On a Stochastic Bilevel Programming Problem with Knapsack Constraints

Stefanie Kosuch* Pierre Le Bodic* Janny Leung◇ Abdel Lisser*

*Laboratoire de Recherche en Informatique (LRI), Université Paris Sud - Paris XI
91405 Orsay Cedex - France

◇Department of Systems Engineering & Engineering Management, Chinese University of Hong Kong
Shatin, NT - Hong Kong SAR - The People's Republic of China

Abstract

In this paper we propose a mixed integer bilevel problem having a probabilistic knapsack constraint in the first level. The problem formulation is mainly motivated by practical pricing and service provision problems as it can be interpreted as a model for the interaction between a service provider and clients. We assume the probability space to be discrete which allows us to reformulate the problem as a deterministic equivalent bilevel problem. Via a reformulation as linear bilevel problem, we obtain a quadratic optimization problem, the so called Global Linear Complementarity Problem. Based on this quadratic problem, we finally propose a procedure to compute upper bounds on the initial problem by using a Lagrangian relaxation and an iterative linear minmax scheme.

Keywords: *Bilevel, Stochastic, Optimization*

1 Introduction

In network markets, service providers are increasingly presented with optimization problems including not only usual network constraints but also market interactions with competitors [2]. These problems are hardly ever separated: on the one hand, one cannot fully optimize the capacity or reliability of a given network without knowing the demand. On the other hand, one cannot properly adapt its prices to the market if its network is fixed, since the demand will most likely vary according to the prices. It is therefore natural to formulate the problem of pricing one's services under the network constraints at hand as a bilevel optimization problem [3]: while the *leader* (e.g. a service provider) maximizes its profit, the *follower* (e.g. customers) minimizes the cost of the services by choosing among a set of competitors. The follower problem is hence a constraint of the leader problem.

Our study of this problem is restricted to the linear case. In order to reflect uncertainty on the actual use of the services provided and, in doing so, further optimize the profit of the leader, we model the leader's capacity to serve the clients by a probabilistic constraint:

$$\text{(SLBP)} \quad \max_{x} \quad c_1^t x + d_1^t y \tag{1a}$$

$$\text{s.t.} \quad \mathbb{P}\{w^t(\omega)x \leq s(\omega)\} \geq (1-\alpha), \tag{1b}$$

$$0 \leq x \leq \mathbb{1}_{n_x}, \tag{1c}$$

$$y \in \arg\max_{y} c_2^t x + d_2^t y, \tag{1d}$$

$$\text{s.t.} \quad Ax + By \leq b, \tag{1e}$$

$$y \geq 0. \tag{1f}$$

where x (resp. y) is the vector of n_x (resp. n_y) decision variables of the leader (resp. follower), $c_1, c_2, \in \mathbb{R}^{n_x}$, $d_1, d_2 \in \mathbb{R}^{n_y}$, $A \in \mathbb{R}^{m \times n_x}$, $B \in \mathbb{R}^{m \times n_y}$, $b \in \mathbb{R}^m$ and $0 < \alpha \leq 1$. $\mathbb{1}_n$ denotes the vector of ones of dimension n. The components of $w(\omega) \in \mathbb{R}_+^{n_x}$ and the right hand side $s(\omega) \in \mathbb{R}$ are random variables that all depend on the realization ω of the probability space Ω.

Both the objective function of the leader (1a) and the follower (1d) depend on the leader's strategy x *and* the follower's strategy y. The probabilistic knapsack constraints (1b) ensure that customers will be served with a α risk. Constraints (1e) are relative to the demand of the customers.

