

# Discretized reformulations for a capacitated network loading problem arising in a facility location context

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## Abstract

In this paper we present a discretization reformulation technique in the context of a facility location problem with modular link costs. We present a ‘traditional’ model and a straightforward discretized model with a general objective function. The linear relaxation bounds provided by both formulations are compared. Furthermore, we state that a restricted version of the discretized model gives an extended description of the convex hull of a “small” polytope that arises in the original model. Several valid inequalities induced by the discretized variables are also presented. We summarize the computational experience performed, highlighting that in the context of facility location problems with modular costs, a lot can be gained by making use of the proposed discretized models instead of the ‘traditional’ models.

**Keywords:** Capacitated location, modular distribution costs, reformulations

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## 1. Introduction

In this paper we address a facility location problem with modular links. The goal is to decide the best location for a set of facilities to serve a set of demand points so that the total cost is minimized. This cost includes the cost for establishing the facilities and the cost for building the links between the operating facilities and the customers that will be served by them. For building these links we assume the existence of a set of different modules each of which with a specific capacity. Therefore, for each link, we have to decide the number of modules of each size that will be used. The facilities are assumed to be capacitated.

The problem is motivated by the fact that in many situations, the costs associated with the links between facilities and customers are much more complex than the straightforward linear term that is often considered in the literature. Examples can be found, for instance, in telecommunications when different cables are available to build the link between one demand point and a concentrator.

For the problem described above we propose a ‘traditional’ formulation and a reformulation using the so-called discretization technique. This technique has been fundamental in solving some combinatorial problems (see, for instance, Gouveia [6], Gouveia

and Saldanha-da-Gama [7] and Correia *et al.* [4]). As noted in these papers, one advantage of the discretized models is that the *discretized* variables suggest quite intuitive valid inequalities, which help to considerably improve the LP relaxation of the non-discretized model.

Not many papers in the literature have provided complete characterizations of reformulated discretized models with additional inequalities in the space of the original set of variables. An exception to this rule is the paper by Constantino and Gouveia [2], where it is shown that an enhanced discretized model gives an extended description of the convex hull of the feasible solutions of the original model. In the present paper we state a similar type of result arising in a different context namely, we state that a restricted version of the discretized model gives an extended description of the convex hull of the integer solutions of a subproblem that usually arises in network design problems with modular costs (see, for instance, Magnanti *et al.* [10] and the references in Frangioni and Gendron [5]).

The literature in locational analysis is rather scarce regarding models involving more general structures for the distribution costs. Holmberg [8], considers convex distribution/assignment costs for an uncapacitated location problem and Melkote and Daskin [11] consider a problem that includes set-up shipment costs for the links as well as a linear cost depending on the quantity to be shipped.

The remainder of this paper is organized as follows. In section 2 we present a formulation for the network loading problem with modular distribution costs. In section 3 we present a discretized formulation for the problem. A comparison between the formulations is presented in section 4. In section 5 we briefly present some valid inequalities for both the ‘traditional’ and the discretized models. In section 6 we summarize the computational experience that was performed on randomly generated data. The paper ends with some conclusions.

## 2. Definition of the problem

Denote by  $I$  the set of candidate locations for the facilities, each with capacity  $Q$ , and let  $J$  be the set of customers. For each customer  $j \in J$  let  $d_j$  be its (integer) demand. We also let  $f$  represent the (location-independent) fixed-cost for establishing a facility. We assume that several modules, of several sizes, can be installed on each facility-customer link. We denote by  $\{1, \dots, L\}$  the set of different types of modules. Let  $C^l$  designate the capacity of module type  $l$  and  $g^l$  the corresponding cost ( $l \in \{1, \dots, L\}$ ). Finally, let  $c_{ij}$  denote the unitary distribution cost on the link  $(i, j)$ .

