# The Complexity of the Node Capacitated In-Tree Packing Problem


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**Abstract**

In this paper, we deal with a node capacitated in-tree packing problem. The input consists of a directed graph, a root node, a node capacity function and edge consumption functions. The problem is to find the maximum number of rooted in-trees such that the total consumption of in-trees at each node does not exceed the capacity of the node. The problem is one of the network lifetime problems that are among the most important issues in the context of sensor networks. We reveal the computational complexity of the problem under various restrictions on consumption functions and graphs. For example, we consider general graphs, acyclic graphs and complete graphs embedded in the $d$-dimensional space $\mathbb{R}^d$ having edge consumption functions depending on distances between end nodes.



**Keywords**: *Network design, combinatorial optimization, computational complexity, wireless ad hoc network, sensor network.*


## 1 Introduction

In this paper, we consider a *node capacitated in-tree packing problem.* The input consists of a directed graph, a root node, a node capacity function and edge consumption functions. The problem is to find the maximum number of rooted in-trees such that the total consumption of in-trees at each node does not exceed the capacity of the node.

Formally, let $G = (V, E)$ be a directed graph. We call a subset $T \subseteq E$ *in-tree* if $(V, T)$ is a directed spanning rooted in-tree (i.e., incoming arborescence). Let $\mathbb{R}_+$ be the set of nonnegative real numbers. Let $h : E \to \mathbb{R}_+$ and $t : E \to \mathbb{R}_+$ be a head and a tail consumption function on directed edges, respectively. The consumption $c(T, v)$ of an in-tree $T$ at a node $v \in V$ is defined as

$$c(T, v) := \sum_{e \in \delta_T^-(v)} h(e) + \sum_{e \in \delta_T^+(v)} t(e),$$

where $\delta_{E'}^-(v)$ (resp., $\delta_{E'}^+(v)$) is the set of edges in $E'$ entering (resp., leaving) $v$. We call the first term of the above equation *head consumption*, and the second term *tail consumption*. Let $b : V \to \mathbb{R}_+$ be a node capacity function. The node capacitated in-tree packing problem is to find a maximum size set of in-trees $\mathcal{T}$ rooted at the given root $r \in V$ such that

$$\sum_{T \in \mathcal{T}} c(T, v) \le b(v), \ \forall v \in V.$$

Note that the set of in-trees $\mathcal{T}$ is a multiset, i.e., it may include same in-trees.

The aim of this paper is to reveal the computational complexity of several variations of the problem. We deal with general graphs, acyclic graphs and complete graphs embedded in the $d$-dimensional space $\mathbb{R}^d$ as inputs. When the graph is embedded in $\mathbb{R}^d$, we assume that the consumptions of each edge depend only on the distance between its end nodes, and we call such a problem *metric*. More precisely, let $(x_1(v), \ldots, x_d(v)) \in \mathbb{R}^d$ be the $d$-dimensional coordinate of a node $v$. We denote the $L^p$ norm between $v$ and $w$ as $L^p(v, w) = (|x_1(v) - x_1(w)|^p + \cdots + |x_d(v) - x_d(w)|^p)^{1/p}$. In this paper, the parameter $p \in \{1, \ldots, \infty\}$ is a fixed constant. The head and tail consumption functions are called metric if their values for each edge are proportional to the $L^p$ norm between its end nodes (i.e., there exist constants $\alpha$ and $\beta$ such that for each edge $e = (v, w)$ in $E$, $h(e) = \alpha L^p(v, w)$ and $t(e) = \beta L^p(v, w)$ hold), and a problem is metric if the consumption functions are metric. Our results are summarized as follows:

- packing one in-tree
  - without head consumptions: polynomially solvable
  - with head consumptions on acyclic graphs: strongly NP-hard
  - with metric head consumptions: strongly NP-hard
- packing in-trees (in general)
  - without head consumptions
    - on acyclic graphs: polynomially solvable
    - on general graphs: strongly NP-hard
    - with metric tail consumptions: strongly NP-hard
  - with head consumptions: strongly NP-hard