Let us now illustrate this formulation with a real-world application: a mobile phone or internet provider grants capacities on a network to multiple customers, who maximize their profit (1d) while routing their demand on the provider's network (1e). The follower's maximization reduces to choosing between the leader and the competition. Each client is entitled to a given bandwidth, which is modeled by x. The leader charges clients so as to obtain an optimal trade-off between the number of clients and prices (1a). To do so, the leader must take into account the prices of the competition but also the capacity $s(\omega)$ of its network. However, typically, customers only use a fraction $w(\omega)$ of the granted capacity. Therefore it is often the case that the network is not fully utilized. To further optimize the load of the network, it is possible to use stochastic knapsack constraints (1b) to model an *overbooked* network with a given risk of *overload*.

Throughout, we will assume the probability space Ω to be discrete, i.e. having only a finite number of realizations, called scenarii. From a practical point of view, one can imagine the service provider to lend capacity in packages rather than continuously, i.e. customers have to choose between several options of maximal needed capacity. If, in contrary, the customers are free to use the network to route their commodities, i.e. if the actual probability space is continuous, our assumption and the resulting problem reformulations might nevertheless be helpful as one could approximate the probability space by generating a finite number of representative scenarii. From a theoretical point of view, assuming a finite sample space has the advantage that in most cases the problem can be reformulated as a deterministic equivalent problem by treating the constraints for every scenario separately. We will make use of this property in section 2. In section 3, we further transform the problem so as to obtain a quadratic mixed integer program. Finally, in section 4, we suggest a method to compute upper bounds on the initial problem using Lagrange relaxation and an iterative minmax scheme.

2 From $SLBP$ to the (Deterministic Equivalent) Linear Bilevel Problem (LBP)

As we consider the case where the sample space Ω is finite, ω has only a finite number of scenarii $\omega_1, ..., \omega_K$. Let us define $p^k := \mathbb{P}\{\omega = \omega_k\}$, then

$$\sum_{k=1}^{K} p_k = 1, \quad p_k \geq 0$$

For each scenario $\omega_k, k = 1, \ldots, K$ we introduce an auxiliary binary variable z_k as follows:

$$z_k = \begin{cases} 0 & \text{if the scenario is considered} \\ 1 & \text{otherwise} \end{cases}$$

We shall simplify the notations by defining for all $k = 1, \ldots, K$:

$$w_k := w(\omega_k), \; s_k := s(\omega_k), \; w^k := \left(w_1^k, ..., w_n^k\right)$$

For all $k = 1, \ldots, K$, we define M_k such that

$$M_k := \sum_{i=1}^{n_x} w_i^k - s_k$$

Thus, problem (1) can be reformulated as the following mixed integer optimization problem:

$$
\begin{aligned}
(\text{MILBP}) \quad \max_{x,z} \quad & c_1^t x + d_1^t y \\
\text{s.t. } & w_k^t x \le s_k + M_k z_k, \ k = 1, \dots, K, && \text{(2a)} \\
& p^t z \le \alpha, && \text{(2b)} \\
& 0 \le x \le \mathbb{1}_{n_x}, && \text{(2c)} \\
& z \in \{0,1\}^K, && \text{(2d)} \\
& y \in \arg\max_y c_2^t x + d_2^t y, && \text{(2e)} \\
& \text{s.t. } Ax + By \le b, && \text{(2f)} \\
& \qquad y \ge 0. && \text{(2g)}
\end{aligned}
$$

Constraints (2a) ensure that, if scenario ω_k is not covered (i.e. $z_k = 1$), then the adopted strategy x does not have to respect the knapsack constraint for this scenario. However, as per constraint (2b), the probability of occurrence of the uncovered scenarii must be below the risk α.
Note that in (2a), it is equivalent to use a single M such that $M = \max_{k=1,\dots,K} M_k$, but when relaxing (2d) into (3d), using multiple M_k should prove more efficient from a polyhedral point of view.
As shown in [4], we can now reformulate the mixed integer bilevel problem (2) as a linear one:

