Projected Perspective Reformulations for NonLinear Network Design Problems

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Abstract

The *Perspective Relaxation* (PR) is a general approach for constructing tight approximations to Mixed-Integer NonLinear Problems with semicontinuous variables. The PR of a MINLP can be formulated either as a Mixed-Integer Second-Order Cone Program (provided that the original objective function is SOCP-representable), or as a Semi-Infinite MINLP. While these reformulations significantly improve the lower bound and the running times of the corresponding enumerative approaches, they may spoil some valuable structure of the problem, such as the presence of network constraints. In this paper, we show that under some further assumptions the PR of a Mixed-Integer Quadratic Program can also be reformulated as a piecewise linear-quadratic problem, ultimately yielding a QP relaxation of roughly the same size of the standard continuous relaxation and where the (network) structure of the original problem is safeguarded. We apply this approach to a quadratic-cost single-commodity network design problem, comparing the newly developed algorithm with those based on both the standard continuous relaxation and the two usual variants of PR relaxation.

Keywords: *Mixed-Integer NonLinear Problems, Semicontinuous Variables, Perspective Relaxation, Nonlinear Network Design Problem*

1 Introduction

Semi-continuous variables are very often found in models of real-world problems such as distribution and production planning problems [6, 9], financial trading and planning problems [7], and many others [1, 11, 12]. These are variables which are constrained to either assume the value 0, or to lie in some given polyhedron $\mathcal{P}$; when 0 belongs to $\mathcal{P}$, one incurs in a *fixed cost* to allow the variable to have a nonzero value. We will consider Mixed-Integer NonLinear Programs (MINLP) with n semi-continuous variables $p_i \in \mathbb{R}^{m_i}$ for each $i \in N = \{1, \ldots, n\}$. Assuming that each $\mathcal{P}_i = \{p_i : A_i p_i \leq b_i\}$ has the property that $\{p_i : A_i p_i \leq 0\} = \{0\}$, each p_i can be modeled by using an associated binary variable u_i, leading to problems of the form

$$\min \quad g(z) + \textstyle\sum_{i \in N} f_i(p_i) + c_i u_i \tag{1}$$

$$A_i p_i \leq b_i u_i \qquad i \in N \tag{2}$$

$$(p, u, z) \in \mathcal{O} \quad , \quad u \in \{0,1\}^n \quad , \quad p \in \mathbb{R}^m \quad , \quad z \in \mathbb{R}^q \tag{3}$$

where all f_i and g are closed convex functions, z is the vector of all the "other" variables, and $\mathcal{O}$ is any subset of $\mathbb{R}^{m+n+q}$ (with $m = \sum_{i \in N} m_i$), representing all the "other" constraints of the problem.

It is known that the convex hull of a (possibly disconnected) domain such as $\{0\} \cup \mathcal{P}$ can be conveniently represented in a higher-dimensional space, which allows to derive *disjunctive cuts* for the problem [13]; this leads to defining the *Perspective Reformulation* of (1)—(3) [4, 6]

$$\begin{array}{rc} \min & g(z) + \sum_{i \in N} u_i f_i(p_i/u_i) + c_i u_i \\ & (2) \quad , \quad (3) \end{array} \tag{4}$$

