

Flow-based formulations for the Mixed Capacitated Arc Routing Problem – Lower bounds¹

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Abstract

Mixed Capacitated Arc Routing Problem (MCARP) models arise in distribution or collecting problems where activities are performed by vehicles, with limited capacity, and are continuously distributed along some pre-defined links (routes, streets) of an associated network. Considering directed as well as undirected required links, it is a generalization of the well known Capacitated Arc Routing problem (CARP). A compact flow based and valid model for the MCARP is presented. Due to its large number of variables and constraints we develop an aggregated version of this model. Although not valid, its linear programming bound coincides with the linear programming bound of the original model. During the presentation, computational experiments will be reported.

Keywords: *arc routing, flow-based formulations.*

1 Introduction

Capacitated Arc Routing (CARP) models arise in distribution or collection problems where activities are performed by vehicles, with limited capacity, and are continuously distributed along some pre-defined links (routes, streets) of an associated network. The CARP can be either undirected or directed. In the undirected case, the required links can be served in any direction. The Mixed Capacitated Arc Routing Problem (MCARP) describes a more realistic scenario since it considers directed as well as undirected required links. The MCARP is NP-hard as it generalizes the CARP (Golden and Wong [8]) which is known to be NP-hard.

Many real-world applications can be studied in the context of CARP or MCARP models. References on arc routing models until 2000 can be found in Assad and Golden [1], Eiselt et al. [5] and [6] and Dror's book [4]. Recently Wøhlk [10] surveyed a decade of capacitated arc routing.

The MCARP study reported in this paper is motivated by a household refuse collection problem in a quarter of Lisbon. Each quarter can be planed separately as a fleet of identical vehicles is assigned to its refuse collection. Vehicles depart from a special point, the depot, where they should return after completing their collecting period and then empty at the dumpsite. It is assumed that depot and dumpsite coincide, in order to define a problem similar to the ones in the CARP literature. For simplicity, each vehicle performs only one trip compatible with its capacity.

The first study on lower bounds for the MCARP based on a formulation using only one variable per edge is due to Belenguer et al. [3]. This model is not valid for the MCARP and is similar to models presented for other mixed arc routing problems, as the mixed Chinese postman problem [9] and the mixed general routing problem [7]. The authors use this model and several valid inequalities, in a cutting plane fashion, to get lower bounds for the MCARP that outperform the existing procedures.

We give a compact formulation for the MCARP. Two well known modeling ideas are used: i) flow variables to guarantee the connectivity of the solutions; and ii) variables indexed by vehicle to guarantee the matching between trips and vehicles. The model will be used within an ILP package to

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solve medium sized problems and to produce lower bounds on larger instances. Lower bounds are also obtained from the associated linear programming relaxation.

Essentially, our model differs from the first presented by Golden and Wong [8] for the CARP in several aspects: i) it formulates the mixed case while theirs was developed for the undirected CARP; ii) the flow variables have a different interpretation (here they are related with the demands to be served and in their paper flows are associated with the number of edges to be served); iii) here additional constraints are included to ensure that trips start at the depot; iv) extra valid inequalities are considered to strengthen the corresponding linear programming formulation.

We will also present and discuss an aggregated model, where links and flow variables are not disaggregated by vehicle, which although not valid, it is attractive for three reasons. First, the integer optimal solution, providing good lower bounds, is easier to compute than the optimal integer solution of the previous model. Second, for some instances, the optimal solution of the aggregated model is also optimal for the original problem. Finally, the linear programming relaxation values of the two models are equal, providing a faster alternative to obtain the same lower bounds.

The paper is organized as follows. In section 2.1, we define the MCARP and set notation. A valid formulation and some inequalities are given in section 2.2, while its aggregated version is presented in section 2.3. Final remarks conclude the paper.

