Lagrangean Decomposition for the Fixed-Charge Multicommodity Capacitated Network Design Problem

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Abstract

Traditional Lagrangean relaxations for the fixed-charge multicommodity capacitated network design problem (MCNDP) involve dualizing either capacity or flow conservation constraints. The former (shortest-path relaxation) results in loosing the capacity structure whereas the latter (knapsack relaxation) does not maintain any information related to the network structure. Furthermore, both relaxations yield bounds that are at best equal to the value of the linear programming (LP) relaxation. This paper describes a new relaxation for the MCNDP, based on Lagrangean decomposition, through which the resulting subproblems partially preserve both the network and the capacity structure of the original problem. This is, to the best of the authors' knowledge, the first Lagrangean relaxation proposed for the MCNDP that may yield stronger bounds than the LP relaxation.

Keywords: *Multicommodity network design, Lagrangean decomposition, LP relaxation, Lower bound.*

1 Introduction

We are concerned with the fixed-charge multicommodity capacitated network design problem (MCNDP) defined on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ where $\mathcal{V}$ is the set of nodes and $\mathcal{A}$ is the set of links. Let $\mathbf{y} = \{y_{ij} | (i,j) \in \mathcal{A}\}$ denote the vector of design variables with $\mathbf{y} \in \mathcal{Y} = \{0,1\}^{|\mathcal{A}|}$, where y_{ij} is an indicator of whether or not link (i,j) is established in the network. For each link activated in the network, there is a fixed-charge cost denoted by f_{ij}. There exists a set of commodities denoted by $\mathcal{P}$. Each commodity has one origin $o(p)$ and one destination $d(p)$, and the quantity of commodity p that is to be sent from $o(p)$ to $d(p)$ is denoted by w^p. If a commodity has more than one origin or destination, this can be modelled by splitting the commodity into several commodities, each with a single origin and destination (see [4]). Let $\mathbf{x} = \{x_{ij}^p | (i,j) \in \mathcal{A}, p \in \mathcal{P}\}$ denote the vector of variables with $x \in \mathcal{X} = \mathbb{R}_+^{|\mathcal{A}||\mathcal{P}|}$, where x_{ij}^p denotes the amount of commodity p flowing on (i,j). We denote by c_{ij}^p the (nonnegative) unit cost of routing the demand for commodity p over link (i,j). Each link has a nonnegative capacity, restricting the total amount of flow on this link by an amount u_{ij}. For each $(i,j) \in \mathcal{A}$, an upper bound $b_{ij}^p = \min\{w^p, u_{ij}\}$ may be imposed on the flow of commodity $p \in \mathcal{P}$. For each node $i \in \mathcal{N}$, we define the sets $\mathcal{N}_i^+ = \{j \in \mathcal{N} | (i,j) \in \mathcal{A}\}$ and $\mathcal{N}_i^- = \{j \in \mathcal{N} | (j,i) \in \mathcal{A}\}$. The MCNDP consists of activating links in the network, and determining the amount of flow to be routed on each activated link such that the demand of each node is met and the total cost (composed of the total fixed cost of the selected links and the variable cost of transporting commodities over links) is minimized. The formulation for the MCNDP is given as follows:

$$(\mathcal{F}) \qquad \text{Minimize} \sum_{(i,j)\in\mathcal{A}} f_{ij} y_{ij} + \sum_{(i,j)\in\mathcal{A}} \sum_{p\in\mathcal{P}} c_{ij}^p x_{ij}^p \tag{1}$$

subject to

$$\sum_{j\in\mathcal{N}_i^+} x_{ij}^p - \sum_{j\in\mathcal{N}_i^-} x_{ji}^p = d_i^p \qquad \forall i \in \mathcal{N}, p \in \mathcal{P} \tag{2}$$

$$\sum_{p\in\mathcal{P}} x_{ij}^p \leq u_{ij} y_{ij} \qquad \forall (i,j) \in \mathcal{A} \tag{3}$$

$$x_{ij}^p \leq b_{ij}^p y_{ij} \qquad \forall (i,j) \in \mathcal{A}, p \in \mathcal{P} \tag{4}$$

$$\mathbf{y} \in \mathcal{Y} \tag{5}$$

$$\mathbf{x} \in \mathcal{X}, \tag{6}$$

where

$$d_i^p = \begin{cases} w^p, & \text{if node } i = o(p) \\ -w^p, & \text{if node } i = d(p) \\ 0, & \text{otherwise.} \end{cases}$$

In $\mathcal{F}$, equalities (2) are named as flow constraints which ensure that the demand of each node is met. Constraints (3) ensure that link capacities are respected. Constraints (4) make sure that the flow of any commodity on a link is zero when that link is not activated. We refer to the latter two as capacity constraints.