whose continuous relaxation is significantly stronger than that of (1)—(3), and that therefore a more convenient starting point to develop exact and approximate solution algorithms [6, 7, 9, 12]. However, an issue with (4) is the high nonlinearity in the objective function due to the added fractional term. Two alternative reformulations of (4) have been proposed; one as a Mixed-Integer Second-Order Cone Program [14, 2, 12] (provided that the original objective function is SOCP-representable), and the other as a Semi-Infinite MILP [6]. In several cases, the latter outperforms the former in the context of exact or approximate enumerative solution approaches [8], basically due to the much higher reoptimization efficiency of active-set (simplex-like) methods for Linear and Quadratic Programs w.r.t. the available Interior Point methods for Conic Programs. However, both reformulations of (4) require the solution of substantially more complex continuous relaxations than the original formulation of (1)—(3); furthermore, they may spoil the valuable structure of the problem, such as the presence of network constraints. In this paper, we show that under some further assumptions, the PR of a Mixed-Integer *Quadratic* Program can also be reformulated as a piecewise linear-quadratic problem, ultimately yielding a QP relaxation of roughly the same size of the standard continuous relaxation; this is discussed in Section 2. Furthermore, if the original problem has some exploitable structure, then this structure is preserved in the reformulation, thus allowing to construct specialized approaches for solving the PR. We apply this approach to a Quadratic-cost (single-commodity) network design problem (Section 3), reporting numerical experiments comparing state-of-the-art, off-the-shelf MIQP solvers with taylor-made, specialized solution approaches.

2 A piecewise description of the convex envelope

We here analyze the properties of the Perspective Reformulation under three further assumptions on the data of the original problem (1)—(3):

A1) each p_i is a *single variable* (i.e., $m_i = 1$) and each $\mathcal{P}_i$ is a bounded real interval $[0, p_{max}]$;

A2) the variables u_i *only* appear each in the corresponding constraint (2), i.e., the "other" constraints $\mathcal{O}$ do not concern the u_i;

A3) all functions are *quadratic*, i.e., $f_i(p_i) = a_i p_i^2 + b_i p_i$ (and since they are convex, $a_i > 0$).

While these assumptions are indeed restricting, they are in fact satisfied by most of the applications of the PR reported so far [6, 7, 11, 2, 12]. Since in this paragraph we will only work with *one* block at a time, to simplify the notation in the following we will drop the index "i". We will therefore consider the (fragment of) Mixed-Integer Quadratic Program (MIQP)

$$\min \left\{ \, ap^2 + bp + cu \; : \; 0 \le p \le p_{max} u \; , \; u \in \{0,1\} \, \right\} \tag{5}$$

and its Perspective Relaxation

$$\min \left\{ \, f(p,u) = (1/u)ap^2 + bp + cu \; : \; 0 \le p \le p_{max} u \; , \; u \in \{0,1\} \, \right\} \quad . \tag{6}$$

The basic idea behind the approach is to recast (6) as the minimization over $p \in [0, p_{max}]$ of the following function:

$$z(p) = \min_u f(p,u) = bp + \min \left\{ \, (1/u)ap^2 + cu \; : \; 0 \le p \le p_{max} u \; , \; u \in [0,1] \, \right\} \quad . \tag{7}$$

It is well-known that $z(p)$ (partial minimization of a convex function) is convex; furthermore, due to the specific structure of the problem $z(p)$ can be algebraically characterized. In particular, due to convexity of $f(p,u)$, the optimal solution $u^*(p)$ of the inner optimization problem in (7) is easily obtained by the solution $\tilde{u}$ (if any) of the first-order optimality conditions of the unconstrained version of the problem, i.e., $\partial f(p,u)/\partial u = c - ap^2/u^2 = 0$. In fact, if $\tilde{u}$ is feasible for the problem, then it is optimal ($u^*(p) = \tilde{u}$); otherwise, $u^*(p)$ is the projection of $\tilde{u}$ over the feasible region, i.e., the extreme of the interval nearer to $\tilde{u}$ (this is where assumption A1 is used). Thus, by developing the different cases, one can construct an explicit algebraic description of $z(p) = f(p, u^*(p))$.

