Approximation Algorithms for the Mixed Postman Problem with Restrictions on the Arcs

Francisco Javier Zaragoza Martínez*

*Departamento de Sistemas, Universidad Autónoma Metropolitana Unidad Azcapotzalco
Av. San Pablo 180, Col. Reynosa Tamaulipas, Del. Azcapotzalco, Mexico City, 02200 Mexico

Abstract

The mixed postman problem consists of finding a minimum cost tour of a mixed graph traversing all its vertices, edges, and arcs at least once. We consider the variant of the mixed postman problem where all arcs must be traversed exactly once and we transform it into a minimum cost feasible flow on an undirected graph. The decision version of our problem is NP-complete even if restricted to planar graphs. We present a $\frac{4}{3}$-approximation algorithm for our problem, as well as a relaxation that can be solved exactly in polynomial time.

Keywords: *postman problem, mixed graph, approximation algorithm*

1 Preliminaries

A *mixed graph* $M = (V, E, A)$ is a triple consisting of a nonempty set V of *vertices*, a set E of *edges*, and a set A of *arcs*. We assume these three sets to be finite. Each edge e joins two vertices u and v called its *ends*, and each arc a is oriented from a vertex u to a vertex v called its *tail* and *head*, respectively. A *walk* from v_0 to v_n is an ordered tuple $W = (v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ on $V \cup E \cup A$ such that $n \in \mathbb{Z}_+$, $v_i \in V$ for all $0 \leq i \leq n$, and, for all $1 \leq i \leq n$, either $e_i \in E$ and v_{i-1}, v_i are its ends, or $e_i \in A$ and it is oriented from v_{i-1} to v_i. We say that W *traverses* all of $v_0, v_1, \ldots, v_n$ and $e_1, \ldots, e_n$. If $v_0 = v_n$ and W traverses all vertices of M, then W is said to be a *tour*. If for every $u, v \in V$ there is a walk from u to v and from v to u, we say that M is *strongly connected*. Let $S \subseteq V$. The *undirected cut* $\delta_E(S)$ is the set of edges with one end in S and the other end in $\bar{S} = V \setminus S$. The *directed cut* $\delta_A(S)$ is the set of arcs with tails in S and heads in $\bar{S}$. For $v \in V$ we write $\delta_E(v)$ and $\delta_A(v)$ instead of $\delta_E(\{v\})$ and $\delta_A(\{v\})$, respectively. We define the *degree* $d_E(S) = |\delta_E(S)|$, the *outdegree* $d_A(S) = |\delta_A(S)|$, and the *indegree* $d_A(\bar{S}) = |\delta_A(\bar{S})|$. Given a nonempty set S, a subset $T \subseteq S$, and a vector $x \in \mathbb{R}^S$, we denote $\sum_{t \in T} x_t$ by $x(T)$. Given a directed graph $D = (V, A)$, a vector $l \in \mathbb{Z}^A$ of *lower bounds*, and a vector $b \in \mathbb{Z}^V$ with $b(V) = 0$, we say that a vector $x \in \mathbb{Q}^A$ is a *feasible flow* if $x_a \geq l_a$ for all $a \in A$ and $x(\delta_A(\bar{v})) - x(\delta_A(v)) = b_v$ for all $v \in V$.

2 The Mixed Postman Problem with Restrictions on the Arcs

Given a mixed graph M, a tour T of M is a *postman tour* if it traverses all edges and arcs of M at least once. T is an *edges postman tour* if it traverses each arc of M exactly once. In Figure 1 we show a mixed graph and one of its edges postman tours: $(u, e, v, b, w, f, u, e, v, b, w, c, x, d, z, g, v, b, w, c, x, h, y, i, z, i, y, a, u)$.

Given a mixed graph $M = (V, E, A)$, for each edge e of M a nonnegative rational cost c_e, and an edges postman tour T of M, the *cost* of T is the sum of the costs of the edges traversed by T. Note that since arcs must be traversed exactly once, we do not assign them a cost in the minimization version of the problem. This is a variant of the well-known *mixed postman problem* introduced by Minieka [5].

Problem: Minimum edges postman problem.
Input: A strongly connected mixed graph $M = (V, E, A)$ and a vector $c \in \mathbb{Q}_+^E$.
Output: The minimum cost $\text{MEPT}(M, c)$ of an edges postman tour of M.

