M(u, v)$ be a magnet. Then if we apply the MR on $M(u, v)$, the resulting graph is C_k-free ($k \geq 5$).*

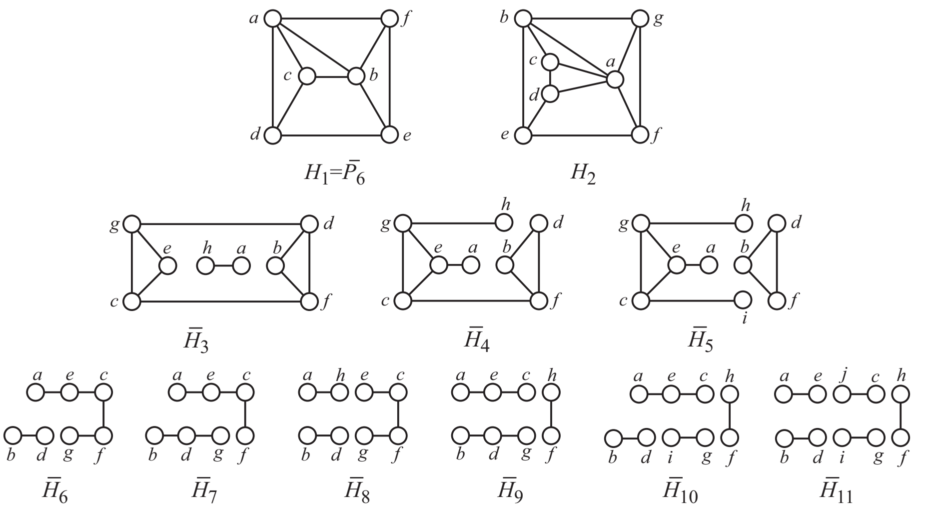

Figure 1: *The forbidden graphs H_1 and H_2 and the complements of $H_3, \cdots, H_{11}$.*

A graph is *S-demagnetizable* if it admits a demagnetization with an empty edge set (i.e., if G can be reduced to a stable set by a sequence of magnet reductions). G is *perfectly S-demagnetizable* if all its subgraphs are S-demagnetizable. It has been conjectured by Lévêque [4] that a graph is perfectly S-demagnetizable if and only if it contains no induced cycle of length at least 5 nor the complement of any odd cycle of length at least 7. We want to characterize the class of graphs for which all demagnetizations of all subgraphs of G have an empty edge set. Such graphs will be called *strongly S-demagnetizable.*

Let us introduce a collection of subgraphs $H_3, \cdots, H_{11}$ which are obtained by taking the complements of graphs constructed by vertex splitting either of the complement of a cycle C_6 or of a path P_6. The complements of $H_3, \cdots, H_{11}$ are given in Figure 1. It is easy to see that applying MR in a graph G is equivalent to contracting a pair of nonadjacent vertices not linked by any path of 3 edges in the complement of G. It is then simpler to observe (in the complement of G) that some contractions in the complements of $H_3, \cdots, H_{11}$ may produce the complements of some H_i $(1 \leq i \leq 11)$. One can show the following by examining a collection of cases :

Lemma 2 *If G is a $\{H_1, \cdots, H_{11}, C_k(k \geq 5)\}$-free graph and if G' is obtained from G by a MR, then G' is $\{H_1, \cdots, H_{11}\}$-free.*

3 The main result

From Lemmas 1 and 2 we derive immediately the following :

Theorem 1 *A graph is strongly S-demagnetizable if and only if it is $\{H_1, \ldots, H_{11}, C_k(k \geq 5)\}$-free.*

As a direct consequence we have :

Proposition 1 *Let G be a $\{H_1, \cdots, H_{11}, C_k(k \geq 5)\}$-free graph with n vertices and m edges. Then one can reduce G to a stable set by repeated magnet reductions. So the stability number of G (maximum size of a stable set) can be determined by a combinatorial algorithm in $O(n^3m)$.*

It is interesting to notice that the algorithm of reductions by MR can be applied to any graph. It will stop if at some stage one cannot find any magnet. From the above observations it will stop only if G is a stable set or if it contains some cycle C_k with $k \geq 5$. If G is strongly S-demagnetizable it will necessarily stop with a stable set. But if it is S-demagnetizable (but not strongly S-demagnetizable) it may possibly not stop with a stable set as can be seen easily by considering the case of the complement of a path P_6 : we may end up with a C_5.

One should finally observe that the class of strongly S-demagnetizable graphs belongs to the perfect graphs. It is however different from the class of weakly triangulated graphs (graphs containing no induced cycle C_k ($k \geq 5$) nor their complements). For instance the complement of the cycle C_6 is strongly S-demagnetizable but it is not weakly triangulated while the complement of a path P_6 is weakly triangulated but not strongly S-demagnetizable.

References

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