Network flow methods for electoral systems

Friedrich Pukelsheim* Federica Ricca⋆

Andrea Scozzari◇ Paolo Serafini+ Bruno Simeone⋆

*Institut für Mathematik, Universität Augsburg
Universitätstrasse 14, 86159 Augsburg, Germany

⋆Dipartimento di Statistica, Probabilità e Statistiche Applicate, Università di Roma "La Sapienza"
Piazzale Aldo Moro 5, 00185 Roma, Italy

◇Dipartimento di Matematica per le Decisioni Economiche, Finanziarie ed Assicurative,
Università di Roma "La Sapienza", Via del Castro Laurenziano 9, 00169 Roma, Italy

+Dipartimento di Matematica e Informatica, Università di Udine
Via delle Scienze 206, 33100 Udine, Italy

Abstract

Researchers in the field of electoral systems have recently turned their attention to network flow techniques, with the intention to resolve certain practically relevant problems of contemporary electoral systems. Here we review some of this work, with a focus on biproportional apportionment methods and on the give-up problem.

In the biproportional apportionment problem, the whole electoral region is subdivided into several electoral districts. The input data consists of a matrix of the vote counts a party receives in a district. The task is to convert the vote matrix into a (integer) seat matrix, maintaining proportionality "as much as possible". Moreover each district must be allocated its pre-specified number of seats, and each party must receive the number of seats it is entitled to on the basis of the aggregate, national vote counts.

The give-up problem arises in the current electoral law for the Italian Parliament. For each region each party submits a blocked, ordered lists of candidates. Candidates may run on more than one list (that is, in more than one region), and many of them do in order to advertise their national standing. Candidates winning a seat in more than one region must give-up all of them but one. The give-up problem finds a schedule of give-ups, for all lists of a party, that results in a globally best, in some sense, choice of deputies for that party.

Keywords: *electoral systems, biproportional seat apportionment, network flows, matrix scaling.*

1 Introduction

The use of network flow models and methods is widespread in Operations Research, with plenty of applications in a large variety of areas [1]. In recent years, researchers in the field of electoral systems have turned their attention to network flow techniques in order to deal with biproportional apportionment and other decision problems. We feel that the time has come to give an account of this research direction. For an optimization-oriented general introduction to electoral systems, the reader may refer to Grilli di Cortona et al. [12].

A transportation procedure is already featured in Hess et al. [14], the earliest Operations Research paper in political districting. In their transportation model the origins are the territorial units with supplies equal to their populations, the destinations are the districts with demands all equal to the average district population, that is, the total population of the territory divided by the (prescribed) number of districts. The transportation model tries to assign the territorial units to the districts so that the populations of the districts are equal to each other. In general, the resulting assignment is fractional, against the integrity criterion which forbids splitting any territorial unit between two or more districts. The Authors then devise a heuristic to recover an integral assignment from the fractional one. While the constraints of the transportation model reflect the criterion of population equality, the objective function takes into account district compactness, a powerful criterion for keeping gerrymandering malpractices at bay.

This early reference is not meant to be the beginning of a full historical account of network flow techniques in the electoral system literature. We shall rather focus on recent work intended to improve on certain technical, but significant, aspects of contemporary electoral systems which are either ill-solved, at least in some countries, or not dealt with in a quantitative way.

2 Biproportional seat apportionment

Perhaps the main area of electoral systems where network flow techniques are brought to bear is biproportional seat apportionment. The problem arises in situations where the entire electoral region, usually the whole nation, is subdivided into electoral districts. By constitutional or legal requirements, the electoral districts are to receive a share of seats proportional to their population counts. At the same time, political parties are to be allocated a number of seats that mirrors their nationwide electoral performance.

Thus let H be the *house size* of a national parliament. Firstly, the H parliamentary seats are apportioned among m electoral districts proportionally to population counts, allocating r_i seats to district $i = 1, \ldots, m$. Secondly, the H seats are apportioned among n lists of candidates of the contending parties, proportionally to the number of votes each party list has received. Let c_j be the nationwide seats of party $j = 1, \ldots, n$. Both steps, of apportioning the H seats among the districts on the one hand, and among the parties on the other, form the *super-apportionment*. Balinski and Young [5] is the ultimate comprehensive reference on proportional seat apportionment, its mathematical aspects, and its history.