The success of the solution algorithms proposed for the MCNDP, such as branch-and-bound (see, e.g., [4]) and branch-and-cut (see, e.g., [1]) is, in part, due to efficient calculation of lower bounds of the problem, for which Lagrangean relaxation is a popular approach. Traditional Lagrangean relaxations for the MCNDP are performed through dualizing either capacity or flow constraints. As for the former, constraints (4) and (3) are relaxed, as a result of which the problem decomposes into smaller subproblems, one for each commodity, where each subproblem can be solved by a shortest path algorithm. Under this so-called *shortest-path relaxation*, the relaxed problems do not maintain any information related to link capacities. The latter relaxation, named as the *knapsack relaxation*, lies in penalizing constraints (2), through which one obtains a decomposition of the relaxed problem into a set of knapsack-like subproblems. These subproblems do not, however, maintain any information related to the structure of the network. Both of the above-mentioned relaxations yield bounds that are at best equal to the value of the LP relaxation (for more details see [3, 4, 2]). We describe a new relaxation in the next section.

2 A New Relaxation Based on Lagrangean Decomposition

In this section, we describe a new relaxation for the MCNDP by duplicating the link and the flow variables in a special manner, giving way to a decomposition where the subproblems partially preserve *both* flow balance and knapsack constraints. We first make the following observation related to variable fixing, which will be used in the ensuing exposition. Since all the costs are nonnegative in formulation $\mathcal{F}$, it follows that for all $p \in \mathcal{P}$, one can fix $x_{ij}^p = 0$ if $o(p) = j$ and similarly, $x_{ij}^p = 0$ if $d(p) = i$.

We duplicate the flow and split the link variables in the following manner. We define a set of new variables as z_{ij}^p and v_{ij}^p for every $(i,j) \in \mathcal{A}$ and $p \in \mathcal{P}$ such that $z_{ij}^p = x_{ij}^p$ and $v_{ij}^p = y_{ij}$. These new variables are used in the following extended (equivalent) formulation for the MCNDP.

$$(\mathcal{L}) \qquad \text{Minimize} \quad \sum_{(i,j)\in\mathcal{A}} f_{ij} y_{ij} + \sum_{(i,j)\in\mathcal{A}} \sum_{p\in\mathcal{P}} \frac{c_{ij}^p}{2} x_{ij}^p + \sum_{(j,i)\in\mathcal{A}} \sum_{p\in\mathcal{P}} \frac{c_{ji}^p}{2} z_{ji}^p \tag{7}$$

subject to

$$\sum_{j\in\mathcal{N}_i^+} x_{ij}^p - \sum_{j\in\mathcal{N}_i^-} z_{ji}^p = d_i^p \qquad \forall i \in \mathcal{N}, p \in \mathcal{P} \tag{8}$$

$$\sum_{p\in\mathcal{P}} \frac{1}{2} x_{ij}^p + \sum_{p\in\mathcal{P}} \frac{1}{2} z_{ij}^p \leq u_{ij} y_{ij} \qquad \forall (i,j) \in \mathcal{A} \tag{9}$$

$$z_{ij}^p = x_{ij}^p \qquad \forall (i,j) \in \mathcal{A}, p \in \mathcal{P} \tag{10}$$

$$v_{ij}^p = y_{ij} \qquad \forall (i,j) \in \mathcal{A}, p \in \mathcal{P} \tag{11}$$

$$x_{ij}^p \leq b_{ij}^p v_{ij}^p \qquad \forall (i,j) \in \mathcal{A}, p \in \mathcal{P} \tag{12}$$

$$z_{ji}^p \leq b_{ji}^p v_{ji}^p \qquad \forall (j,i) \in \mathcal{A}, p \in \mathcal{P} \tag{13}$$

$$\mathbf{y} \in \mathcal{Y}, \mathbf{x}, \mathbf{z} \in \mathcal{X}, \mathbf{v} \in \mathcal{V},$$

where $\mathbf{z} = \{z_{ij}^p | (i,j) \in \mathcal{A}, p \in \mathcal{P}\}$ and $\mathcal{V} = \{0,1\}^{|\mathcal{A}||\mathcal{P}|}$.