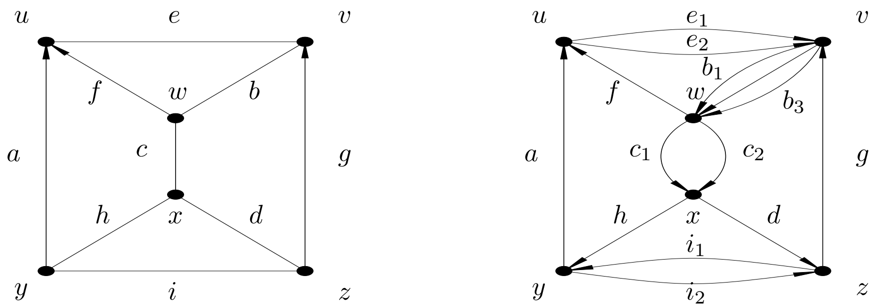

Figure 1: A mixed graph and an edges postman tour.

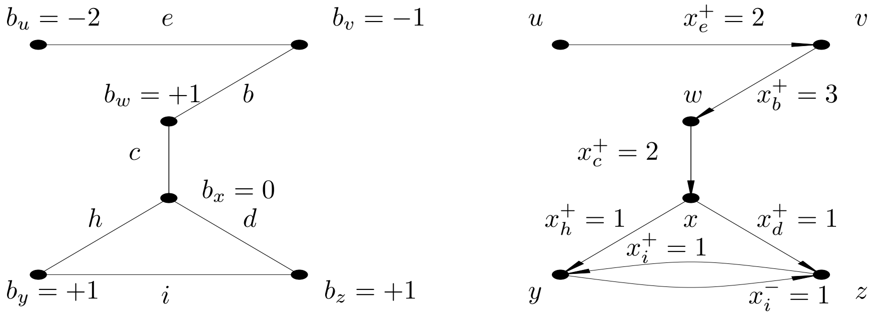

Figure 2: A feasible flow on an undirected graph.

We can easily remove the arcs from the description of the problem. Let $G = (V, E)$, let $\vec{G} = (V, E^+ \cup E^-)$ be its *associated directed graph* (that is, the directed graph obtained from G by replacing each of its edges by two arcs oriented in opposite directions between its ends) and, for each $v \in V$, let $b_v = d_A(v) - d_A(\bar{v})$ be the *demand* at vertex v. Then the problem of finding an edges postman tour on the mixed graph M is equivalent to the problem of finding a feasible flow x on the directed graph $\vec{G}$, with vector of demands b and vector of lower bounds $\mathbf{0}$, such that $x_{e^+} + x_{e^-} \geq 1$ for all $e \in E$. See Figure 2 for an example. Hence, we can reformulate the above problem as follows.

Problem: Minimum edges postman problem.
Input: An undirected graph $G = (V, E)$ and vectors $b \in \mathbb{Z}^V$ and $c \in \mathbb{Q}_+^E$.
Output: The minimum cost $\text{MEPT}(G, b, c)$ of an edges postman tour of (G, b).

From this alternate formulation, it is easy to see that (G, b) has an edges postman tour if and only if the vertex set S of each connected component of G satisfies $b(S) = 0$. Equivalently, M has an edges postman tour if and only if the vertex set S of each connected component of G satisfies $d_A(S) = d_A(\bar{S})$. Veerasamy conjectured that the decision version of the minimum edges postman problem is NP-complete [7] and this is true even por planar graphs [8].

We say that a family F of edges is an *edges postman set* of (G, b) if there exists an edges postman tour of (G, b) using each edge e of G once more than the number of times e appears in F.

Problem: Minimum edges postman set problem.
Input: An undirected graph $G = (V, E)$ and vectors $b \in \mathbb{Z}^V$ and $c \in \mathbb{Q}_+^E$.
Output: The minimum cost $\text{MEPS}(G, b, c)$ of an edges postman set of (G, b).

A natural generalization of the minimum edges postman problem is to require that each edge $e \in E$ is traversed at least l_e and at most u_e times ($u_e \geq l_e \geq 0$). Such an edges postman tour is said to be *bounded*.