The biproportional apportionment problem is a "matrix problem". It arises in the final *sub-apportionment* step of the procedure, of allocating the nationwide party-seats to those electoral districts where the party was running. Let v_{ij} be the number of votes in district i for party j. That is to say, the vote counts are the input data and form an $m \times n$ matrix. So does the output, the seat numbers x_{ij}. These seat numbers are to satisfy four properties:

$$
\begin{aligned}
&1) \quad \text{their sum within district } i \text{ exhausts the pre-specified district-seats } r_i;\\
&2) \quad \text{their sum for a each party } j \text{ exhausts the nation-wide party-seats } c_j;\\
&3) \quad x_{ij} = 0 \text{ if } v_{ij} = 0;\\
&4) \quad \text{the seat numbers } x_{ij} \text{ are ``as proportional as possible'' to the vote counts } v_{ij}.
\end{aligned}
\tag{1}
$$

In terms of the seat matrix $x = [x_{ij}]$ items (1)-1 and (1)-2 demand that its rows must sum to the pre-specified row marginals r_i, while column sums must be equal to the given column marginals c_j. It is less obvious, and more of a challenge, to turn the proportionality requirement (1)-4 into an operational concept.

3 Scaling in biproportional seat apportionment

A first proposal that makes biproportionality manageable is due to Balinski and Demange [3]. They set up a system of axioms justifying a procedure obeying the motto *Scale and round!* To obtain the seat numbers x_{ij}, find "row multipliers" $\rho_i > 0$ and "column multipliers" $\gamma_j > 0$ to scale the vote counts into the quantities $\rho_i \, v_{ij} \gamma_j$, and then round the scaled quantities to the nearest integer,

$$x_{ij} = \langle \, \rho_i \, v_{ij} \, \gamma_j \, \rangle.$$

The $m+n$ multipliers ρ_i and γ_j are determined in such a way that the resulting seat numbers x_{ij} fulfill the $m+n$ linear side conditions inherent in (1)-1 and (1)-2. The angle brackets $\langle t \rangle$ denote standard rounding, whereby a positive real number t is rounded to the nearest integer. A *tie* emerges whenever t happens to lie half-way between its neighboring integers; then there is the choice of rounding up or of rounding down. Balinski and Demange [4] proposed an out-of-kilter type algorithm, which got implemented under the name *Tie-and-Transfer* (TT) algorithm in the public domain software BAZI (see Maier and Pukelsheim [17]).

It is tempting, of course, to try and embed the Balinski and Demange procedure into an optimization approach. Following Carnal [6] and Helgason, Jörnsten, and Migdalas [13], Gaffke and Pukelsheim [8, 9] propose a problem formulation that is not restricted to standard rounding, but admits more general rounding rules. Any such rule equips an integer interval $[z-1, z]$ with a decision point $s(z)$, called *signpost*. Below $s(z)$ a positive $t \in [z-1, z]$ gets rounded down to $z-1$, above $s(z)$ it is rounded up to z. For instance, standard rounding uses the signpost sequence $s(z) = z - 1/2$, the midpoints of the integer intervals $[z-1,\ z]$. Let $N_v^{m\times n}$ denote the set of $m \times n$ integer matrices x inheriting all zeros that appear in the vote matrix v, that is, $v_{ij} = 0 \Rightarrow x_{ij} = 0$. The optimization formulation then reads as follows:

$$\text{Minimize} \quad \prod_{(i,j)\,:\,v_{ij}>0} \quad \prod_{z \le x_{ij}\,:\,s(z)>0} \frac{s(z)}{v_{ij}},$$
$$\sum_{j\le n} x_{ij} = r_i \text{ for all } i \le m$$
$$\sum_{i\le m} x_{ij} = c_j \text{ for all } j \le n,$$
$$x \in N_v^{m\times n}.$$

Gaffke and Pukelsheim point out that the above formulation can be reduced to a min-cost flow problem with convex separable objective function. From this perspective they prove that the Balinski and Demange procedure qualifies as a dual algorithm producing an optimal solution. Rote and Zachariassen [20] carry this approach further on, providing an efficient implementation and showing that its complexity is quite satisfactory.