Recently, several kinds of graph packing problems are studied in the context of ad hoc wireless networks and sensor networks. These problems are called *network lifetime problems*. Included among the important problems in this category are the node capacitated spanning subgraph packing problems considered in [1, 6, 9]. For sensor networks, for example, a spanning subgraph corresponds to a communication network topology for collecting information from all nodes (sensors) to the root (base station) or for sending information from the root to all other nodes. Sending a message along an edge consumes energy at end nodes, usually depending on the distance between them. Battery capacities of sensor nodes are severely limited. It is therefore important to design the topologies for communication to save energy consumption and make sensors operate as long as possible. The merit of designing such topologies in a centralized manner was indicated in [6], in which a cluster-based network organization algorithm, called LEACH-C (low energy adaptive clustering hierarchy centralized), was proposed. As the topology for communication, they considered in-trees with a bounded height. For more energy efficient communication networks, a multi-round topology construction problem was formulated as a mathematical programming problem, and a heuristic solution method was proposed [9]. The objective of the problem is to maximize the number of packed in-trees (n.b., spanning in-trees without any additional constraints are considered). In the formulation of [1], head consumptions are not considered, and the consumption at each node is the maximum tail consumption among the edges leaving the node. There are variations of the problem with respect to additional conditions on the spanning subgraph such as strong connectivity, symmetric connectivity, and being a directed out-tree rooted at a given node. The authors discussed the hardness of the problem and proposed several approximation algorithms.

These network lifetime problems, including our problem, are similar to the well-known *edge-disjoint spanning arborescence packing problem*: Given a directed graph $G = (V, E)$ and a root $r$ in $V$, find the maximum number of edge-disjoint arborescences rooted at $r$. Note that the given graph $G$ may have parallel edges. The edge-disjoint arborescence packing problem is solvable in polynomial time [3, 7]. Its capacitated version is also solvable in polynomial time [4, 8, 10].

The rest of this paper is organized as follows. In Section 2, we show the computational complexity of a special case of our problem, packing one in-tree. In Section 3, we show the complexity of the problem in general.

In this paper, we assume that the given root node is reachable from all other nodes, otherwise the node capacitated in-tree packing problem is obviously infeasible.

## 2 Packing One In-Tree

In this section, we consider the decision problem of the node capacitated one in-tree packing problem: Given an instance of the node capacitated in-tree packing problem, decide whether the instance has a packing with one in-tree.

We first consider the problem without head consumptions, i.e., $h(e) = 0, \ \forall e \in E$. In this case, the consumption of a spanning in-tree at each node is caused by exactly one edge leaving the node. Let $E' \subseteq E$ be the set of all edges whose tail consumptions are at most the capacity of their tail nodes, i.e., $\forall e' \in E', \ t(e') \leq b(\mathrm{tail}(e'))$, where $\mathrm{tail}(e)$ is the tail node of a directed edge $e$. Because the node capacitated one in-tree packing problem on $(V, E)$ is equivalent to the problem of finding an in-tree in $(V, E')$, the following lemma holds.

**Lemma 1** *The node capacitated one in-tree packing problem without head consumptions is solvable in* $O(|E|)$ *time.*

Next we consider the problem with head consumptions. In this case, the decision problem of the node capacitated one in-tree packing problem is strongly NP-hard.

**Theorem 1** *The decision problem of the node capacitated one in-tree packing problem is strongly NP-hard. The problem is still strongly NP-hard, even when the given graph is acyclic and there are no tail consumptions.*

**Proof (Outline):** We show that the bin-packing problem polynomially transforms to the above decision problem without tail consumptions. The decision problem of the bin-packing is known to be strongly NP-complete [5].

Let $\{1, \ldots, k\}$ be the set of bins whose capacities are a positive integer $B$. Let $n$ be the number of items to be packed into bins and $s_i$ $(i = 1, \ldots, n)$ be the sizes of items satisfying $s_i \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of nonnegative integers. We note that, as the bin-packing problem is strongly NP-complete, the bin-packing problem is NP-complete even when $\max\{s_i : i \in I\}$ and $B$ are bounded by a polynomial of $n$.