Problem: Minimum bounded edges postman problem.
Input: An undirected graph $G = (V, E)$ and vectors $l, u \in \mathbb{Z}_+^E$ with $l \leq u$, $b \in \mathbb{Z}^V$, and $c \in \mathbb{Q}_+^E$.
Output: The minimum cost $\text{MBEPT}(G, l, u, b, c)$ of a bounded edges postman tour of (G, l, u, b).

3 Integer Programming Formulation and Relaxation

Let $M = (V, E, A)$ be a strongly connected mixed graph, let $c \in \mathbb{Q}_+^E$, let $G = (V, E)$, and let $b_v = d_A(v) - d_A(\bar{v})$ for every $v \in V$. An integer programming formulation for the edges postman problem based on Ford and Fulkerson's characterization of mixed Eulerian graphs [3, page 60] is as follows.

$$
\begin{aligned}
\mathrm{MEPT}(G, b, c) &= \min c^\top x & (1)\\
\text{subject to} & \\
x(\delta_E(S)) &\geq b(S) \text{ for all } S \subseteq V & (2)\\
x(\delta_E(v)) &\equiv b_v \pmod 2 \text{ for all } v \in V & (3)\\
x_e &\geq 1 \text{ for all } e \in E & (4)\\
x_e &\quad \text{integral for all } e \in E. & (5)
\end{aligned}
$$

We can obtain an integer programming formulation $\mathrm{MBEPT}(G, l, u, b, c)$ for the bounded edges postman problem replacing the constraints (4) by $u_e \geq x_e \geq l_e$ for all $e \in E$. We can obtain linear programming relaxations $\mathrm{LMEPT}(G, b, c)$ and $\mathrm{LMBEPT}(G, l, u, b, c)$ by deleting the parity constraints (3) and the integrality constraints (5).

We say that $S \subseteq V$ is an *odd set* and that $\delta_E(S)$ is an *odd cut* if $b(S) + l(\delta_E(S))$ is odd. We also say that $v \in V$ is *odd* if the set $\{v\}$ is odd. It is easy to see that an integral solution x to $\mathrm{LMBEPT}(G, l, u, b, c)$ must satisfy the odd-cut constraints

$$x(\delta_E(S)) \geq l(\delta_E(S)) + 1 \text{ for each odd set } S \subseteq V. \qquad (6)$$

Let $\mathcal{O}_{BEPT}(G, l, u, b)$ be the set of solutions of $\mathrm{LMBEPT}(G, l, u, b, c)$ that satisfy the odd-cut constraints (6). Since the decision version of the bounded edges postman problem is NP-complete, we cannot expect that $\mathcal{O}_{BEPT}(G, l, u, b)$ is the convex hull of solutions of $\mathrm{MBEPT}(G, l, u, b, c)$. In fact, $\mathcal{O}_{BEPT}(G, l, u, b)$ is not an integral polyhedron.

4 The Minimum b-Join Problem

Let $G = (V, E)$ be an undirected graph, and let $T \subseteq V$ with $|T|$ even. A *T-join* of G is a vector $x \in \mathbb{Z}_+^E$ such that for each $v \in V$, $x(\delta_E(v))$ is odd if and only if $v \in T$. For $S \subseteq V$, we say that S is *T-odd* and that $\delta_E(S)$ is a *T-cut* if $|S \cap T|$ is odd. Let $b \in \mathbb{Z}^V$ be a vector with $b(V)$ even, and let $T = \{v \in V : b_v \text{ is odd}\}$. Note that $|T|$ is even. We say that $x \in \mathbb{Z}_+^E$ is a *b-join* of G if x is a T-join of G, and $x(\delta_E(v)) \geq b_v$ for all $v \in V$.

Problem: Minimum b-join.
Input: An undirected graph $G = (V, E)$, a vector $b \in \mathbb{Z}^V$, and a vector $c \in \mathbb{Q}_+^E$.
Output: The minimum cost $\mathrm{MBJ}(G, c)$ of a b-join of G.