In order to formulate the capacitated network loading problem introduced before, we consider the following decision variables: i) integer variables  $x_{ij}$  indicating the demand of customer  $j$  supplied from facility  $i$  ( $i \in I, j \in J$ ), ii) variables  $u_{ij}^l$  which represent the number of modules of type  $l$  to be installed in link  $(i, j)$  and iii) binary variables  $y_i$  indicating whether a facility is established at  $i$  ( $i \in I$ ). The problem can be formulated as follows:

$$(M) \quad \text{Min} \quad \sum_{i \in I} f y_i + \sum_{i \in I} \sum_{j \in J} \sum_{l=1}^L g^l u_{ij}^l + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \quad (1)$$

$$\text{s.t.} \quad \sum_{i \in I} x_{ij} = d_j \quad j \in J \quad (2)$$

$$\sum_{j \in J} x_{ij} \leq Q y_i \quad i \in I \quad (3)$$

$$x_{ij} \leq \sum_{l=1}^L C^l u_{ij}^l \quad i \in I, j \in J \quad (4)$$

$$u_{ij}^l \geq 0 \text{ and integer} \quad i \in I, j \in J, l \in \{1, \dots, L\} \quad (5)$$

$$x_{ij} \geq 0 \text{ and integer} \quad i \in I, j \in J \quad (6)$$

$$0 \leq y_i \leq 1 \text{ and integer} \quad i \in I \quad (7)$$

The objective function (1) minimizes the total cost, which includes the cost for installing the facilities and the cost associated with the links. Constraints (2) guarantee that the demand of each customer is satisfied while constraints (3) are the capacity constraints for the facilities. Constraints (4) state that the modules installed on link  $(i,j)$  have enough combined capacity to support the traffic on that link. Constraints (5), (6) and (7) are domain constraints.

### 3. The Discretized Model

A reformulation of (M) can be obtained by using the discretized variables  $z_{ij}^1, \dots, z_{ij}^Q$  for each link  $(i,j)$  (see, for instance, Gouveia [6]).  $z_{ij}^q$  ( $q = 1, \dots, Q$ ) indicates whether  $q$  units are shipped through link  $(i,j)$ . The “discretized” formulation is as follows

$$(DM+) \quad \text{Min} \quad \sum_{i \in I} f y_i + \sum_{i \in I} \sum_{j \in J} \sum_{q=1}^Q (\alpha_{ij}^q + q c_{ij}) z_{ij}^q \quad (8)$$

$$\text{s.t.} \quad \sum_{i \in I} \sum_{q=1}^Q q z_{ij}^q = d_j \quad j \in J \quad (9)$$

$$\sum_{j \in J} \sum_{q=1}^Q q z_{ij}^q \leq Q y_i \quad i \in I \quad (10)$$

$$\sum_{q=1}^Q z_{ij}^q \leq 1 \quad i \in I, j \in J \quad (11)$$

$$0 \leq z_{ij}^q \leq 1 \text{ and integer} \quad i \in I, j \in J, q = 1, \dots, Q \quad (12)$$

$$0 \leq y_i \leq 1 \text{ and integer} \quad i \in I \quad (13)$$

In this formulation,  $\alpha_{ij}^q$  denotes the lowest set-up cost for link  $(i,j)$  conditional to an installed capacity of at least  $q$  units, that is,  $\alpha_{ij}^q$  corresponds to the optimal value of the following knapsack problem

$$(Kp_{ij}^q) \quad \text{Min} \quad \sum_{l=1}^L g^l \mu_{ij}^l(q) \quad (14)$$

$$\text{s.t.} \quad \sum_{l=1}^L C^l \mu_{ij}^l(q) \geq q \quad (15)$$

$$\mu_{ij}^l(q) \geq 0 \text{ and integer} \quad l \in \{1, \dots, L\} \quad (16)$$

We denote by  $\mu_{ij}^{1*}(q), \mu_{ij}^{2*}(q), \dots, \mu_{ij}^{L*}(q)$  the optimal solution values of the variables of  $(Kp_{ij}^q)$ . In formulation (DM+), constraints (9) and (10) correspond to constraints (2) and (3). Constraints (11) ensure that in each link  $(i,j)$  no more than one variable  $z_{ij}^q$  can take the value one.

