A Node Rooted Flow-Based Model for the Local Access Network Expansion Problem

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Abstract

In the local access expansion problem we need to expand a given tree network by increasing the capacity of the edges or by installing concentrating devices in some nodes of the network, in order to satisfy the increase in node demands. It was shown in [4] that the problem can be viewed as an extension of the well-known Capacitated Minimum Spanning Tree Problem (CMSTP) and two flow-based models (a single and a multi-commodity model) with additional constraints were introduced. In this work we present a new flow-based model that includes additional information on the first node of each feasible path. We will show that this additional information permits us to write models with a tighter linear programming relaxation which, in turn, would permit us to solve, in an easier way, instances from the literature. We test and compare the new and old models for instances with 100, 200 and 500 nodes in order to show the advantages of the new proposed model.

Keywords: *Telecommunications, Local Access Network, CMSTP, Flow Models, Linear Programming Relaxation*

1 Introduction

A local access network is a set of customer nodes, a central office and several links that allow the transmission of the traffic from one customer to another, usually by a tree structure (for an overview of several problems arising in the context of access network planning, see, for instance, [1, 3]). Concentrating devices (called concentrators in the sequel) are electronic devices that compress the traffic and can be installed in the customer nodes. When the forecast of future demand indicates that the current network is inadequate, the network should be upgraded, either by adding capacity to the existing links or by installing concentrators in some nodes ([2, 4, 5, 7, 9]).

The problem we address here is the expansion of a local access network with a fixed tree topology. We consider the single time period version of the problem, where the local access network expansion problem is to determine which edges need to be expanded and where the concentrators should be located in order to guarantee that the node demands can be sent to (or from) the central office with the minimum cost. We consider the problem with uncapacitated concentrators. This version has been considered and studied in [4], where the problem was shown to be an extension of the well-known Capacitated Minimum Spanning Tree Problem (CMSTP) (see, for instance, [6, 8]) and two flow-based models were introduced and tested. These two models, a single and a multi-commodity flow model, are augmented versions of models developed for the CMSTP. We present a new flow-based model that considers additional information on the first node (which corresponds to the central office or to a concentrator) of each feasible path. The new information permits us to create new interesting valid inequalities and to propose new reduction tests. We will show that this additional information permits us to write models with a tighter linear programming relaxation that the previously proposed models.

2 Problem description and graph transformation

The problem studied here considers the following assumptions (already stated in [2, 4, 5]):

- the information is concentrated once, either by one concentrator or by the central office (the central office can be viewed as a concentrator with zero costs);
- the demand of each node is unsplitted, that is, all the demand is processed by only a single concentrator or by the central office;
- if the demand of a node i is processed by a given concentrator, the demand of every node in the path from node i to the concentrator is also processed by the same concentrator (called contiguity restriction);
- the demand processed by a given concentrator is transmitted to the central office through the original links of the network or by a direct link to the central node. In this case, the corresponding transmission cost is incorporated into the fixed cost of the concentrator.

Given the demand in each node, the available and the maximum capacities that can be installed in each edge of the tree, the expansion costs of the edges and the concentrators costs, the local access expansion problem is to determine, with minimum cost, which edges need to be expanded and where the concentrators should be located in order to guarantee that the node demands can be sent to (or from) the root node. The available capacity identifies the amount of information that can circulate initially on each edge and the maximum capacity corresponds to the maximum amount of traffic that will be able to circulate on the edge after expanding the network.

Let $T = (N, E)$ be the tree defining the local access network, let $N = \{1, \ldots, n\}$ be the corresponding node set (node 1 is the root of the tree and represents the central office node) and let E be the edge set of the tree (corresponding to the existing links between customer nodes). For each node j, let d_j be the corresponding demand and let F_j and c_j be, respectively, the fixed and variable costs of installing one concentrator at that node. For each edge (i, j) we associate the available capacity B_{ij} and the maximum capacity M_{ij} (this value is equal to the total demand in the tree). We also define a fixed cost G_{ij} and a variable cost e_{ij} related to the expansion of the link (i, j).

With the inclusion of an additional node 0 and edges $(0, j)$, for each node $j \in N$, we transform the problem under study into an extension of the CMSTP. Given the tree T, N and E as presented before, we define a new graph T_0, obtain from T by adding the new node and the new edges, as described in Fig. 1. Let N_0 denote the node set in T_0 and let E_0 denote the corresponding edge set.