We can generalize b-joins by giving a vector $l \in \mathbb{Z}^E$ of lower bounds, and requiring that $x \geq l$. Let $\mathcal{P}_{BJ}(G, l, b) \subseteq \mathbb{R}^E$ be the polyhedron defined by

$$
\begin{aligned}
x(\delta_E(v)) &\geq b_v \text{ for all } v \in V & (7)\\
x(\delta_E(S)) &\geq l(\delta_E(S)) + 1 \text{ for all odd } S \subseteq V & (8)\\
x_e &\geq l_e \text{ for all } e \in E, & (9)
\end{aligned}
$$

where $S \subseteq V$ is odd if $b(S) + l(\delta_E(S))$ is odd. Observe that $\mathcal{P}_{BJ}(G, l, b)$ is a relaxation of $\mathcal{O}_{BEPT}(G, l, u, b)$ with $u_e = +\infty$ for all $e \in E$. Let $\mathcal{P}_{BJ}(G, b)$ be the polyhedron $\mathcal{P}_{BJ}(G, \mathbf{0}, b)$.

Lemma 1 *If $\mathcal{P}_{BJ}(G, b)$ is integral for all choices of b, then $\mathcal{P}_{BJ}(G, l, b)$ is integral for all choices of l and b.*

The following can be proven in a similar way to the proof given by Cook et al. [1, Section 6.1] for a description of the *b-factor polytope*.

Theorem 1 *The polyhedron $\mathcal{P}_{BJ}(G, b)$ is integral.*

Since $\mathcal{P}_{BJ}(G, l, b)$ is integral, and since we can separate in polynomial time all its defining constraints [6], the equivalence between separation and optimization [4] implies that we can optimize over $\mathcal{P}_{BJ}(G, l, b)$ in polynomial time, that is, we can solve the minimum b-join problem in polynomial time, even in the presence of lower bounds. We ask whether there is such an algorithm that does not depend on the ellipsoid method.

5 Approximation Algorithms

In the following sections we describe two approximation algorithms for the minimum edges postman problem. Our main contribution is an algorithm that has both a guarantee of $\frac{4}{3}$ for the minimum edges postman problem, and a guarantee of 2 for the minimum edges postman set problem. To the best of our knowledge, this is the first positive result about approximating the cost of an optimal postman set for any NP-hard postman problem.

In the analysis of these algorithms, we use the following lemma.

Lemma 2 (Edmonds and Johnson [2]) *Let $G = (V, E)$ be a 2-edge-connected undirected graph, let $T \subseteq V$ be an even set, and let $c \in \mathbb{Q}_+^E$. Then G has a T-join J with cost $c(J) \leq \frac{1}{2}c(E)$.*

The next lemma allows us to consider only 2-edge-connected undirected graphs.

Lemma 3 (Veerasamy [7]) *Let (G, b, c) be an instance of the minimum edges postman problem with $G = (V, E)$ connected. Assume that $e \in E$ is a cut-edge, and that $S \subseteq V$ is such that $\delta(S) = \{e\}$. Then any optimal solution $x^* \in \mathbb{Z}_+^E$ for (G, b, c) has $x_e^* = |b(S)|$ if $b(S) \neq 0$, and $x_e^* = 2$ if $b(S) = 0$.*

Therefore, before applying any of the algorithms that we describe later, we apply the following procedure due to Veerasamy [7] to a given instance (G, b): As long as G has a cut-edge $e = \{u, v\}$, let S be as in the above lemma, with $u \in S$, $v \in \bar{S}$, let $G' = G \setminus e$, define b' as $b'_u = b_u - b(S)$, $b'_v = b_v + b(S)$, and $b'_w = b_w$ for all $w \in V \setminus \{u, v\}$, and redefine (G, b) as (G', b'). After at most $|V|$ iterations, all connected components of G are 2-edge-connected, and we apply any of our algorithms to each of them.

To show the workings of the algorithms we describe next, we apply each of them to the instance (G, c) shown in Figure 3, with MEPT$(G, c) = 26$ and MEPSP$(G, c) = 6$.

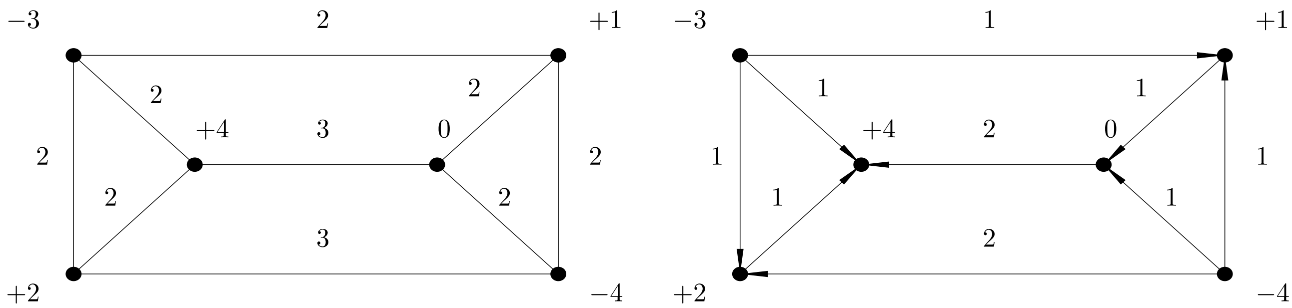

Figure 3: An example solved optimally.