2.1 The piecewise description of $z(p)$

We start by rewriting the constraints in (7) as

$$(0 \leq)\ p/p_{max} \leq u \leq 1 \tag{8}$$

(since $p_{max} \geq p \geq 0 \Rightarrow p/p_{max} \geq 0$). We must now proceed by cases:

1) If $c \leq 0$, then $\tilde{u}$ is undefined: the derivative is always negative. Thus, there is no global minima of the unconstrained problem, and therefore $u^*(p) = 1$, yielding

$$z(p) = ap^2 + bp + c \tag{9}$$

2) If, instead, $c > 0$, then $\tilde{u} = p\sqrt{a/c}$ (note that we have used $p \geq 0$, $c > 0, a > 0$). In general, two cases can arise:

   2.1) $\tilde{u} \leq p/p_{max} \Leftrightarrow p_{max} \leq \sqrt{c/a} \Leftrightarrow u^*(p) = p/p_{max} \Rightarrow$

$$z(p) = \big(\ b + ap_{max} + c/p_{max}\ \big)p \tag{10}$$

   2.2) $0 \geq \tilde{u} \geq p/p_{max} \Leftrightarrow p_{max} \geq \sqrt{c/a}(\geq p_{min})$. This gives two further subcases

   * $(p_{max} \geq)\ p \geq \sqrt{c/a}\ (\geq 0) \Rightarrow \tilde{u} \geq 1 \Rightarrow u^*(p) = 1$;
   * $0 \leq p \leq \sqrt{c/a}\ (\leq p_{max}) \Rightarrow \tilde{u} \leq 1 \Rightarrow u^*(p) = \tilde{u}$.

   finally showing that $z(p)$ is the piecewise linear-quadratic function

$$z(p) = \begin{cases} \big(\ b + 2\sqrt{ac}\ \big)p & \text{if } 0 \leq p \leq \sqrt{c/a} \\ ap^2 + bp + c & \text{if } \sqrt{c/a} \leq p \leq p_{max} \end{cases} \tag{11}$$

   Note that (11) is continuous and differentiable even at the (potential) breakpoint $p = \sqrt{c/a}$, and therefore convex (as expected).

In all the cases, $z(p)$ is a convex differentiable piecewise-quadratic function with at most 2 pieces.

3 Quadratic-cost network design

Assume that a directed graph $G = (N, A)$ is given. For each node $i \in N$ a *deficit* $b_i \in \mathbb{R}$ is given indicating the amount of flow that the node requires (negative deficits indicate source nodes). Each arc (i, j) of the graph is either constructed at a fixed cost c_{ij}, and therefore flow is allowed to pass through the arc up to a given maximum capacity u_{ij}, or it is not constructed, thus paying no cost but also losing the possibility of sending any flow through the arc. If x_{ij} units of flow are sent through a (constructed) arc (i, j), then a quadratic flow cost $b_{ij}x_{ij} + a_{ij}x_{ij}^2$ is also incurred. The problem is that of deciding which arcs have to be constructed and how the flow has to be routed in such a way that demands are satisfied and the total (construction + routing) cost is minimized. The problem can be written as

$$\begin{array}{ll} \min & \sum_{(i,j)\in A} c_{ij}y_{ij} + b_{ij}x_{ij} + a_{ij}x_{ij}^2 \\ & \sum_{(j,i)\in A} x_{ji} - \sum_{(i,j)\in A} x_{ij} = b_i \quad i \in N \\ & 0 \leq x_{ij} \leq u_{ij}y_{ij} \;\; , \;\; y_{ij} \in \{0,1\} \quad (i,j) \in A \end{array} \tag{12}$$

This network design problem is $\mathcal{NP}$-hard, and therefore enumerative techniques are required if an optimal solution is sought for; a recent application of this general model in a Facility Location setting is given e.g. in [11, 12]. Since we can assume $c_{ij} > 0$ (for otherwise y_{ij} can surely be fixed to 1), in the continuous relaxation of (12) the "design" variables y_{ij} can be *projected* onto the x_{ij}; that is, since at optimality it surely is $y_{ij} = x_{ij}/u_{ij}$, the y_{ij} variables can be eliminated and c_{ij}/u_{ij} added to the linear cost term b_{ij}. Such a problem can be efficiently solved by means of (convex) Quadratic Min-Cost Flow (QMCF) algorithms; however, the bound provided by the continuous relaxation is usually weak, leading to a large number of nodes in the enumeration tree and therefore to a large solution time.