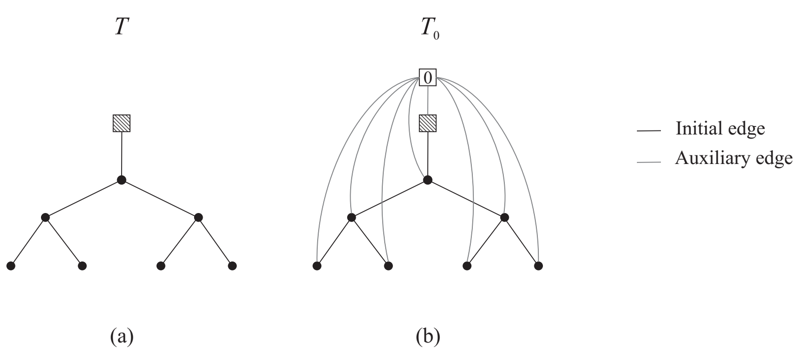

Figure 1: *Structure of:* (a) T *and* (b) T_0.

When an edge $(0, j)$ is included in the solution it means that a concentrator is located in node j. The total demand served by that concentrator in T is represented by the flow in edge $(0, j)$ in T_0. Since the inclusion of each one of these edges in T_0 corresponds to the installation of one concentrator in T, the corresponding available edge capacity B_{0j} must be unlimited and these edges are not included in the set of edges to be expanded. Two costs are associated with the new edges $(0, j)$: one corresponding to the fixed cost of installing one concentrator at j, F_j, and the other corresponding to the variable cost c_j, which depends on the amount of flow that circulates through the edge. Fig. 2 illustrates a feasible solution in T and the corresponding feasible solution in T_0. As we can see, this solution is a spanning tree in the new graph.

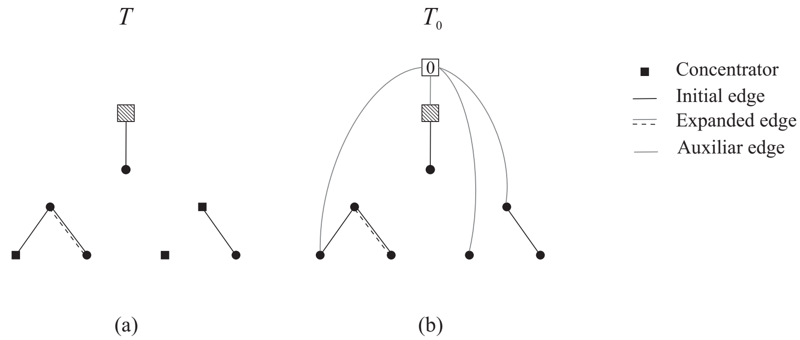

Figure 2: *Feasible solution for:* (a) T *and* (b) T_0.

Since a powerful modelling construct to improve formulations for several network design problems is to "direct the given network", we model our problem in a directed graph $D_0 = (N_0, A_0)$, where N_0 denotes the set of nodes and A_0 the set of all arcs, as described next. Each edge (i, j) in E is replaced by two arcs, $\langle i, j\rangle$ and $\langle j, i\rangle$, with the same parameters as the original edge (the exception are edges of the form $(1, j)$ and $(0, j)$ that are replaced only by one single arc, $\langle 1, j\rangle$ and $\langle 0, j\rangle$, respectively; arcs $\langle 0, j\rangle$ are designated as *auxiliary arcs*). A denotes the set of arcs without the auxiliary arcs. We consider that traffic flows from the central office to the nodes.

Based on this construction, we have previously defined two flow-based models for the problem ([4]), denoted by *FA* and *FD*. These two models can be seen as single and multi-commodity flow-based models previously developed for the CMSTP, augmented with new inequalities to model the edge capacities expansion.

3 Node rooted aggregated flow model - *NRFA*

The idea of the new model is to add information about the first node on each feasible path leaving the root. Recall that for any two nodes p and j $(p, j \neq 0)$, if the demand of node j is served by the concentrator located at node p in T (Fig. 3(a)), the demand of every node in the path from the concentrator to node j is also processed by the same concentrator (by the contiguity restriction) and the path between p and j must be in the solution in D_0 (Fig. 3(b)).

The model uses binary variables x^p_{ij} indicating whether or not arc $\langle i, j\rangle \in A$ is included in a directed path rooted at node p (if $x^p_{ij} = 1$, nodes i and j are assigned to the concentrator located at node p). These variables are defined for each pair $(p, \langle i, j\rangle)$ only if the directed path from p to j traverses arc $\langle i, j\rangle$. For the auxiliary arcs, we define the binary variables x_{0p} indicating whether or not a concentrator is located at node p. In a similar way, we define nonnegative flow variables y^p_{ij} indicating the amount of flow that is sent from the concentrator at node p and that circulates in arc $\langle i, j\rangle$. To model the expansion of the links, we define the binary variables z^p_{ij} indicating whether or not arc $\langle i, j\rangle$, belonging to the directed path rooted at p, is expanded and the nonnegative variables s^p_{ij} denoting the corresponding expanded