6 Veerasamy's Algorithm

The first approximation algorithm we describe (called Edges1) was given by Veerasamy in his doctoral thesis [7]. Given an instance (G, b, c) of the minimum edges postman problem where G is a 2-edge-connected undirected graph, find a minimum cost feasible flow x^F on the directed graph $\vec{G}$, with demands b and costs $c_{e^+} = c_{e^-} = c_e$ for all $e \in E$. Let $U = \{e \in E : x^F_{e^+} + x^F_{e^-} = 0\}$, which we call the set of *unused* edges by the flow x^F. Let $T = \{v \in V : d_U(v) \text{ is odd}\}$, and let J be a minimum cost T-join of (G, T, c). Note that U together with J is an even subgraph of G, and hence each of its connected components is Eulerian. The output of Edges1 is x^1, the incidence vector of the edges postman tour of (G, b) obtained from adding U and J to the flow x^F. On the left side of Figure 4 we indicate with arrows the flow x^F, with cost 19, and with bold edges the set U, with cost 11. On the right side we show with dashed edges the set J, with cost 4. The edges postman tour found by Edges1 has cost 34.

Veerasamy claimed that Edges1 has a guarantee of $\frac{3}{2}$ for the minimum edges postman problem. However, we prove that the guarantee of Edges1 is somewhere between 2 and $\frac{5}{2}$.

Theorem 2 *Algorithm* Edges1 *is a $\frac{5}{2}$-approximation algorithm for the minimum edges postman problem. The guarantee of* Edges1 *is not better than 2 and it has no guarantee for the minimum edges postman set problem.*

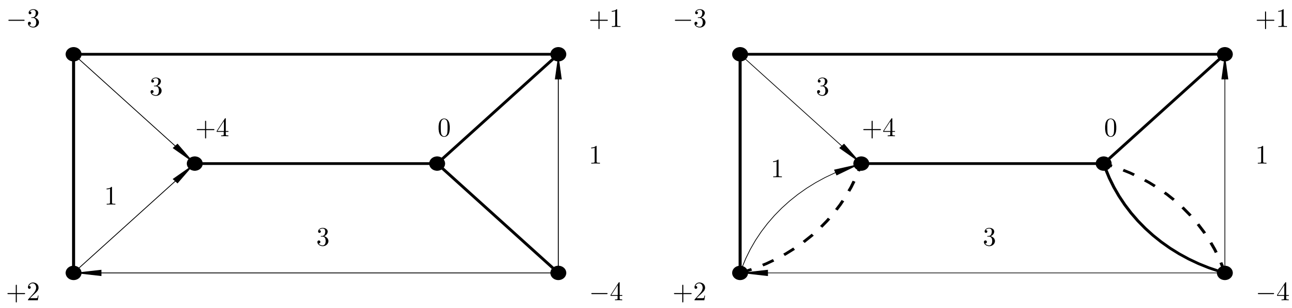

Figure 4: Veerasamy's algorithm applied to an example.

Proof. Let $G = (V, E)$ be a 2-edge-connected undirected graph, and let (G, b, c) be an instance of the minimum edges postman problem with optimal value $C^* \geq c(E)$. Since any edges postman tour of (G, b) corresponds with a feasible flow of $(\vec{G}, b)$, it follows that C^* is at least the cost C^F of x^F. In the worst case, all edges are left unused by x^F, and hence $c(U) \leq c(E)$. By Lemma 2, $c(J) \leq \frac{1}{2}c(E)$. Hence, the cost C^1 of x^1 satisfies

$$C^1 = C^F + c(U) + c(J) \leq C^* + \tfrac{3}{2}c(E) \leq \tfrac{5}{2}C^*. \tag{10}$$