2 Formulations

2.1 Definitions and Notation

The refuse collection problem is to plan the collection of garbage in a city with minimum total cost. The street network is described by a mixed graph. Edges characterize two way streets where zig-zag collection is allowed. Arcs represent one way streets or large two way streets with no zig-zag collection, in the former case both opposite arcs should be included in the network. Nodes characterize the street crossings or dead-end streets. A special node, called *depot*, is the starting and ending point for the vehicle trips. The depot is also the dumpsite. A *vehicle trip* stands for the path that can be performed by a vehicle from and back to the depot while servicing the streets (network links), compatible with its capacity. The streets to be served, where there is refuse to be collected, are the *required links* or *tasks*. Every street (task or not) traversed by a vehicle without serving it is named a *deadheading link*. For simplicity, it is assumed that each vehicle performs only one trip. Capacity and number of vehicles, demands on each street, service and deadheading costs and dump cost at depot are known.

Consider, then, the following notation:

- $\Gamma = (N, A' \cup E)$ is the mixed graph, with $A_R \subseteq A'$ and $E_R \subseteq E$ being the set of required arcs and edges, respectively, and N the set of nodes, representing street crossings, dead-end streets, or the depot, node $0 \in N$, where every vehicle trip must start and end; $|N| = n + 1$.
- $G = (N, A)$ is a directed graph where each edge from E is replaced by two opposite arcs, i.e., $A = A' \cup \{(i, j), (j, i) : (i, j) \in E\}$. $R \subseteq A$ is the set of required arcs in G , also named as tasks; $|R| = |A_R| + 2|E_R|$.
- P is the maximum number of trips allowed; W is the capacity of each vehicle; and λ is the dump cost, paid every time the vehicle is emptied at the depot.
- d_{ij} is the deadheading cost of arc $(i, j) \in A$; c_{ij} is the service cost of arc $(i, j) \in R$.
- q_{ij} is the demand of arc $(i, j) \in R$ and $Q_T = \sum_{(i,j) \in A_R \cup E_R} q_{ij}$ is the total demand.

The problem is to find a set of no more that P vehicle trips, satisfying the vehicles capacity, starting and ending at the depot, and servicing all the tasks at total minimum cost.

Next we present two compact formulations for the MCARP, one that is valid (section 2.2) and the other (section 2.3) that enables to obtain reasonable lower bounds.

2.2 Formulation for the MCARP

The following mixed integer linear programming is a valid model for the MCARP.

For $p = 1, \dots, P$ define:

- $x_{ij}^p = \begin{cases} 1 & \text{if } (i, j) \in R \text{ is served by tour } p \\ 0 & \text{otherwise} \end{cases} \quad \forall (i, j) \in R;$
- y_{ij}^p is the number of times that arc $(i, j) \in A$ is deadheaded during trip p ;
- f_{ij}^p is the flow in arc $(i, j) \in A$, related with the remaining demand in trip p or in a sub-circuit in p .

$$\mathbf{(F1)} \quad \min \sum_{p=1}^P \left[\sum_{(i,j) \in R} c_{ij} x_{ij}^p + \sum_{(i,j) \in A} d_{ij} y_{ij}^p + \lambda \sum_{(i,0) \in A} y_{i0}^p + \lambda \sum_{(i,0) \in R} x_{i0}^p \right] \quad (1.1)$$

subject to:

$$\sum_{j:(i,j) \in A} y_{ij}^p + \sum_{j:(i,j) \in R} x_{ij}^p = \sum_{j:(j,i) \in A} y_{ji}^p + \sum_{j:(j,i) \in R} x_{ji}^p \quad i = 0, 1, \dots, n; \quad p = 1, \dots, P \quad (1.2)$$

$$\sum_{p=1}^P x_{ij}^p = 1 \quad \forall (i, j) \in A_R \quad (1.3)$$

$$\sum_{p=1}^P (x_{ij}^p + x_{ji}^p) = 1 \quad \forall (i, j) \in E_R \quad (1.4)$$