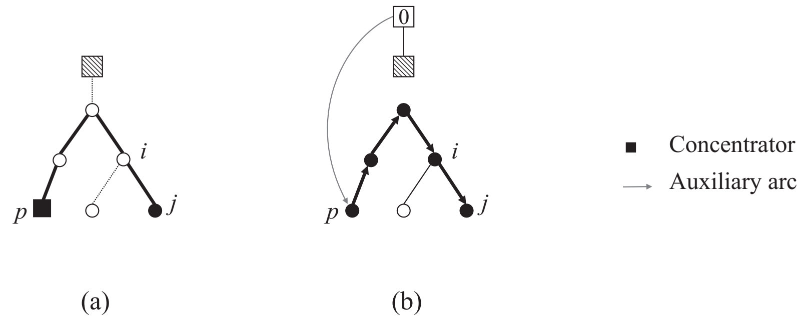

Figure 3: (a) *Concentrator at node p serves node j* (b) *Concentrator at node p* ($\langle 0,p\rangle$ *is in the solution) serves node j and all the nodes from the path between p and j.*

amount of flow.

With these variables, besides including disaggregations by the index p of some of the constraints in the single-commodity flow model *FA* presented in [4], we can introduce two new sets of inequalities to the *NRFA* model, which help to improve the linear programming relaxation, and a new coefficient M_{ij}^p that corresponds to an upper bound on the required capacity for arc $\langle i,j\rangle$, when the path rooted at node p includes arc $\langle i,j\rangle$. This bound is usually stronger than the one used in previous models. Note that if variable x_{ij}^p is equal to 1, it is necessary to have a concentrator at node p (x_{0p} must be equal to 1) which shows that $x_{ij}^p \le x_{0p}, \forall\langle i,j\rangle \in A, \forall p \in N$ are valid inequalities. The second set of constraints $x_{ij}^p \le x_{p_i i}^p, \forall\langle i,j\rangle \in A$, $\forall p \in N \setminus \{i\}$ guarantees that if the path rooted at node p traverses arc $\langle i,j\rangle$, it passes through arc $\langle p_i, i\rangle$ as well (by the contiguity restriction), where node p_i is the predecessor of node i in the path.

The new formulation, denoted by *NRFA*, is defined as follows:

$$\text{Minimize} \sum_{j=1}^{n} F_j\, x_{0j} + \sum_{j=1}^{n} c_j\, y_{0j} + \sum_{\langle i,j\rangle \in A} \sum_{p=1}^{n} G_{ij}\, z_{ij}^p + \sum_{\langle i,j\rangle \in A} \sum_{p=1}^{n} e_{ij}\, s_{ij}^p$$

subject to

$$\sum_{i:\langle i,j\rangle \in A_0} \sum_{p=1}^{n} x_{ij}^p = 1, \quad \forall j \in N \tag{NR1}$$

$$\sum_{i:\langle i,j\rangle \in A_0} y_{ij}^p - \sum_{i:\langle j,i\rangle \in A} y_{ji}^p = d_j \sum_{i:\langle i,j\rangle \in A_0} x_{ij}^p, \quad \forall j \in N, \forall p \in N \tag{NR2}$$

$$z_{ij}^p \le x_{ij}^p, \quad \forall\langle i,j\rangle \in A, \forall p \in N \tag{NR3}$$

$$y_{ij}^p \le B_{ij} x_{ij}^p + s_{ij}^p, \quad \forall\langle i,j\rangle \in A, \forall p \in N \tag{NR4.1}$$

$$y_{0j} \le B_{0j} x_{0j}, \quad \forall j \in N \tag{NR4.2}$$

$$s_{ij}^p \le (M_{ij}^p - B_{ij}) z_{ij}^p, \quad \forall\langle i,j\rangle \in A, \forall p \in N \tag{NR5}$$

$$x_{ij}^p \in \{0,1\}, y_{ij}^p \ge 0, \quad \forall\langle i,j\rangle \in A_0, \forall p \in N \tag{NR6}$$

$$z_{ij}^p \in \{0,1\}, s_{ij}^p \ge 0, \quad \forall\langle i,j\rangle \in A, \forall p \in N$$

The objective function states that we want to minimize the sum of the concentrator costs together with all the link costs. Constraints (NR1) ensure that each node, except node 0, has only one incident arc and (NR2) are the flow conservation constraints that guarantee the demand d_j at each node j. Constraints (NR3) ensure that an arc is expanded only if it is used. Constraints (NR4.1) guarantee that the flow value in each arc is less than or equal to the available capacity plus the expanded capacity. For the auxiliary arcs, constraints (NR4.2) guarantee that the flow value in each arc is less than or equal to the maximum capacity of the arc. (NR5) define an upper bound for the added capacity of each expanded arc. Constraints (NR2), (NR3), (NR4.1) and (NR5) are disaggregations of constraints appearing in the single-commodity flow model *FA* presented in [4]. Note, however, that as (NR5) are considered for each node p, the maximum flow in the arc $\langle i, j \rangle$ can be reduced when it is known that node p (the corresponding concentrator) will not serve some nodes in the tree. This permits us to define the new coefficient M_{ij}^p, which is usually smaller than the corresponding coefficient in the *FA* and *FD* models.