### 4. Comparing the Linear Programming Relaxation of the Models

Consider the following equations relating the  $x_{ij}$  and  $u_{ij}^l$  with the  $z_{ij}^q$  variables:

$$x_{ij} = \sum_{q=1}^Q qz_{ij}^q \quad i \in I, j \in J \quad (17)$$

$$u_{ij}^l = \sum_{q=1}^Q \mu_{ij}^{l*}(q)z_{ij}^q \quad i \in I, j \in J, l \in \{1, \dots, L\} \quad (18)$$

Note that for a given set of costs  $g^l$ , equations (18) give the optimal value of the variables  $u_{ij}^l$  in terms of the optimal values of the  $z_{ij}^q$  variables and vice-versa.

Now, for each pair  $(i, j)$  consider the feasible set

$$x_{ij} \leq \sum_{l=1}^L C^l u_{ij}^l \quad (4 \text{ for } (i, j))$$

$$u_{ij}^l \geq 0 \text{ and integer} \quad l \in \{1, \dots, L\} \quad (5 \text{ for } (i, j))$$

$$x_{ij} \leq Q \quad (19)$$

$$x_{ij} \geq 0 \text{ and integer} \quad (6 \text{ for } (i, j))$$

and denote by  $\text{ARC}(i, j)$  the convex hull of the corresponding set of feasible integer solutions. Note that constraints (19) are redundant when considered in model (M) but are not when defining this subset.

We note that in the case of the problem with just one single class of modules this linear system involves only two variables and it is easily characterized by adding the so-called arc residual inequalities (which were proposed by Magnanti *et al.* [9] for a more complicated problem).

Describing  $\text{ARC}(i, j)$ , becomes much more complicated when three (or more) variables are involved (the  $x_{ij}$  variable and the modular capacity variables  $u_{ij}^1, u_{ij}^2, \dots$ ). The only two other completeness results we know of, are given by Magnanti *et al.* [10] for two module classes (that is  $L = 2$ ) and  $C^l = 1$  where the authors introduce the so-called generalized arc residual inequalities and by Pochet and Wolsey [12] where they characterize the convex hull of  $\text{ARC}(i, j)$ , but where  $C^{l+1}$  is an integer multiple of  $C^l$ ,  $l \in \{1, \dots, L-1\}$ . The characterization given in this paper suggests that for the general case, a description of  $\text{ARC}(i, j)$  would be quite elusive. The readers are also referred to Agra and Constantino [1] which provide a rather complicated characterization for a similar problem involving only two integer variables and one continuous variable.

We let (M+) denote the formulation obtained from (M) by replacing (4), (5) and (6) by  $\text{ARC}(i, j)$  for all  $(i, j)$ . Clearly, by construction, we have that  $v(\overline{M+}) \geq v(\overline{M})$ . Moreover, for some instances this inequality is strict.

Consider now (M+<sub>ij</sub>) to be the formulation

$$\begin{aligned} (\text{M+}_{ij}) \quad \text{Min} \quad & \sum_{l=1}^L g^l u_{ij}^l + c_{ij} x_{ij} \\ \text{s.t.} \quad & \text{ARC}(i, j) \end{aligned} \quad (20)$$

Consider the alternative formulation (DM+<sub>ij</sub>) given by

$$\begin{aligned} (\text{DM+}_{ij}) \quad \text{Min} \quad & (20) \\ \text{s.t.} \quad & (11 \text{ for } (i, j)), (17 \text{ for } (i, j)), (18 \text{ for } (i, j)) \\ & (12 \text{ for } (i, j)), (5 \text{ for } (i, j)), (6 \text{ for } (i, j)) \end{aligned}$$

Considering that for any model  $P$  we denote by  $v(P)$  its optimal value and by  $v(\overline{P})$  the

optimal value of its linear programming relaxation, the following result holds:

**Result:** i)  $v(\overline{M +_{ij}}) = v(\overline{DM +_{ij}}), \forall (i, j).$

ii)  $v(\overline{M +}) = v(\overline{DM +})$

**Proof:** See Correia *et al.* [3].

With this result we show that the lower bound provided by the linear relaxation of (DM+) dominates (strictly, for some instances) the lower bound produced by the linear relaxation of model (M).