Constraints (9), (10) and (11) are relaxed in formulation $\mathcal{L}$ by associating multipliers $\beta_{ij} \geq 0$, ξ_{ij}^p, and θ_{ij}^p, respectively. The resulting relaxed problem takes the following form:

$$\begin{aligned}(\mathcal{L}(\beta,\xi,\theta)) \qquad \text{Minimize} \quad & \sum_{(i,j)\in\mathcal{A}} (f_{ij} - u_{ij}\beta_{ij} - \sum_{p\in\mathcal{P}} \theta_{ij}^p) y_{ij} \\ & + \sum_{(i,j)\in\mathcal{A}} \sum_{p\in\mathcal{P}} \{(\frac{c_{ij}^p}{2} + \frac{\beta_{ij}}{2} - \xi_{ij}^p) x_{ij}^p + \frac{\theta_{ij}^p}{2} v_{ij}^p\} \\ & + \sum_{(j,i)\in\mathcal{A}} \sum_{p\in\mathcal{P}} \{(\frac{c_{ji}^p}{2} + \frac{\beta_{ji}}{2} + \xi_{ji}^p) z_{ji}^p + \frac{\theta_{ji}^p}{2} v_{ji}^p\},\end{aligned} \tag{14}$$

subject to (8), (12), (13) and $\mathbf{y} \in \mathcal{Y}, \mathbf{x}, \mathbf{z} \in \mathcal{X}, \mathbf{v} \in \mathcal{V}$.

This problem decomposes into a subproblem depending on the y variables only, $\mathcal{L}^Y(\beta,\xi,\theta)$, and a subproblem involving the x, z and v variables, $\mathcal{L}^{XV}(\beta,\xi,\theta)$. We first analyze the structure of the last subproblem. To this end, we define the following sets, which define a partition of $\mathcal{P}$ for any node i:

- $\mathcal{P}_i^O = \{p \in \mathcal{P} | i = o(p)\}$,
- $\mathcal{P}_i^D = \{p \in \mathcal{P} | i = d(p)\}$,
- $\mathcal{P}_i^T = \mathcal{P} \setminus \{\mathcal{P}_i^O \cup \mathcal{P}_i^D\}$.

The variable fixing described above extends here in that we must have $x_{ij}^p = 0$, $\forall (i,j) \in \mathcal{A}, p \in \mathcal{P}_i^D$ and $z_{ji}^p = 0$, $\forall (j,i) \in \mathcal{A}, p \in \mathcal{P}_i^O$ in any optimal solution. Using this fact, we can simplify the constraints of $\mathcal{L}^{XV}(\beta,\xi,\theta)$. Together with the observation that the subproblem decomposes by nodes and commodities, this yields the three following formulations, the first one defined for each node i and each commodity $p \in \mathcal{P}_i^O$, the second one relative to each node i and each commodity $p \in \mathcal{P}_i^T$, and the third one corresponding to each node i and each commodity $p \in \mathcal{P}_i^D$:

$$g_i^p(O) = \text{Minimize} \qquad \sum_{j\in\mathcal{N}_i^+} \{(\frac{c_{ij}^p}{2} + \frac{\beta_{ij}}{2} - \xi_{ij}^p) x_{ij}^p + \frac{\theta_{ij}^p}{2} v_{ij}^p\} \tag{15}$$

subject to

$$\sum_{j \in \mathcal{N}_i^+} x_{ij}^p = w^p \tag{16}$$

$$x_{ij}^p \leq b_{ij}^p v_{ij}^p \qquad \forall j \in \mathcal{N}_i^+ \tag{17}$$

$$\mathbf{x} \in \mathcal{X}, \mathbf{v} \in \mathcal{V}.$$

$$g_i^p(T) = \text{Minimize} \qquad \sum_{j \in \mathcal{N}_i^+} \{(\frac{c_{ij}^p}{2} + \frac{\beta_{ij}}{2} - \xi_{ij}^p)x_{ij}^p + \frac{\theta_{ij}^p}{2} v_{ij}^p\} + \sum_{j \in \mathcal{N}_i^-} \{(\frac{c_{ji}^p}{2} + \frac{\beta_{ji}}{2} + \xi_{ji}^p)z_{ji}^p + \frac{\theta_{ji}^p}{2} v_{ji}^p\} \tag{18}$$