$$
\begin{aligned}
(\text{LBP}) \quad \max_{x,z} \quad & c_1^t x + d_1^t y \\
\text{s.t. } & w_k^t x \le s_k + M_k z_k, \ k = 1, \dots, K, && \text{(3a)} \\
& p^t z \le \alpha, && \text{(3b)} \\
& 0 \le x \le \mathbb{1}_{n_x}, && \text{(3c)} \\
& 0 \le z \le \mathbb{1}_K, && \text{(3d)} \\
& v = 0, && \text{(3e)} \\
& (y, v) \in \arg\max_{y,v} c_2^t x + d_2^t y + (\mathbb{1}_K)^t v, && \text{(3f)} \\
& \text{s.t. } Ax + By \le b, && \text{(3g)} \\
& \qquad v \le z, && \text{(3h)} \\
& \qquad v \le \mathbb{1}_K - z, && \text{(3i)} \\
& \qquad y \ge 0. && \text{(3j)}
\end{aligned}
$$

where $\dim(v) = \dim(z) = K$.

Definition 2.1. *We denote $(\tilde{x}, \tilde{z}, \tilde{y})$ (resp. $(\tilde{x}, \tilde{z}, \tilde{y}, \tilde{v})$) a feasible solution for problem (2) (resp. problem (3)) if all first and second level constraints are satisfied. A rational solution of problem (2) (resp. (3)) is a feasible solution such that $\tilde{y}$ (resp. $(\tilde{y}, \tilde{v})$) is optimal for the second level problem with parameters $\tilde{x}$ and $\tilde{z}$.*

Proposition 2.2 (see Proposition 3.2. of [1])**.**

1.) *Let (x^*, z^*, y^*, v^*) be a rational optimal solution of LBP (3). Then $v^* = 0$ and (x^*, z^*, y^*) is a rational optimal solution of MILBP (2).*

2.) *Let (x^*, z^*, y^*) be a rational optimal solution of MILBP (2). Then $(x^*, z^*, y^*, 0)$ is a rational optimal solution of LBP (3).*

Proof. Proof of 1.): By constraint (3e), we have $v^* = 0$. If $(x^*, z^*, y^*, 0)$ is a rational solution, then by its optimality for the second level problem, we have $0 = v^* = \min(1_K - z^*, z^*)$, so $z^* \in \{0,1\}^K$. Thus, (x^*, y^*, z^*) is feasible for problem (2). By optimality of $(x^*, y^*, z^*, v^* = 0)$, (x^*, y^*, z^*) is also optimal for

problem (2).
Proof of 2.): It is easy to see that $(x^*, z^*, y^*, 0)$ is a rational solution of problem (3). From 1.) we know that *every* rational optimal solution (x^*, z^*, y^*, v^*) of problem (3) satisfies $v^* = 0$. It follows that $(x^*, z^*, y^*, 0)$ is an optimal solution of problem (3).

□

3 From LBP to the Global Linear Complementarity Problem ($GLCP$)

We will now continue the transformation process by reformulating the LBP into a $GLCP$ as described in [1]. We therefore need the dual of the follower problem (3f)-(3j):

$$
\begin{aligned}
\text{(DFP)} \min_{\lambda,\mu_1,\mu_2} \quad & \lambda^t(b - Ax) + \mu_1 z + \mu_2(\mathbb{1}_K - z), \\
\text{s.t.} \quad & B^t\lambda \geq d_2, && \text{(4a)} \\
& \mathbb{I}_K\mu_1 + \mathbb{I}_K\mu_2 \geq \mathbb{1}_K, && \text{(4b)} \\
& \lambda, \mu_1, \mu_2 \geq 0. && \text{(4c)}
\end{aligned}
$$

where $\lambda \in \mathbb{R}^m$ (resp. $\mu_1 \in \mathbb{R}^K$, $\mu_2 \in \mathbb{R}^K$) is the dual variable of (3g) (resp. (3h), (3i)). We also need the corresponding complementary slackness conditions to ensure the optimality of DFP:

$$
\begin{aligned}
\lambda^t(b - Ax - By) &= 0 & y^t(B^t\lambda - d_2) &= 0 \\
\mu_1^t(z - v) &= 0 & v^t(\mathbb{I}_K\mu_1 + \mathbb{I}_K\mu_2 - \mathbb{1}_K) &= 0 \\
\mu_2^t(\mathbb{1}_K - z - v) &= 0
\end{aligned}
$$