For each $\epsilon > 0$, consider the undirected graph consisting of two parallel edges e and f, with ends u and v, and with $c_e = 1$, $c_f = 1 + \epsilon$, $b_u = +2$, and $b_v = -2$. It is easy to see that an application of EDGES1 to this instance gives $x^F_e = 2$, $x^F_f = 0$, $U = \{f\}$, and $J = \{e\}$, with cost $C^1 = 2 + (1 + \epsilon) + 1 = 4 + \epsilon$. However, an optimal solution to this instance has $x^*_e = x^*_f = 1$, with cost $C^* = 2 + \epsilon$. Hence $\lim_{\epsilon \to 0} \frac{C^1}{C^*} = 2$. Observe that an optimal edges postman set has cost 0, while EDGES1 outputs an edges postman set of cost 2. □

7 Our algorithm

The previous approximation algorithm satisfies first the demands at the vertices, and then it corrects for those edges that were left unused by adding a certain T-join. The second approximation algorithm that we describe (called EDGES4) performs these two steps in the reverse order.

Let $T = \{v \in V : b_v + d(v) \text{ is odd}\}$. The key observation is that any edges postman set of (G, b) must contain a T-join. Let J be a minimum cost T-join of (G, T, c). The output of EDGES4 is x^4, an extreme point optimal solution with cost C^4 of the linear programming relaxation LMBEPT(G, l, u, b, c) of the minimum bounded edges postman problem, with lower bounds $l_e = 2$ if $e \in J$, and $l_e = 1$ otherwise and upper bounds $u_e = +\infty$ for all $e \in E$. Since $b_v + l(\delta(v))$ is even for all $v \in V$, it follows that x^4 is an integral vector, and hence an edges postman tour of (G, b). On the left side of Figure 5 we indicate with bold edges the set J. On the right side we indicate with arrows the edges postman tour x^4 found by EDGES4, with cost 28.

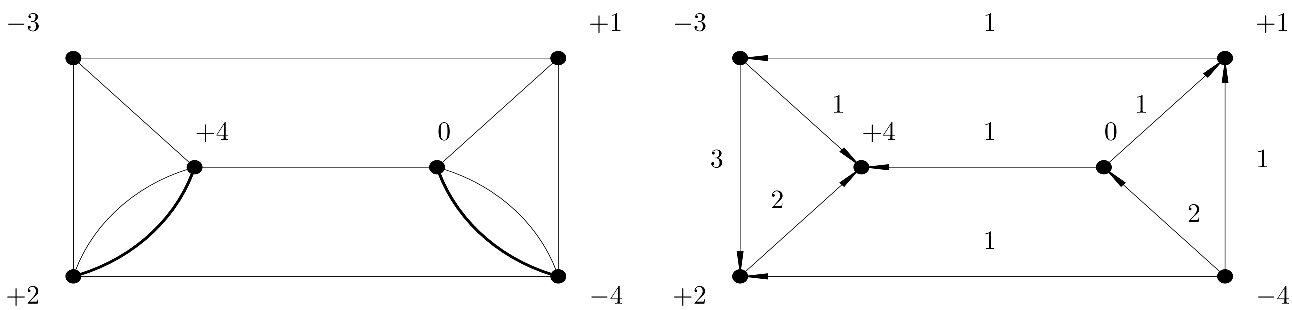

Figure 5: Algorithm EDGES4 applied to an example.

Theorem 3 *Algorithm* EDGES4 *is a tight* $\frac{4}{3}$*-approximation algorithm for the minimum edges postman problem, and also a tight* 2*-approximation algorithm for the minimum edges postman set problem.*

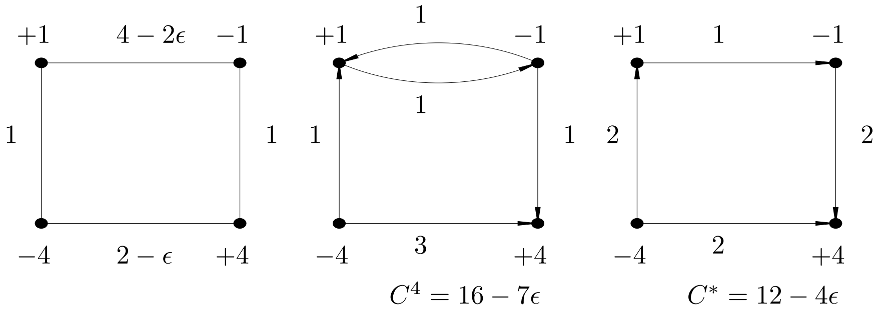

Figure 6: Worst case example for Edges4.