Practical viewpoints call for a slight generalization of the problem formulation. The seat matrix x that results from the biproportional formulation, when held against the input vote matrix v, may feature *discordant seat assignments*. When comparing two cells (i, j) and (a, b), that is, party j in district i and party b in district a, it may happen that fewer votes go along with more seats, $v_{ij} < v_{ab}$ and $x_{ij} > x_{ab}$. Discordant seat assignments represent local adjustments that are unavoidable in order to achieve global biproportionality, as already observed by Gassner [11].

A particular irritation occurs when a single-seat district is struck by a discordant seat assignment, so that the one and only seat does not go to the district candidate who performed best, but someone else who did less well. To overcome this obstacle, Maier [16] proposes a Winner-Take-One (WTO) amendment stipulating that in each district the strongest party is allocated at least one seat. The network flow formulation from above is clearly powerful enough to support the additional district-wise WTO amendment.

Balinski and Demange's biproportional apportionment scheme has a continuous counterpart of long standing, where the fitted quantities x_{ij} are not restricted to be integers, but may be arbitrary nonnegative numbers. In Statistics it is called the *Iterative Proportional Fitting* (IPF) procedure, and is used for the analysis of contingency tables and the adjustment of doubly stochastic matrices. In Economics it is a tool for input-output analysis. In Computer Science it is called *matrix scaling*, or the *RAS* method, whose complexity was recently analyzed by Kalantari, Lari, Ricca, and Simeone [15]. The reader may wish also to consult the classical monograph on this method by Bacharach [2].

Two-way seat apportionment solutions, to districts as well as to parties, may alternatively be based on the outcome $q = [q_{ij}]$ of the continuous IPF procedure,

$$q_{ij} = \lambda_i \, v_{ij} \, \mu_j \ge 0.$$

That is, the row multipliers λ_i and column multipliers μ_j are such that the row sums of the matrix $q = [q_{ij}]$ match the row marginals r_i, and the column sums verify the pre-specified column marginals c_j. The fractional quantities q_{ij} may be taken as the "ideal share" (sometimes called "fair share" or "exact quota") of seats to which party j in district i would be entitled if fractional seats were allowed, see Balinski and Demange [3].

It is then a natural approach to approximate the unattainable quota matrix q by an integer matrix x with entries $x_{ij} = \lfloor q_{ij} \rfloor$ or $x_{ij} = \lceil q_{ij} \rceil$, by imposing the row and column restrictions on x and, at the same time, by minimizing a reasonable distance measure to q. This approach is followed by Cox and Ernst [7], and also in Gassner [11].

4 Error minimization in biproportional seat apportionment

Serafini and Simeone [22] observe that the idea of approximating a given quota matrix $q = [q_{ij}]$ by an integer matrix $x = [x_{ij}]$ works fine also with other target quotas, and it is not restricted to the ideal shares from the IPF procedure. They point out that, since Italy became a republic (1946), district-wise (i.e., row-wise) quotas have always been traditionally adopted for the Chamber elections,

$$q_{ij} = \frac{v_{ij}}{\sum_{b \le n} v_{ib}} r_i.$$

The seat matrix x thus obtained is close to, but of course not identical with, the biproportional seat matrix à la Balinski-Demange. Row-wise quotas are used also in other legislations, such as the Belgian one, but one should be aware that they may give rise to disturbing anomalies, as pointed out in Gassner [10] and in Pennisi [18].

It is conceivable also to use party-wise (i.e., column-wise) quotas. However, both in the Belgian and Italian legislation, districts and parties are not dealt with in a symmetric fashion, and proportionality within districts is felt to be prevalent on proportionality within parties. This asymmetry is confirmed by the fact that in the Italian system seats are assigned to parties by an ordinary law and to districts by the very Constitution. In any case, we believe that it is preferable to use the above defined fair shares; the more so since, in contrast with row- or column-wise quotas, they have the additional property that one can always obtain an integral feasible apportionment by rounding them down or up, in view of the Integrality Theorem of Flows.

In any case Serafini and Simeone do not make any particular assumption on how the numbers q_{ij} are defined and computed. It can be assumed that these numbers are given as input to the biproportional seat apportionment problem. They are called *target quotas* because the seats x_{ij} should adhere to them (in a proportional sense) as much as possible. The only constraint for the target quotas is that $\sum_{ij} q_{ij} = H$. The error w.r.t. to the target quotas in assigning the actual seats can be defined in different ways. The *absolute error* is defined as $\tau := \max_{ij} |x_{ij} - q_{ij}|$, and the *relative error* is $\sigma := \max_{(ij) \notin Z} |x_{ij} - q_{ij}|/q_{ij}$, where $Z = \{(ij) : v_{ij} = 0\}$.