We obtain the following equivalent $GLCP$ which is no longer a bilevel problem[1]:

$$
\begin{aligned}
\text{(GLCP)} \quad \max_{x,y,z,\lambda,\mu_1,\mu_2} \quad & c_1^t x + d_1^t y \\
\text{s.t.} \quad & w_k^t x \leq s_k + M_k z_k, \; k = 1, \dots, K, && \text{(5a)} \\
& p^t z \leq \alpha, && \text{(5b)} \\
& Ax + By \leq b, && \text{(5c)} \\
& B^t\lambda \geq d_2, && \text{(5d)} \\
& \mathbb{I}_K\mu_1 + \mathbb{I}_K\mu_2 \geq \mathbb{1}_K, && \text{(5e)} \\
& \lambda^t(b - Ax - By) = 0, && \text{(5f)} \\
& \mu_1^t z = 0, && \text{(5g)} \\
& \mu_2^t(\mathbb{1}_K - z) = 0, && \text{(5h)} \\
& y^t(B^t\lambda - d_2) = 0, && \text{(5i)} \\
& 0 \leq x \leq \mathbb{1}_{n_x}, 0 \leq z \leq \mathbb{1}_K && \text{(5j)} \\
& y, \lambda, \mu_1, \mu_2 \geq 0. && \text{(5k)}
\end{aligned}
$$

Note that in this formulation the decision variable v has been eliminated due to the fact that $v = 0$.
Up to this point, all reformulations have been equivalent, i.e. by solving the quadratic problem (5) we get an optimal solution of the initial, stochastic bilevel problem (2) (provided that the probability space is discrete). Solving a generally nonconvex problem such as (5) directly is hard. Instead, we propose a method to compute upper bounds by relaxing it into a linear minmax problem.

4 Calculating upper bounds using Lagrangian relaxation

We relax the quadratic terms (5f), (5g), (5h) and (5i) of the (GLCP) into the objective function, using λ, μ_1 and μ_2 as a kind of Lagrange multipliers:

$$\mathcal{L}(x, y, z, \lambda, \mu_1, \mu_2) = c_1^t x + d_1^t y + \lambda^t(b - Ax - By) + \mu_1^t z + \mu_2^t(\mathbb{1}_K - z) + y^t(B^t\lambda - d_2)$$

Then the Lagrangian relaxation of problem (5) becomes

$$\text{(LGN)} \quad \min_{\lambda,\mu_1,\mu_2} \max_{x,y,z} \quad \mathcal{L}(x, y, z, \lambda, \mu_1, \mu_2)$$

$$\text{s.t. } w_k^t x \leq s_k + M_k z_k, \; k = 1, \ldots, K, \tag{6a}$$

$$p^t z \leq \alpha, \tag{6b}$$

$$Ax + By \leq b, \tag{6c}$$

$$B^t\lambda \geq d_2, \tag{6d}$$

$$\mathbb{I}_K \mu_1 + \mathbb{I}_K \mu_2 \geq \mathbb{1}_K, \tag{6e}$$

$$0 \leq x \leq \mathbb{1}_{n_x}, 0 \leq z \leq \mathbb{1}_K, \tag{6f}$$

$$y, \lambda, \mu_1, \mu_2 \geq 0. \tag{6g}$$

Proposition 4.1. *Let $(x^*, y^*, z^*, \lambda^*, \mu_1^*, \mu_2^*)$ be an optimal solution of problem (6). Then $\mathcal{L}(x^*, y^*, z^*, \lambda^*, \mu_1^*, \mu_2^*)$ is an upper bound on the optimal solution value of problem (5)*

In order to practically solve problem (6), we use an iterative minmax scheme. More precisely, in iteration $N \geq 1$ we solve one after another the following two linear problems:

- The lagrangian subproblem (LGNs(N)), maximized over the primal variables. In each iteration, an auxiliary constraint is added to enforce the decrease of its optimal solution.
- Problem (LGNd(N)), which is composed of (4) with additional constraints to enforce the increase of its optimal solution.