For the set of $k$ bins, we introduce a set of nodes $U := \{u_1, \ldots, u_k\}$. For each item $i$, we introduce a node $w_i$. Let $W := \{w_1, \ldots, w_n\}$. We also introduce a root node $r$. Edges are emanating from each node in $W$ to all nodes in $U$, and emanating from each node in $U$ to the root $r$. We set a head consumptions $h(e) := s_i$ if the edge $e$ is leaving $w_i \in W$, and 0 otherwise. We set the capacity of a node to $B$ if the node corresponds to a bin, and 0 otherwise. See Figure 1 for an example. The size of the resulting instance of the in-tree packing problem is bounded by a polynomial of $n$, and the largest number in the instance is the same as that of bin-packing. It is obvious that the in-tree packing has a feasible packing of size one if and only if the bin-packing has a feasible packing. □

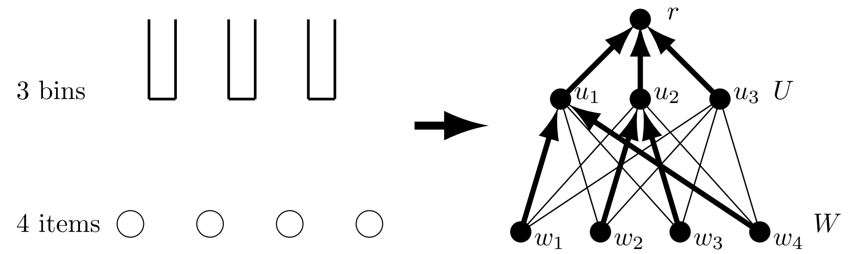


Figure 1: An example of the transformation from the bin-packing problem

The metric case of the problem is also NP-hard even when the graph is embedded in the 1-dimensional space $\mathbb{R}^1$ and there are no tail consumptions. Though we showed that the bin-packing problem polynomially transforms to this special case, we omit the details in this short paper.

# 3 Packing In-Trees (in General)

In this section, we deal with the general case of the node capacitated in-tree packing problem. From the results in the previous section, the node capacitated in-tree packing problem with head consumptions is strongly NP-hard. We therefore concentrate on the case without head consumptions in this section.

We first consider the case where the given graph is acyclic. When a given graph is acyclic, an in-tree of the graph is easily found.

**Proposition 1** *When a graph $G = (V, E)$ is acyclic, a set of edges $E' \subseteq E$ is an in-tree if the outdegree in $(V, E')$ is one for all nodes except for the root.*

From Proposition 1, we can easily see that the node capacitated in-tree packing problem without head consumptions is solvable in $O(|E|)$ time.

**Lemma 2** *When the given graph is acyclic and there are no head consumptions, the node capacitated in-tree packing problem is solvable in $O(|E|)$ time.*

**Proof:** Let $G = (V, E)$ be the given graph. Let $r \in V$ be the given root node. For all $v \in V \setminus \{r\}$, we let $e_{\min}(v)$ be an edge leaving $v$ such that the tail consumption is minimum among all edges leaving $v$, i.e.,

$$e_{\min}(v) := e \in \delta^+_E(v) \quad \text{such that} \quad t(e) \leq t(e'),\ \forall e' \in \delta^+_E(v).$$

Then the set of edges $\{e_{\min}(v) : v \in V \setminus \{r\}\}$ is an in-tree from Proposition 1. An optimal in-tree packing can be obtained by packing this tree as many as possible. □

**Theorem 2** *The following problem is strongly NP-hard: Given an instance of the node capacitated in-tree packing problem and a number $n$, decide whether the instance has a packing of size $n$. The problem is still strongly NP-hard, even when there are no head consumptions.*

**Proof:** We show that the bin-packing problem (known to be strongly NP-complete) polynomially transforms to the above decision problem without head consumptions.

Let $\{1, \dots, k\}$ be the set of bins whose capacities are $B \in \mathbb{Z}_+$. Let $n$ be the number of items to be packed into bins and $s_i$ $(i = 1, \dots, n)$ be the sizes of items satisfying $s_i \in \mathbb{Z}_+$.