The approach proposed by Serafini and Simeone calls for finding a seat apportionment satisfying (1) and minimizing either the absolute error or the relative error. To this aim, the Authors note that, given a bound $\tau > 0$ on the absolute error or a bound $\sigma > 0$ on the relative error, the problem can be modelled as a feasible flow problem with lower and upper arc capacities. The source nodes are the districts and the sink nodes are the parties. From each source i there is an outgoing flow equal to r_i, and into each sink j there is an incoming flow equal to c_j. In case we measure the absolute error, each arc (i, j) has a capacity interval

$$[\lceil q_{ij} - \tau \rceil^+ \ , \ \lfloor q_{ij} + \tau \rfloor] := [c_{ij}^- \ , \ c_{ij}^+],$$

where, by definition $a^+ := \max\{a, 0\}$. The bipartite graph of the flow problem is not complete in general. If $v_{ij} = 0$ the nodes i and j are not joined by an arc. For the relative error case lower and upper capacities are defined analogously.

A feasible flow x_{ij} satisfies $c^-_{ij} \le x_{ij} \le c^+_{ij}$ and, by flow properties, if there is a feasible flow x there is also an integral flow since the capacity values are integers. Hence one wants to find the minimum τ^* or σ^* such that a feasible flow exists.

The existence of a feasible flow can be easily established through the solution of a max-flow problem. The optimal τ^* or σ^* can be found by binary search. Since only a finite number of values for τ or for σ is relevant to the solution, the binary search is carried over this finite set, instead of the real range of possible values. The relevant values for the absolute error minimization are those values for τ such that either $q_{ij} - \tau$ or $q_{ij} + \tau$ is integral for some (i, j). Similarly, one can define the relevant values for the relative error minimization. Since the number of relevant values is at most $(H + 1) \cdot |N|$, binary search over these values takes $O(\log(H \cdot |N|))$ max flow computations, i.e. it works in polynomial time. A more detailed analysis of the binary search for the absolute error minimization reveals that the algorithm can be refined into a strongly polynomial algorithm.

5 Other issues

Serafini and Simeone [22] are also concerned with two other practical issues: uniqueness of the optimal assignment and possibility of providing the layman with a certificate of optimality. As for the first problem it is clear that any sound seat assignment method which takes as input the votes, must output a unique assignment. On the other hand, optimization problems usually admit multiple optimal solutions. Therefore it is crucial to develop a method that outputs a unique assignment. As for the second problem, it can be argued that sound assignment procedures are generally too complex to be fully understood by the general public. A voting system cannot be based on the simple trust that the persons involved in the computations are honest and do not make mistakes. Therefore a way to check the election outcome which does not call for difficult mathematical concepts should be provided.

One way to overcome the difficulty of non unique solutions consists in finding unordered lexico minima, as defined in Schrage [21]. In this case the vectors to be ordered consist of the absolute errors for all pairs (ij). In order to find unordered lexico minima, once a minimax solution has been found with relevant error τ_k for the *blocking pair* $(i(k)j(k))$ (k is the index of the relevant τ), a solution minimizing

$$\max_{(ij)\neq(i(k)j(k))} |x_{ij} - q_{ij}|$$

is found. This can be done as before with the only difference that the capacity for the pair $(i(k)j(k))$ is no longer changed. Once a second solution with error τ_h ($h < k$) and blocking pair $(i(h)j(h))$ has been found, one proceeds recursively by fixing the capacities of the blocking pairs one at a time. In this procedure one has to use relevant errors that are less than τ_{min} or σ_{min}. If for the current relevant τ $\tau < \min\{q_{ij} - \lfloor q_{ij} \rfloor ; \lceil q_{ij} \rceil - q_{ij}\} \le 1/2$ holds, one simply fixes the capacity interval for the arc (i, j) to $[\bar{q}_{ij}, \bar{q}_{ij}]$ with $\bar{q}_{ij}$ equal to q_{ij} rounded to the nearest integer and the computation is finished because there cannot be any better error. Hence it turns out that the blocking pairs are those for which the absolute error is larger than one half.