Applying the results of §2.1 to (12) gives a *Separable Convex-cost NonLinear MCF* problem, where the flow cost function on each arc is a piecewise linear-quadratic convex cost function. In turn, this can be rewritten as a QMCF problem

$$\begin{array}{ll} \min & \sum_{(i,j)\in A'} b'_{ij}x_{ij} + a'_{ij}x_{ij}^2 \\ & \sum_{(j,i)\in A'} x_{ji} - \sum_{(i,j)\in A'} x_{ij} = b_i \quad i \in N \\ & 0 \leq x_{ij} \leq u'_{ij} \qquad\qquad\qquad\quad (i,j) \in A' \end{array}$$

on a graph $G' = (N, A')$ with the same node set and *at most* 2 times the number of arcs. For each of the original arcs (i,j), at most two "parallel" copies are constructed. If $u_{ij} \leq \sqrt{c_{ij}/a_{ij}}$ (case 2.1), then only one representative of (i,j) is constructed in G', with $b'_{ij} = b_{ij} + a_{ij}u_{ij} + c_{ij}/u_{ij}$, $a'_{ij} = 0$ and $u'_{ij} = u_{ij}$. If, instead, $u_{ij} < \sqrt{c_{ij}/a_{ij}}$ (case 2.2), then two parallel copies of the arc (i,j) have to be constructed in G'; the first has $b'_{ij} = b_{ij} + 2\sqrt{a_{ij}c_{ij}}$, $a'_{ij} = 0$ and $u'_{ij} = \sqrt{c_{ij}/a_{ij}}$, while the second has $b'_{ij} = b_{ij}$, $a'_{ij} = a_{ij}$ and $u'_{ij} = u_{ij} - \sqrt{c_{ij}/a_{ij}}$. For this kind of "partitioned" NonLinear MCF problems (where some of the arcs have strictly convex cost functions, while the other have linear cost functions) specialized algorithms have been proposed [5]; in general, any algorithm for Convex (Quadratic) MCF problems (e.g. [3]) can be used. While codes implementing these algorithms are either not available, or not very efficient in practice, the efficient off-the-shelf solver `Cplex` has a specialized solution approach for (convex) QMCFs.

A possible alternative is to further linearize the quadratic part of the objective function for those arcs that has one (i.e., for which $u_{ij} < \sqrt{c_{ij}/a_{ij}}$). Thus, arbitrarily fixing some integer $k \geq 1$ (in principle different for each arc) we can partition the "quadratic portion" of each arc flow into k disjoint intervals and construct the corresponding k-pieces lower linearization (from below) of the quadratic objective function. This amounts at constructing k "parallel" copies of each arc (i,j) with appropriate upper bounds u_{ij}^h and cost coefficients b_{ij}^h for $h = 1, \ldots, k$, to be added to the "linear copy" of the arc with its appropriate upper bound and cost coefficient, as discussed above. Of course, the disadvantage of this approach is that the lower bound will be worse, and only increasing k the difference can be reduced; on the other hand, the resulting relaxation is a linear (as opposed to Quadratic) MCF problem, for which several extremely efficient solvers are available that can also be easily interchanged due to the availability of an abstract `C++` interface [10]. We thus have four different approaches to compare:

1. a B&C on the PR (6), using either the Semi-Infinite MILP or the MI-SCOP formulation;
2. a specialized B&B where the continuous relaxation is solved with a QMCF solver;
3. a specialized B&B with approximated continuous relaxation (for various k) solved with a linear MCF solver.

4. a standard B&C on the continuous relaxation.

We remark that current B&C solvers like `Cplex` provide all means for implementing all four options in the same framework, thus allowing for a fair assessments of the computational advantages of each.

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