For the set of $k$ bins, we introduce a set of nodes $U := \{u_1, \ldots, u_k\}$. For an item $i$, we introduce a node $w_i$. Let $W := \{w_1, \ldots, w_n\}$. We also introduce a root node $r$. Edges are emanating from each node in $W$ to the root $r$, connecting nodes in $U$ and $W$ with each other, and connecting all nodes in $U$ with each other. We set a tail consumption $s_i$ to edges entering $w_i$, 1 to edges entering the root, and 0 to others. We set the capacity of each node in $W$ to 1, each node in $U$ to $B$, and the root to 0. See Figure 2 for an example. The size of the resulting instance is bounded by a polynomial of $n$, and the largest number in it is the same as that of bin-packing.

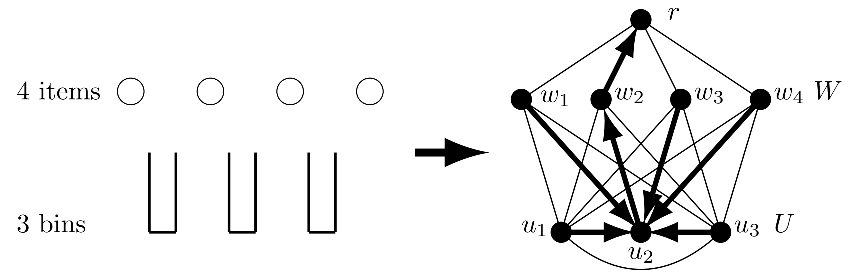


Figure 2: An example of the transformation from the bin-packing problem

Suppose the in-tree packing has a packing $\mathcal{T} := \{T_1, \ldots, T_n\}$. Because the capacity of each node in $W$ is the same as the tail consumption of each edge entering the root, each edge $(w_i, r)$ appears in exactly one tree in $\mathcal{T}$. Let $T_i$ be the in-tree containing an edge $(w_i, r)$. In the tree $T_i$, every edge leaving from $W \setminus w_i$ must enter a node in $U$. Because $T_i$ is an in-tree, there must be at least one edge leaving $U$ and entering $w_i$. Let $(u_j, w_i)$ be such an edge. Then, a feasible solution to the bin-packing problem is obtained by packing each item $i$ into the bin $j$. Similarly, the opposite holds. □

In the following, we show that the node capacitated in-tree packing problem is still hard when the problem is *metric*. The metric problem is quite natural in the context of wireless radio networks, because the radio intensity decreases according to the distance [6]. In other words, as the distance between nodes becomes larger, a radio transmitter consumes more energy.

**Theorem 3** *The following problem is strongly NP-hard: Given a number $n$ and an instance of the node capacitated metric in-tree packing problem, decide whether the instance has an in-tree packing of size $n$. The problem is still strongly NP-hard, even when there are no head consumptions and the given graph is embedded in the 1-dimensional space $\mathbb{R}^1$.*

**Proof:** We show that the bin-packing problem polynomially transforms to the above decision problem such that there are no head consumptions and the given graph is embedded in $\mathbb{R}^1$.

Let $\{1, \ldots, k\}$ be the set of bins whose capacities are $B \in \mathbb{Z}_+$. Let $n$ be the number of items to be packed into bins and $s_i \in \mathbb{Z}_+$ $(i = 1, \ldots, n)$ be the sizes of items. We assume that $\max\{s_i : i \in I\}$ and $B$ are bounded by a polynomial of $n$; even with this restriction, the bin-packing problem is NP-complete [5].