Serafini and Simeone [22] also point out that it is possible to exploit the max flow-min cut theorem to get a certificate of optimality whereby anybody can check through simple elementary calculations that no solution can be better than the given one.

6 The Give-up Problem

Ricca, Scozzari and Simeone [19] discuss network flow techniques in the context of seat give-ups of multiple winners in a system with blocked regional lists, such as the Italian one. The current Italian electoral law requires that each party presents, in each region, an *ordered* list of candidates. It also allows for the same person to be present in more than one list. Voters can cast their ballots for parties, but not for candidates. If a party receives w seats in a region, the winners of that party will be exactly the first w of its list in that region, but if a candidate is a winner in more than one region, he or she must give-up all the seats won but one. The decisions about give-ups are usually centralized. Clearly, central decisions must be based on inter-regional comparisons of preferences. To this purpose, the Authors consider two (possibly combined) models: 1) a strict linear order $\succ$ is defined over the set of all candidates. It is assumed that, if $i \succ j$, then i precedes j in all the lists L_k where both i and j are present; 2) for each region k and each candidate i in the list of that region, a *disutility* or *cost* c_{ki} of letting i win in region k is defined.

Actually, consider a party and suppose that $C = \{1, 2, \dots, i, \dots, n\}$ is the set of its candidates, with $|C| = n$. Let $R = \{1, 2, \dots, k, \dots, m\}$ be the set of regions. Let $L = (L_1, \dots, L_k, \dots, L_m)$ be a set of m ordered lists such that $L_k \subseteq C$ for every $k = 1, 2, \dots, m$. Finally, let $(s_1, \dots, s_k, \dots, s_m)$ be a vector of integers, where, for all k, s_k $(1 \leq s_k \leq |L_k|)$ denotes the number of seats obtained by the party in the k-th region, with $S = s_1 + s_2 + \dots + s_m$. W.l.o.g., one may assume that $n \geq S$.

The Authors make a clear distinction between the dynamic notion of 'give-up schedule', viewed as a sequence of seat acceptance/give-up decisions, and the static notion of 'feasible assignment', which is the outcome of the decision process.

Formally, a *give-up schedule* is any finite sequence of seat acceptance/rejection decisions by candidates that can be obtained through a procedure of the following type. Initially all the lists are *unscanned*. During the execution of the procedure, it may happen that the number of candidates remaining in an unscanned list is equal to, or even smaller than, the number of currently available seats for that list. In the former case, the list will be called *tight*, in the latter case *thin*. Each iteration consists of the following steps:

1. If all lists have been scanned then stop;
2. else if there exists some thin list, then stop;
3. else an unscanned list k is chosen;
4. the top candidate i of the list is examined and i either accepts or rejects (gives up) the seat in that region, subject to the following constraints:
   (a) if i has previously accepted a seat in another region, then i must refuse the seat of region k; moreover, if the list was tight it becomes thin.
   (b) if the list is tight or the seat in region k is the last available to i, then i must accept it;
5. candidate i is removed from list L_k;
6. if there are no more seats available in region k, then the list L_k is declared scanned.

Notice that such procedure halts if either all lists become scanned, or some unscanned list becomes thin.

A *feasible seat assignment* is an assignment of seats to candidates such that (i) each candidate gets at most one seat; (ii) the number of candidates who win a seat in region k is exactly s_k.

Consider the bipartite graph $B = (R, C, E)$, where there is an edge between $k \in R$ and $i \in C$ if candidate i is included in the list L_k. The edge-set is denoted E. Let $N(u)$ be the set of all vertices adjacent to vertex u. A feasible assignment is described by a binary matrix x=$[x_{ki}]$ such that $\sum_{k \in N(i)} x_{ki} \leq 1$, $\forall i \in C$, and $\sum_{i \in N(k)} x_{ki} = s_k$, $\forall k \in R$.

Proposition 1 Every feasible seat assignment is realized by some give-up schedule. Conversely, if a give-up schedule stops with all the lists scanned, then it produces a feasible assignment.