Although the new model includes apparently many more variables and constraints than the *FA* model, the proposed pre-processing permits us to eliminate several directed paths and consequently eliminate many variables and constraints as well as reduce the value of the M_{ij}^p coefficient (in fact, it was the pre-processing leading to a reduction in the M_{ij}^p coefficients that has motivate this model). The valid inequalities based on the new variables described before, and valid inequalities similar to the ones introduced in *FA* and *FD* models (see [4]), are included in the model in order to improve the linear programming relaxation of formulation *NRFA*.

4 Computational tests and some conclusions

To compare the three classes of flow-based formulations developed for the local access network expansion problem (single commodity and multi-commodity models previously presented in [4] and the new model), we present several computational results to assess their efficiency in solving instances of 100, 200 and 500 nodes. We have generated several instances with two types of tree structure and with different parameters in order to represent different alternatives to expand the network, denoted by A, B and C, where A represents the situation that favours concentrators installation, B favours the link expansion and C represents a more balanced alternative. For each case, we have also considered several values for the node demands, the costs and the links available capacities. We have also tested the models with instances with 100, 200 and 500 nodes taken from [5]. Table 1 and Table 2 correspond to the average results for the first and second set of instances, respectively. Due to lack of space we have grouped together the different classes of instances, maintaining only an indication by size (number of nodes) and alternative (A, B, C). For each model, we present two formulations, the initial one (index 0) and the one obtain after the inclusion of valid inequalities, which are similar to the three models (index 1). Thus, the tables depicts the gaps given by the optimal linear programming bound of formulations FA_0, FD_0, $NRFA_0$, FA_1, FD_1 and $NRFA_1$.

Instance	FA_0	FD_0	$NRFA_0$	FA_1	FD_1	$NRFA_1$
100A	0.63	0.16	0.16	0.22	0.15	0.15
200A	0.64	0.15	0.15	0.20	0.14	0.13
500A	0.59	0.14	0.14	0.17	0.13	0.12
100B	1.25	0.66	0.64	0.90	0.64	0.63
200B	0.95	0.51	0.49	0.72	0.48	0.47
500B	1.03	0.47	0.46	0.74	0.46	0.45
100C	21.36	6.53	6.26	9.25	6.02	5.80
200C	16.94	5.25	5.08	7.11	4.86	4.71
500C	17.89	4.84	4.61	7.29	4.44	4.22

Table 1: *Average Gaps - generated instances with 100, 200 and 500 nodes.*

The results show that the new $NRFA_1$ model gives, in general, the best gaps. The three models permit us to obtain, in general, the optimal solutions very quickly for the alternative A, alternative B with the 100 and 200 nodes instances and for the instances taken from [5]. For the others instances, the *NRFA* model uses, in general, more CPU time to obtain the linear programming relaxation solution but, in many cases, this model uses less CPU time to obtain the optimal solution than the others models

Instance	FA_0	FD_0	$NRFA_0$	FA_1	FD_1	$NRFA_1$
100	2.04	0.62	0.59	0.90	0.56	0.53
200	2.22	0.67	0.55	1.07	0.58	0.57
500	2.18	0.64	0.62	1.11	0.58	0.56

Table 2: *Average Gaps - instances taken from [5].*

(more details will be given at the conference). We have also compared the use of these formulations with the pseudo-polynomial dynamic programming algorithm of Flippo et al. [5], that has time complexity $O(nB^2)$, where B is an upper-bound for the concentrator capacity (for the case with uncapacitated concentrators, we set B equal to the total demand in the network, as the authors suggest). In most cases, the flow models presented use less CPU time to obtain the optimal solution than the dynamic programming algorithm. Note, however, that the problem under study with uncapacitated concentrators corresponds to the worst-case behaviour of the dynamic programming algorithm.

The characterization of the local access expansion problem studied as an extension of the CMSTP has permitted us to use flow based models with extra constraints to model the capacity expansion. Here we have presented a tighter flow model that has better performance than the previous flow models discussed in [4].

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