## 5. Valid Inequalities

The use of a discretized model is also very interesting because the discretized variables suggest new sets of valid inequalities. One set of such inequalities that result quite well in practice, can be obtained by dividing each term in the equalities (9) by an integer value  $p$  ( $p > 1$ ) and by rounding down each coefficient in the left-hand side term and then by rounding down the right-hand side term. We obtain:

$$\sum_{i \in I} \sum_{q=1}^Q \left\lfloor \frac{q}{p} \right\rfloor z_{ij}^q \leq \left\lfloor \frac{d_j}{p} \right\rfloor \quad j \in J, p = 2, \dots, Q \quad (21)$$

By using a similar reasoning, but rounding up the terms involved, we obtain:

$$\sum_{i \in I} \sum_{q=1}^Q \left\lceil \frac{q}{p} \right\rceil z_{ij}^q \geq \left\lceil \frac{d_j}{p} \right\rceil \quad j \in J, p = 2, \dots, Q \quad (22)$$

Two well known inequalities in the literature of location problems that can also be used are:

$$\sum_{i \in I} y_i \geq \left\lceil \frac{\sum_{j \in J} d_j}{Q} \right\rceil \quad (23)$$

$$x_{ij} \leq \min\{d_j, Q\} y_i \quad i \in I, j \in J \quad (24)$$

Another set of valid inequalities that are appropriate for the modular case is:

$$\sum_{i \in I} \sum_{l=1}^L \left\lfloor \frac{C^l}{p} \right\rfloor u_{ij}^l \geq \left\lfloor \frac{d_j}{p} \right\rfloor \quad j \in J, p = 2, \dots, C^L \quad (25)$$

These inequalities can be derived from

$$\sum_{i \in I} \sum_{l=1}^L C^l u_{ij}^l \geq d_j \quad j \in J \quad (26)$$

Any inequality in the space of the model (M) can be translated to the discretized model by using (17) and (18).

## 6. Computational Experiments

Some computational experience was performed on randomly generated data. Instances with 1, 2 and 3 different modules and up to 20 locations and 200 demand points were considered. The results were produced with the commercial package ILOG CPLEX 10.1. All the tests were

performed in a PC with a Pentium IV processor, 2.6 GHz and 512 Mb of RAM. More detailed computational results will be presented in the talk presented at INOC 2009 (see also Correia et al. [3]).

Here we summarize the main conclusions that were drawn:

i) A substantial gain is achieved both in terms of the linear programming relaxation bound and in terms of the time to obtain the integer optimum when we use model (DM+) instead of model (M).

ii) The linear programming relaxation bound of model (M) can be greatly enhanced by making use of the ‘classical’ inequalities (23), (24) together with the inequalities (25).

iii) The inclusion of the inequalities (23) to (DM+) leads to a reduction in the gaps. These reductions are even greater when constraints (24) are added to the model. Nevertheless, it is with the inclusion of the new constraints (21) and (22) that a decisive and substantial reduction in gaps is obtained, without a significant increase in the computational effort.

iv) For all models, the time required to solve the linear programming relaxation is not significant.

v) When obtaining the optimal integer solutions, the performance of the models considered (namely (DM+) augmented with inequalities (21) and (22)), does not deteriorate when the complexity of the instances increases, in particular when the number of modules changes from 1 to 2 or 3.

Summing up, the computational experience carried out show that model (DM+) enhanced with the valid inequalities is extremely competitive and suggests, in our opinion, that these types of models may play a significant role in solving network design problems with modular costs.

## 7. Conclusions

In this paper we have studied discretized formulations for a capacitated network loading problem with modular costs in the links arising in a facility location context.

We have presented a so-called ‘traditional’ model and a straightforward discretized model with a general objective function and whose linear programming relaxation is better than the linear programming relaxation of the original model. In order to explain this dominance, we have shown that a restricted version of the discretized model gives an extended description of the convex hull of a “small” polytope that is included in the original model. Several classes of valid inequalities are presented. The models were tested on randomly generated data. The results showed that the discretized model enhanced with the valid inequalities is extremely robust for solving modular cost problems since: i) it produces very tight linear relaxation bounds for many of the instances tested and ii) it solves to optimality many instances that the ‘traditional’ models could not solve.

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