In general, one would like the final set of winners to be concentrated in the top part of the ranking $\succ$. This broad goal can be formalized in several ways. The simplest formulation is to find a feasible seat assignment whose winners are precisely the first S candidates in the linear order $\succ$. A feasible assignment satisfying this property will be called *perfect*. However, such assignment may not exist. A more general possibility is to define, for any given instance of the give-up problem, and for a given ranking $\succ$, the *height* of a feasible assignment as the smallest positive integer h such that all the candidates after the h-th in $\succ$ get no seats (of course, this definition does not imply that all the first h candidates are necessarily winners.) One then looks for a feasible assignment minimizing the height. A third option is to define a notion of *lex-best* assignment. For a given feasible assignment x, let $\mathcal{I}(x) = \{i_1, \dots, i_n\}$ be a binary indicator vector such that $|\mathcal{I}(x)| = n$ and for $\nu = 1, \ldots, n$, $i_\nu = \sum_{k \in R} x_{k\nu}$. Let x and y be two feasible assignments, $x \neq y$, and let $\mathcal{I}(x)$ and $\mathcal{J}(y)$ be the associated binary indicator vectors. One says that x is *lexicographically better* than (or *dominates*) y if $\mathcal{I}(x)$ is lexicographically greater than $\mathcal{J}(y)$. A feasible assignment x is called *lexicographically best* if it is not dominated by any other feasible assignment.

Let Q be the set of the first S candidates in $\succ$. Let B' be the subgraph of B induced by $R \cup Q$. Add to B' a source s and a sink t; then connect the sink s to each region-node $k = 1, \ldots, m$, and each candidate-node $i = 1, \ldots, S$ to the sink t. Finally, direct all edges in B' from R to Q. Let $\mathcal{N}$ be the resulting network. Each arc (s, k), $k = 1, \ldots, m$, is assigned both an upper and a lower capacity equal to s_k, while all the other arcs in $\mathcal{N}$ are assigned a lower capacity and an upper capacity equal to 0 and 1, respectively.

Proposition 2 A perfect seat assignment exists if and only if there exists a feasible flow in the network $\mathcal{N}$ with the above defined lower and upper capacities.

The Authors also show that one can find a feasible assignment minimizing the height by solving a *bottleneck transportation* problem. As a consequence, one can solve the height minimization problem in strongly polynomial time by $O(\log(n))$ max-flow computations.

As an alternative, one can use appropriate minimum cost network flow models. Namely, starting from the bipartite graph B, build the network $\mathcal{M}$ in a way similar to the above network $\mathcal{N}$. Moreover, let a nonnegative cost function $c(\cdot)$ be defined on the edges of $\mathcal{M}$. Next, assume that the costs on the arcs $\{(s, k) : k \in R\} \cup \{(i, t) : i \in C\}$ are all equal to zero. One can consider different cost functions c_{ki}, $\forall k \in R,\ i \in C$, e.g. :

i) $c_{ki} = i \quad \forall k, i$ (*arithmetic model*);

ii) $c_{ki} = 2^i \quad \forall k, i$ (*geometric model*).

Proposition 3 A perfect seat assignment exists if and only if there exists an optimal solution x^* to the arithmetic minimum cost flow problem on $\mathcal{M}$ with cost $c(x^*) = S\,(S+1)/2$.

Proposition 4 Any optimal solution to the geometric minimum cost flow model on $\mathcal{M}$ provides a feasible assignment with minimum height.

Given an instance of the give-up problem and the corresponding bipartite graph (R, C, E), for any given feasible assignment $\mathbf{x}$ an *illegitimate* path w.r.t. $\mathbf{x}$ is an even path from a non-winner candidate i to a winner candidate j, with $i \succ j$, and formed alternately by edges with flow 0 and flow 1 in $\mathbf{x}$.

The following characterization result holds:

Proposition 5 Given an instance of the give-up problem, the following three statements are equivalent:

1) x is an optimal solution to the arithmetic min cost flow model;

2) there exists no illegitimate path w.r.t. x;

3) x is lexicographically best.

Let M be the number of edges of $\mathcal{M}$. On the grounds of the above result, an $O(Mn)$ algorithm for solving the arithmetic min cost network flow problem can be easily derived.

References

[1] R.K. Ahuja, T.L. Magnanti and J.B. Orlin, *Network flows. Theory, algorithms and applications*, Prentice Hall, New Jersey, 1993.