subject to

$$\sum_{j \in \mathcal{N}_i^+} x_{ij}^p - \sum_{j \in \mathcal{N}_i^-} z_{ji}^p = 0 \tag{19}$$

$$x_{ij}^p \leq b_{ij}^p v_{ij}^p \qquad \forall j \in \mathcal{N}_i^+ \tag{20}$$

$$z_{ji}^p \leq b_{ji}^p v_{ji}^p \qquad \forall j \in \mathcal{N}_i^- \tag{21}$$

$$\mathbf{x}, \mathbf{z} \in \mathcal{X}, \mathbf{v} \in \mathcal{V}.$$

$$g_i^p(D) = \text{Minimize} \qquad \sum_{j \in \mathcal{N}_i^-} \{(\frac{c_{ji}^p}{2} + \frac{\beta_{ji}}{2} + \xi_{ji}^p)z_{ji}^p + \frac{\theta_{ji}^p}{2} v_{ji}^p\} \tag{22}$$

subject to

$$\sum_{j \in \mathcal{N}_i^-} z_{ji}^p = w^p \tag{23}$$

$$z_{ji}^p \leq b_{ji}^p v_{ji}^p \qquad \forall j \in \mathcal{N}_i^- \tag{24}$$

$$\mathbf{z} \in \mathcal{X}, \mathbf{v} \in \mathcal{V}.$$

The optimal value of subproblem $\mathcal{L}^{XV}(\beta, \xi, \theta))$ is then computed as:

$$\sum_{i \in \mathcal{N}} \sum_{p \in \mathcal{P}_i^O} g_i^p(O) + \sum_{i \in \mathcal{N}} \sum_{p \in \mathcal{P}_i^T} g_i^p(T) + \sum_{i \in \mathcal{N}} \sum_{p \in \mathcal{P}_i^D} g_i^p(D). \tag{25}$$

The first and third structures correspond to the single-node fixed-charge network design problem with only outbound or inbound flows, which has first been studied in [5]. The second structure is the single-node fixed-charge network design problem with both outbound and inbound flows, which was introduced in [6]. Both problems do not have the integrality property. It follows that the lower bound obtained by our Lagrangean decomposition can be strictly better than the LP relaxation bound.

Now, we look into the structure of subproblem $\mathcal{L}^Y(\beta, \xi, \theta)$, which can be written as follows:

$$(\mathcal{L}^Y(\beta, \xi, \theta)) \qquad \text{Minimize} \sum_{(i,j) \in \mathcal{A}} (f_{ij} - u_{ij}\beta_{ij} - \sum_{p \in \mathcal{P}} \theta_{ij}^p) y_{ij}, \tag{26}$$

subject to $\mathbf{y} \in \mathcal{Y}$. Clearly, this problem can be solved by simple inspection of the signs of the costs. We can also strengthen this subproblem, and hence the Lagrangean bound, by adding valid inequalities involving only the y variables. The simplest of these inequalities are the following single-node cutset inequalities:

$$\sum_{j \in \mathcal{N}_i^+} u_{ij} y_{ij} \geq \sum_{p \in \mathcal{P}_i^O} w^p \qquad \forall i \in \mathcal{N}, \tag{27}$$

$$\sum_{j \in \mathcal{N}_i^-} u_{ji} y_{ji} \geq \sum_{p \in \mathcal{P}_i^D} w^p \qquad \forall i \in \mathcal{N}. \tag{28}$$

It is even possible to gradually add to subproblem $\mathcal{L}^Y(\beta, \xi, \theta)$) other valid inequalities involving only the y variables, thus further strengthening the quality of the Lagrangean lower bound. This is the approach we suggest in conjunction with a heuristic based on our Lagrangean decomposition, which is described next.

3 A Lagrangean Heuristic

Each time subproblem $\mathcal{L}^Y(\beta, \xi, \theta)$ is solved, we have at hand a tentative design $\overline{y}$ over which we can solve the following multicommodity network flow problem:

$$h(\overline{y}) = \text{Minimize} \sum_{(i,j) \in \mathcal{A}} \sum_{p \in \mathcal{P}} c_{ij}^p x_{ij}^p \tag{29}$$

subject to

$$\sum_{j \in \mathcal{N}_i^+} x_{ij}^p - \sum_{j \in \mathcal{N}_i^-} x_{ji}^p = d_i^p \qquad \forall i \in \mathcal{N}, p \in \mathcal{P} \tag{30}$$

$$\sum_{p \in \mathcal{P}} x_{ij}^p \leq u_{ij} \overline{y}_{ij}, \qquad \forall (i,j) \in \mathcal{A} \tag{31}$$

$$\mathbf{x} \in \mathcal{X}. \tag{32}$$

There are two possible outcomes:

- The problem has a feasible solution $\widehat{x}$, in which case we have identified a feasible solution to the MCND of value
$$h(\overline{y}) + \sum_{(i,j) \in \mathcal{A}} f_{ij} \widehat{y}_{ij},$$
where $\widehat{y}_{ij} = \lceil \sum_{p \in \mathcal{P}} \widehat{x}_{ij}^p / u_{ij} \rceil \forall \ (i,j) \in \mathcal{A}$. This upper bound replaces the value of the best known feasible solution if it improves upon it.

- The problem is infeasible, in which case there exists a dual ray (π, α) such that
$$\sum_{p \in \mathcal{P}} \pi_{d(p)}^p w^p - \sum_{(i,j) \in A} \alpha_{ij} u_{ij} y_{ij} \leq 0, \tag{33}$$
is valid for the MCND and cuts off $\overline{y}$, where π_i^p and α_{ij} are the dual variables associated to constraints (30) and (31), respectively. This Benders cut can be added to subproblem $\mathcal{L}^Y(\beta, \xi, \theta)$), therefore strengthening the Lagrangean bound.

3.1 An illustrative example

We illustrate an application of the proposed relaxation on a sample MCNDP instance I with three nodes and three links, for which the corresponding network is depicted on the left hand side of Figure 1. This instance contains a single commodity p with $o(p) = 1$, $d(p) = 3$ and $w^p = 3$. All routing (flow) costs c_{ij}^p are equal to zero and all fixed-charge costs y_{ij} are equal to one. All links have a common capacity u_{ij} of two units. The optimal solution of the corresponding network design problem on this instance is shown as follows, $\tilde{y}_{12} = \tilde{y}_{23} = \tilde{y}_{13} = 1$, $\tilde{x}_{12}^p = \tilde{x}_{23}^p = 1$ and $\tilde{x}_{13}^p = 2$, and has a value $v(\mathcal{F}_I) = 3$.

The right hand side of Figure 1 shows the solution of the LP relaxation of the MCNDP instance I. In this solution, the dashed links have an LP relaxation value of 0.5, whereas the solid links have an LP

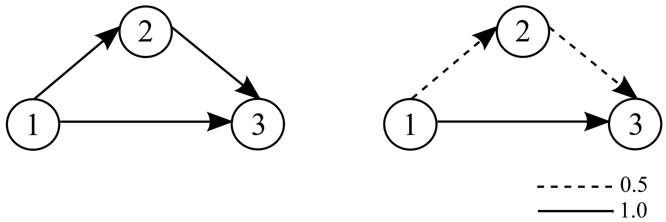

Figure 1: Sample instance with three nodes (on the left) and the solution of the LP relaxation of the corresponding MCNDP (on the right).

relaxation value of 1.0. The amount of flows in this solution are the same as that of the integer solution. The optimal value of the LP relaxation is 2.

When the proposed decomposition is applied to this instance, it is easy to see that with all the Lagrange multipliers set equal to zero, the solution of subproblem $\mathcal{L}^{XV}(\beta, \xi, \theta))$ yields the optimal value of 0. Subproblem $\mathcal{L}^{Y}(\beta, \xi, \theta))$, on the other hand, can be strengthened using the following single-node cutset inequalities, $2y_{12} + 2y_{13} \geq 3$ and $2y_{13} + 2y_{23} \geq 3$. Solved with these two inequalities added, Subproblem $\mathcal{L}^{Y}(\beta, \xi, \theta))$ yields an optimal solution $y_{12} = y_{13} = y_{23} = 1$ with a value of 3. As a result, one obtains a lower bound value of 3 for instance I through the proposed decomposition, which is strictly greater than the optimal value of the LP relaxation. In this case, the Lagrangean heuristic yields a feasible solution to the MCND of value 3, which corresponds to the optimal solution of instance I.

4 Concluding Remarks

Though the results obtained through the small instance as shown above proves the desired result, it serves for illustration purposes only. The effectiveness of the proposed approach is yet to be seen through computational experiments on a wider set of (larger-size) instances. Testing is currently being conducted and the results will be presented at the conference.

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