Proof. Let $G = (V, E)$ be a 2-edge-connected undirected graph, and let (G, b, c) be an instance of the minimum edges postman problem whose optimal solution x^* has value C^*. In order to satisfy the parity constraints at the vertices, any edges postman set of (G, b) must contain a T-join of (G, T), and hence $C^* \geq c(E) + c(J)$. By Lemma 2, $c(J) \leq \frac{1}{2}c(E)$, and hence $C^* \geq 3c(J)$. We can obtain a feasible solution to LMBEPT$(G, l, +\infty, b, c)$ by adding $\frac{1}{2}$ to each component of x^* corresponding with an edge in J. Hence

$$C^4 \leq C^* + c(J) \leq \tfrac{4}{3}C^* \text{ and } C^4 - c(E) \leq C^* - c(E) + c(J) \leq 2(C^* - c(E)). \tag{11}$$

For each $\epsilon > 0$, consider the instance in Figure 6, consisting of an undirected circuit $(v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1)$, with $c_1 = 4 - 2\epsilon$, $c_2 = 1$, $c_3 = 2 - \epsilon$, $c_4 = 1$, $b_1 = +1$, $b_2 = -1$, $b_3 = +4$, $b_4 = -4$. An application of Edges4 to this instance gives $T = \{v_1, v_2\}$, $J = \{e_1\}$, $x^4_{1+} = x^4_{1-} = 1$, $x^4_{2+} = 1$, $x^4_{2-} = 0$, $x^4_{3+} = 0$, $x^4_{3-} = 3$, $x^4_{4+} = 1$, $x^4_{4-} = 0$, with cost $C^4 = 16 - 7\epsilon$. However, an optimal solution has $x^*_{1+} = 1$, $x^*_{1-} = 0$, $x^*_{2+} = 2$, $x^*_{2-} = 0$, $x^*_{3+} = 0$, $x^*_{3-} = 2$, $x^*_{4+} = 2$, $x^*_{4-} = 0$, with cost $C^* = 12 - 4\epsilon$. Hence $\lim_{\epsilon\to 0} \frac{C^4}{C^*} = \frac{4}{3}$, and $\lim_{\epsilon\to 0} \frac{C^4 - c(E)}{C^* - c(E)} = \lim_{\epsilon\to 0} \frac{8-4\epsilon}{4-\epsilon} = 2$. □

8 Further Work and Acknowledgments

We would like to obtain a version of our algorithm that runs in polynomial time without using the ellipsoid method. We are also interested on improving its guarantee. We have studied a similar problem where the edges must be traversed exactly once and found that feasibility is NP-complete. A problem that remains open is whether there exists an approximation algorithm (or even a polynomial-time algorithm to decide feasibility) for the minimum bounded edges postman problem, other than for the case $l < u$ for which there is a 2-approximation algorithm.

This work has been partially supported by Universidad Autónoma Metropolitana Azcapotzalco grant 2270314 and by Sistema Nacional de Investigadores CONACyT grant 33694.

References

[1] W. J. Cook, W. H. Cunningham, W. R. Pulleyblank, and A. Schrijver. *Combinatorial Optimization.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1998.

[2] J. Edmonds and E. L. Johnson. Matching, Euler tours and the Chinese postman. *Math. Prog.*, 5:88–124, 1973.

[3] L. R. Ford, Jr. and D. R. Fulkerson. *Flows in Networks.* Princeton University Press, Princeton, N.J., 1962.

[4] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

[5] E. Minieka. The Chinese postman problem for mixed networks. *Management Sci.*, 25(7):643–648, 1979/80.

[6] M. W. Padberg and M. R. Rao. Odd minimum cut-sets and b-matchings. *Math. Oper. Res.*, 7(1):67–80, 1982.

[7] J. Veerasamy. *Approximation Algorithms for Postman Problems.* PhD thesis, University of Texas, Dallas, 1999.

[8] F. J. Zaragoza Martínez. Complexity of the mixed postman problem with restrictions on the arcs. In *Proceedings of the 3rd International Conference on Electrical and Electronics Engineering*, pages 92–95. IEEE, 2006.