$$\sum_{j:(0,j) \in A} y_{0j}^p + \sum_{j:(0,j) \in R} x_{0j}^p \leq 1 \quad p = 1, \dots, P \quad (1.5)$$

$$\sum_{j:(j,i) \in A} f_{ji}^p - \sum_{j:(i,j) \in A} f_{ij}^p = \sum_{j:(j,i) \in R} q_{ji} x_{ji}^p \quad i = 1, \dots, n; \quad p = 1, \dots, P \quad (1.6)$$

$$\sum_{j:(0,j) \in A} f_{0j}^p = \sum_{(i,j) \in R} q_{ij} x_{ij}^p \quad p = 1, \dots, P \quad (1.7)$$

$$\sum_{i:(i,0) \in A} f_{i0}^p = \sum_{i:(i,0) \in R} q_{i0} x_{i0}^p \quad p = 1, \dots, P \quad (1.8)$$

$$f_{ij}^p \leq W (y_{ij}^p + x_{ij}^p) \quad \forall (i, j) \in A \quad p = 1, \dots, P \quad (1.9)$$

$$x_{ij}^p \in \{0, 1\} \quad \forall (i, j) \in R \quad p = 1, \dots, P \quad (1.10)$$

$$f_{ij}^p \geq 0 \quad \forall (i, j) \in A \quad p = 1, \dots, P \quad (1.11)$$

$$y_{ij}^p \geq 0 \text{ integer} \quad \forall (i, j) \in A \quad p = 1, \dots, P \quad (1.12)$$

The objective function, (1.1), represents the total cost (service, deadheading and dump costs). Conditions (1.2) impose trips continuity at each node; the service in each required arc and edge is guaranteed by (1.3) and (1.4), respectively; (1.5) implies that the dump cost is adequately charged in the objective function; (1.6), (1.7) and (1.8) are flow conservation constraints that together with the linking constraints (1.9) force trips connectivity. Conditions (1.9) also impose upper bounds on the flow variables needed to guarantee the capacity constraints.

In F1 the number of variables is equal to $P(2|A|+|R|)$, and the number of functional restrictions is $P(2|N|+|A|+2)+|A_R|+|E_R|$. The validity of model F1 may be proved by showing that every feasible solution represents a set of trips, thus satisfying the vehicles capacity, and forming paths (thus connected) from and back to the depot, that serves every required link.

The following sets of valid inequalities are used to improve the linear programming relaxation bounds of the formulation and speed up the integer solver.

- Depot degree constraint:

$$\sum_{p=1}^P \left(\sum_{j:(0,j) \in A} y_{0j}^p + \sum_{j:(0,j) \in R} x_{0j}^p \right) \geq \left\lceil \frac{Q_T}{W} \right\rceil \quad (1.13),$$

as a lower bound on the number of trips results from the total demand divided by the vehicles capacity, and the number of trips in a solution in the number of links leaving the depot.

- Lower bounds on the flow variables:

$$f_{ij}^p \geq q_{ij} x_{ij}^p \quad \forall (i, j) \in R \quad p = 1, \dots, P \quad (1.14)$$

$$f_{ij}^p \geq y_{ij}^p - 1 \quad \forall (i, j) \in A \setminus R \quad p = 1, \dots, P \quad (1.15).$$

- Breaking symmetries:

$$\sum_{j:(0,j) \in A} y_{0j}^p + \sum_{j:(0,j) \in R} x_{0j}^p \geq \sum_{j:(0,j) \in A} y_{0j}^{p+1} + \sum_{j:(0,j) \in R} x_{0j}^{p+1} \quad p = 1, \dots, P-1 \quad (1.16).$$

Conditions (1.14)-(1.16) are used to diminish the number of alternative solutions for the same set of trips.

We denote by F1R the model F1 reinforced with constraints (1.13) to (1.16).

As the number of variables and constraints of F1 is too large, we discuss in the following section an aggregated version of this formulation.