[2] M. Bacharach, *Biproportional Matrices & Input-Output Change*, Cambridge, UK, 1970.

[3] M. Balinski and G. Demange, *An axiomatic approach to proportionality between matrices*, Mathematics of Operations Research **14** (1989) pp. 700–719.

[4] M. Balinski and G. Demange, *Algorithms for proportional matrices in reals and integers*, Mathematical Programming **45** (1989) pp. 193–210.

[5] M. Balinski and H.P. Young, *Fair Representation—Meeting the Ideal of One Man, One Vote. Second Edition*, Washington, DC, 2001.

[6] H. Carnal, *Mathématiques et politique*, Elemente der Mathematik **48** (1993) pp. 27–32.

[7] L.H. Cox and L.R. Ernst, *Controlled rounding*, INFOR—Information Systems and Operational Research **20** (1982) pp. 423–432.

[8] N. Gaffke and F. Pukelsheim, *Divisor methods for proportional representation systems: An optimization approach to vector and matrix apportionment problems*, Mathematical Social Sciences **56** (2008) pp. 166–184.

[9] N. Gaffke and F. Pukelsheim, *Vector and matrix apportionment problems and separable convex integer optimization*, Mathematical Methods of Operations Research **67** (2008) pp. 133–159.

[10] M.B. Gassner, *Two-dimensional rounding for a quasi-proportional representation*, European Journal of Political Economy **4** (1988) pp. 529–538.

[11] M.B. Gassner, *Représentations parlementaires—Méthodes mathématiques biproportionnelles de Répartition des Sièges*, Bruxelles, 2000.

[12] P. Grilli di Cortona, C. Manzi, A. Pennisi, F. Ricca and B. Simeone, *Evaluation and optimization of electoral systems*, SIAM Monographs on discrete mathematics and applications, Philadelphia, SIAM, Society for Industrial and Applied Mathematics, 1999.

[13] T. Helgason, K. Jörnsten and A. Migdalas, *Maxmin formulation of the apportionments of seats to a parliament*, in *Minimax and Applications*, D. Du and P.M. Pardalos, eds., 1995, Dordrecht, pp. 109–118.

[14] W. Hess, J.B. Weaver, H.J. Siegfelatt, J.N. Whelan and P.A. Zitlau, *Non partisan political redistricting by computer*, Operations Research **13** (1965) pp. 998-1006.

[15] B. Kalantari, I. Lari, F. Ricca and B. Simeone, *On the complexity of general matrix scaling and entropy minimization via the RAS algorithm*, Mathematical Programming, Series A **112** (2008) pp. 371–401.

[16] S. Maier, *Biproportional Apportionment Methods: Constraints, Algorithms, and Simulation*, Dissertation. Institut für Mathematik, Universität Augsburg, 2008.

[17] S. Maier and F. Pukelsheim, *Bazi: A Free Computer Program for Proportional Representation Apportionment*, Preprint Nr. 042/2007. Institut für Mathematik, Universität Augsburg, 2007. Internet: `www.opus-bayern.de/uni-augsburg/volltexte/2007/711/`, 2007.

[18] A. Pennisi, *The Italian Bug: a flawed procedure for bi-proportional seat allocation*, in *Mathematics and Democracy. Recent Advances in Voting Systems and Collective Choice, Studies in Choice and Welfare*, 2006, Springer, Berlin, pp. 51–165.

[19] F. Ricca, A. Scozzari and B. Simeone, *The Give-up Problem for blocked regional lists with multi-winners*, Technical Report 16-2008, Dipartimento di Statistica, Probabilità e Statistiche Applicate, Università di Roma “La Sapienza”, 2008.

[20] G. Rote and M. Zachariassen, *Matrix scaling by network flow*, in *Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms, Proceedings in Applied Mathematics, 125*, 2007, Philadelphia, PA, pp. 848–854.

[21] L. Schrage, *LINDO: An Optimization Modeling System*, Fourth Edition. The Scientific Press, San Francisco, Cal, 1991.

[22] P. Serafini and B. Simeone, *Network flow methods for best approximation of target quotas in biproportional apportionment*, Technical Report 15-2008, Dipartimento di Statistica, Probabilità e Statistiche Applicate, Università di Roma “La Sapienza”, 2008.