The process stops when $\gamma = \beta$, i.e. when the optimal is found.

(LGNs(N))

$$\max_{\beta,x,y,z} \quad \beta$$

$$\text{s.t. } \beta \leq \mathcal{L}(x, y, z, \lambda^q, \mu_1^q, \mu_2^q) \quad q = 0, \ldots, N-1, \tag{7a}$$

$$w_k^t x \leq s_k + M_k z_k \quad k = 1, \ldots, K, \tag{7b}$$

$$p^t z \leq \alpha, \tag{7c}$$

$$Ax + By \leq b, \tag{7d}$$

$$0 \leq x \leq \mathbb{1}_{n_x}, \tag{7e}$$

$$0 \leq z \leq \mathbb{1}_K, \tag{7f}$$

$$y \geq 0, \tag{7g}$$

(LGNd(N))

$$\min_{\gamma,\lambda,\mu_1,\mu_2} \quad \gamma$$

$$\text{s.t. } \gamma \geq \mathcal{L}(x^q, y^q, z^q, \lambda, \mu_1, \mu_2) \quad q = 1, \ldots, N, \tag{8a}$$

$$B^t\lambda \geq d_2, \tag{8b}$$

$$\mathbb{I}_K \mu_1 + \mathbb{I}_K \mu_2 \geq \mathbb{1}_K, \tag{8c}$$

$$\lambda, \mu_1, \mu_2 \geq 0. \tag{8d}$$

where (x^q, y^q, z^q) is an optimal solution of problem LGNs(q) $(q = 1, \ldots, N)$, $(\lambda^q, \mu_1^q, \mu_2^q)$ is feasible for problem (6) if $q = 0$ and it is an optimal solution of problem LGNd(q) if $q \geq 1$. The following proposition can be easily verified:

Proposition 4.2. *Let $N \geq 1$. Then*

- *problem $LGNs(N)$ gives an upper bound on the Lagrangian relaxation (6) as well as on the initial problem (5); and*
- *problem $LGNd(N)$ gives a lower bound on the Lagrangian relaxation (6).*

It follows that in every iteration we get an upper bound that never increases due to the addition of an inequality. Unfortunately, this is not sufficient to prove the convergence of the scheme as we might be stuck on an upper bound that is not the optimal solution of problem (6).

5 Conclusion

We study a novel stochastic bilevel problem with probabilistic knapsack constraints, which can be used to jointly optimize network resources and service pricing. The initial problem is transformed into an equivalent quadratic problem, which, in turn, is relaxed into a linear minmax problem. An iterative approach applied to the latter allows us to find upper bounds of the original stochastic bilevel problem. Future work will consist in improving the minmax scheme in order to find better bounds and/or to prove a convergence of this iterative procedure towards the optimal solution of either the linear minmax problem or even the initial problem. Numerical experiments on randomly generated data are in preparation.

References

[1] C. Audet, P. Hansen, B. Jaumard and Gilles Savard, *Links Between Linear Bilevel and Mixed 0-1 Programming Problems* , Journal of optimization theory and applications **2** (1997) pp. 273–300.

[2] M. Bouhtou, G. Erbs and M. Minoux, *Joint optimization of pricing and resource allocation in competitive telecommunications networks* , Networks **1** (2007) pp. 37–49

[3] S. Dempe, *Foundations of Bilevel Programming* , Kluwer Academic Publishers, Dordrecht, 2002.

[4] L. Vicente, Gilles Savard and J. Judice, *Discrete Linear Bilevel Programming Problem* , Journal of optimization theory and applications **3** (1996) pp. 597–614.