2.3 Aggregated Formulation

Aggregating variables over the set of trips, we get:

- $x_{ij} = \begin{cases} 1 & \text{if } (i, j) \in R \text{ is served} \\ 0 & \text{otherwise} \end{cases}$, thus, $x_{ij} = \sum_{p=1}^P x_{ij}^p$, $\forall (i, j) \in A_R \cup E_R$.
- y_{ij} is the number of times that arc $(i, j) \in A$ is deadheaded, thus, $y_{ij} = \sum_{p=1}^P y_{ij}^p$.
- f_{ij} is the flow that traverses arc $(i, j) \in A$. thus $f_{ij} = \sum_{p=1}^P f_{ij}^p$.

$$(F2) \min \sum_{(i,j) \in R} c_{ij} x_{ij} + \sum_{(i,j) \in A} d_{ij} y_{ij} + \lambda \sum_{(i,0) \in A} y_{i0} + \lambda \sum_{(i,0) \in R} x_{i0} \quad (2.1)$$

subject to:

$$\sum_{j:(i,j) \in A} y_{ij} + \sum_{j:(i,j) \in R} x_{ij} = \sum_{j:(j,i) \in A} y_{ji} + \sum_{j:(j,i) \in R} x_{ji} \quad i = 0, 1, \dots, n \quad (2.2)$$

$$x_{ij} = 1 \quad \forall (i, j) \in A_R \quad (2.3)$$

$$x_{ij} + x_{ji} = 1 \quad \forall (i, j) \in E_R \quad (2.4)$$

$$\sum_{j:(0,j) \in A} y_{0j} + \sum_{j:(0,j) \in R} x_{0j} \leq P \quad (2.5)$$

$$\sum_{j:(j,i) \in A} f_{ji} - \sum_{j:(i,j) \in A} f_{ij} = \sum_{j:(j,i) \in R} q_{ji} x_{ji} \quad i = 1, \dots, n \quad (2.6)$$

$$\sum_{j:(0,j) \in A} f_{0j} = Q_T \quad (2.7)$$

$$\sum_{i:(i,0) \in A} f_{i0} = \sum_{i:(i,0) \in R} q_{i0} x_{i0} \quad (2.8)$$

$$f_{ij} \leq W (y_{ij} + x_{ij}) \quad \forall (i, j) \in A \quad (2.9)$$

$$x_{ij} \in \{0, 1\} \quad \forall (i, j) \in R \quad (2.10)$$

$$f_{ij} \geq 0 \quad \forall (i, j) \in A \quad (2.11)$$

$$y_{ij} \geq 0 \text{ integer} \quad \forall (i, j) \in A \quad (2.12)$$

This model is not valid for the MCARP, since trips may be not identifiable. This aggregated model contains only $2|A|+|R|$ variables and $2|N|+|A|+2+|A_R|+|E_R|$ constraints. Our computational experiments confirm that this reduction in the number of variables and constraints is relevant to obtain lower bounds within an ILP solver.

As noted before, an attractive feature of this aggregation derives from the proof of the following result:

Result - The linear programming relaxation values of F1 and F2 are equal.

The linear programming relaxation value of F2 can also be improved by adding some valid inequalities. These inequalities are simply the aggregated version of the inequalities (1.13) to (1.16), and lead to a model with a tighter linear programming bound. Again, it can be shown that the linear programming relaxation value of the two enhanced models, aggregated (F2R) and disaggregated (F1R), are equal.

Note that in this model capacity and connectivity constraints are enforced by using the additional flow variables and constraints linking the two sets of variables, whilst in Belenguer and Benavent [2] model no extra set of variables is used, but in turn, an exponential number of constraints forces connectivity.

3 Computational Results and Final Remarks

Computational experiments were conducted using CPLEX 11.0 to analyse and compare the performance of the models. In the conference we will report on these results together with a comparison with methods of Belenguer et al. [3], the best known from the MCARP literature used for medium and large sized instances.

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