We define $s_0 = 0$. For simplicity, we first assume that all the items have different sizes, and $0 = s_0 < s_1 < s_2 < \cdots < s_n \leq B$ holds. For the set of $k$ bins, we introduce a set of nodes $U := \{u_1, \ldots, u_k\}$ whose coordinates are 0. For each item $i$, we introduce a node $w_i$ whose coordinate is $3s_i$ and a set of nodes $V_i := \{v_{i1}, v_{i2}, \ldots, v_{iN_i}\}$ whose coordinates are $3s_{i-1}+1, 3s_{i-1}+2, \ldots, 3s_i-2$ where $N_i := 3(s_i - s_{i-1}) - 2$. Let $W := \{w_1, \ldots, w_n\}$ and $V := V_1 \cup V_2 \cup \cdots \cup V_n$. We also introduce a root node $r$ whose coordinate $M$ is sufficiently large (e.g., $M = 3B + 2n$). All pairs of nodes are connected with each other. The tail consumption of each edge is the distance between its end nodes. The capacity $b(v)$ of each node $v$ is defined as follows:

$$b(v) := \begin{cases} 0 & (v = r), \\ 3B & (v \in U), \\ M - 3s_i + 2(n-1) & (v = w_i \in W), \\ n & (v \in V). \end{cases}$$

See Figure 3 for an example (4 items and 3 bins). Because we assumed that $s_i$ ($\forall i \in I$) and $B$ are bounded by a polynomial of $n$, the size of the resulting instance and the largest number in it are bounded by a polynomial of $n$.

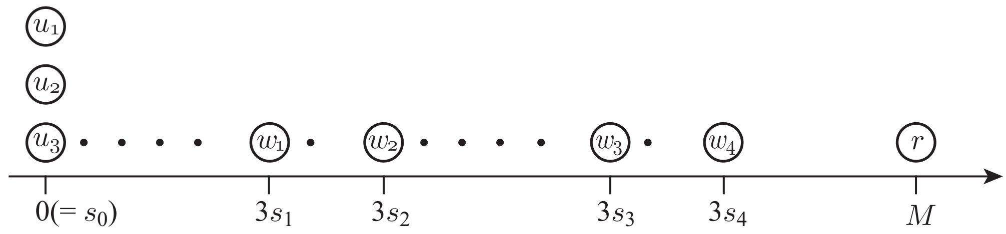


Figure 3: An example of embedded nodes

Suppose the metric in-tree packing has a packing $\mathcal{T} := \{T_1, \ldots, T_n\}$. Each node $v \in V$ must connect to the nearest node on its left side for all $n$ in-trees. Because $M$ is sufficiently large, edges connecting nodes in $U$ and $r$ never appear in the in-trees. Because $b(w_i) = M - 3s_i + 2(n-1)$ ($w_i \in W$) and $M$ is sufficiently large, for any index $i$, the edge $(w_i, r)$ appears at most once in the in-trees. From the setting of $b(w_i)$, once the edge $(w_i, r)$ is used, $(w_i, v_{iN_i})$ must be used in the rest of in-trees. As a result, for each $i \in \{1, \ldots, n\}$, there exists an index $j \in \{1, \ldots, k\}$ such that the in-tree containing $(w_i, r)$ is composed of $(u_j, w_i)$, $(w_i, r)$, $(w_l, v_{lN_l})$ for each $w_l \in W \setminus \{w_i\}$, $(u_l, u_j)$ for each $u_l \in U \setminus \{u_j\}$, and edges leaving from each $v \in V$ to its left neighbor. See Figure 4 for an example. Then edges $(u_j, w_i)$ indicate a feasible solution of the bin-packing problem. Similarly, the opposite holds.

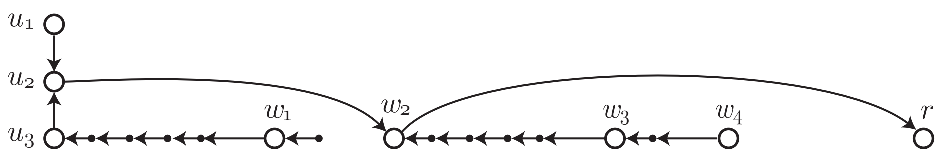


Figure 4: An in-tree of a feasible packing

Using symbolic perturbation technique with additional $\lceil \log_2 n \rceil$ bits [2], a problem instance including items of the same size transforms to the problem such that all the items have different sizes. When the graph is embedded in $\mathbb{R}^d$ and the tail consumption is the $L^p$ norm, the above 1-dimensional case is included. □

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