Arc-Flow Model for the Two-Dimensional Cutting Stock Problem

Rita Macedo* Cláudio Alves** J. M. Valério de Carvalho**

*Centro de Investigação Algoritmi, Universidade do Minho
Escola de Engenharia, Universidade do Minho, 4710-057 Braga, Portugal

*Departamento de Produção e Sistemas, Universidade do Minho
Escola de Engenharia, Universidade do Minho, 4710-057 Braga, Portugal

---

Abstract

We describe an exact model for the two-dimensional Cutting Stock Problem with two stages and the guillotine constraint. It is a linear programming arc-flow model, formulated as a minimum flow problem, which is an extension of a model proposed by Valério de Carvalho for the one dimensional case. In this paper, we explore how this model behaves when it is solved with a commercial software, explicitly considering all its variables and constraints. We also derive a new family of cutting planes, and consider some extensions of the original problem. The model was tested on a set of real instances from the wood industry, with promising results.

Keywords: *two-dimensional cutting stock, network flow, lower bounds*

---

1 Introduction

Cutting stock problems (*CSP*) are combinatorial optimization problems, which occur in many real-world applications of business and industry, motivating several areas of research. Generally speaking, they consist of cutting a given set of small objects, called *items*, out of a given set of larger ones, called *stock sheets*, usually with the objective of reducing the area of generated waste to a minimum. Due to the complexity and extensive nature of these problems, many different optimization formulations and solution approaches arise in the literature, according to their dimension, application field and special constraints and requirements. Therefore, many researchers provided surveys and categorised bibliographies on this subject (Dowsland and Dowsland [1], Dyckhoff and Finke [2], Lodi et al. [3, 4], Dyckhoff et al. [5], among others). Moreover, Dyckhoff [6] defined a formal typology for cutting and packing problems, which systematically integrates the various kinds of problems and notions. This typology was improved by Wäscher et al. [7] with the definition of new categorization criteria.

In this paper, we propose a new model to solve exactly the two-dimensional cutting stock problem (*2D-CSP*), with two stages and the guillotine constraint. It is an extension for the two dimensions of the arc-flow model proposed by Valério de Carvalho in [8]. According to the typology defined by Wäscher et al. [7], this problem can be categorized as a 2D regular SSSCSP (*Single Stock Cutting Stock Problem*).

The *CSP*, with all its extensions and variants, is well known to be NP-hard. Therefore, it is usually approached with methods like branch-and-bound, column generation or dynamic programming. We did not devise any method to find the integer solution of our problem. Instead, as our model has a pseudo-polynomial number of variables and constraints, we considered them all explicitly and solved the problem by using a commercial solver, in order to investigate its performance.

In Section 2, the two-dimensional cutting stock problem is defined. In Section 3, the arc-flow model is presented, as well as a new family of cutting planes. In Section 4, two extensions are described. Some computational results are reported in Section 5, and finally, in Section 6, some conclusions are drawn.

2 Two-Dimensional Cutting Stock Problem

The two-dimensional version of the *CSP* can be stated as follows: a given set of small items, each item $i \in \{1, ...m\}$ of width w_i, height h_i and demand of b_i pieces, has to be cut out of a virtually infinite supply of stock sheets of width W and height H (where $0 < w_i \leq W$ and $0 < h_i \leq H$, $\forall i \in \{1, ..., m\}$), in order to minimize the number of used stock sheets.

The *2D-CSP* can be further classified into several categories, depending on the problem's specific constraints. It can be *regular*, if the shapes of the items to be cut can be described by few parameters, or *irregular*, otherwise. The cutting of irregular shapes is also known as *nesting*. Regular cuts can be *rectangular* or *non-rectangular*, for the cases where the items are rectangles or have a different shape, respectively. The rectangular cutting can be *oriented*, if an item of width w and height h is considered to be different from another one of width h and height w, or *non-oriented*, otherwise. If all cuts must be made straight from one edge to the opposite edge of the stock sheet (or of one of its already cut fragments), dividing it in two, the cutting patterns produced are of *guillotine* type. *Non-guillotine* patterns are not restricted by this rule, being the problems which consider them much harder to solve. A *staged* pattern is a guillotine pattern cut into pieces in a limited number of phases. The direction of the first stage cuts may be either horizontal or vertical (parallel to one side of the stock sheet), and the cuts of the same stage are in the same direction. The cut directions of any two adjacent stages must be perpendicular to each other. If the maximum number of stages is not allowed to exceed n, the problem is called *n-staged*. When there is no such restriction, the problem is called *non-staged*. Whenever a final stage is allowed only to separate small items from waste areas, the problem is called *non-exact*. Otherwise, it is called *exact*. The focus of this paper goes to the exact resolution of the two-staged, non-exact *2D-CSP* with the guillotine constraint.

3 Arc-Flow Model

Valério de Carvalho [8] proposed a linear programming arc-flow model, formulated as a minimum flow problem, for the *1D-CSP* (1)-(4), in which every cutting pattern corresponds to a path in an acyclic directed graph $G = (V, A)$, with $V = \{0, 1, ..., W\}$ as its set of $W + 1$ vertices, which define positions in the stock sheet, and $A = \{(a, b) : 0 \leq a < b \leq W \quad \text{and} \quad b - a = w_i, \forall i = 1, ..., m\}$ as its set of arcs. It is formulated as a minimum flow problem, where variables x_{ab} correspond to the arc's (a, b) flow, i.e., the number of items of width $b - a$ placed at a distance of a units from the beginning of a given stock sheet, and variable z corresponds to the total flow that goes trough the graph and can be seen as the return flow from vertex W to vertex 0.

$$\min \quad z \tag{1}$$

$$\text{s.t.} \quad \sum_{(a,b)\in A} x_{ab} - \sum_{(b,c)\in A} x_{bc} = \begin{cases} -z & , \text{if } b = 0 \\ 0 & , \text{if } b = 1, 2, ..., W - 1 \\ z & , \text{if } b = W \end{cases} \tag{2}$$

$$\sum_{(c,c+w_i)\in A} x_{c,c+w_i} \geq b_i, \quad \forall i \in \{1, ..., m\} \tag{3}$$

$$x_{ab} \geq 0 \text{ and integer}, \quad \forall (a, b) \in A \tag{4}$$

Constraints (2) are related with flow conservation and constraints (3) ensure that all the demands are fulfilled. Valério de Carvalho [8] defined three criteria in order to reduce the symmetry of the solution space and the size of the model, as they reduce the number of allowable cutting patterns. Therefore, for any pattern, the items are sorted by their decreasing widths. The waste of a stock sheet is represented by unit arcs that, therefore, always appear in the last positions of the stock sheet. For example, a pattern cut out of a stock sheet of width $W = 15$, composed by three items of widths 6, 4 and 3 would correspond to a path with one arc of size 6, one arc of size 4, one arc of size 3 and two arcs of size 1.

We extended this formulation for the two-stage *2D-CSP* with the guillotine constraint. In this variant of the problem, we consider that, in the first stage, the stock sheets are cut into horizontal strips, and these strips are, in the second stage, cut into the demanded items. This approach allows us to handle a two dimensional problem as a set of one dimensional ones. In the first stage, we have to cut strips out of the stock sheets. This means that we only need to consider the items' and the stock sheets' heights. In the second stage, the generated strips are cut into the demanded items (with vertical cuts). Again, this means that we only have to consider the items' and stock sheets' widths, once we have defined the set of items that can be cut out of a given strip (the ones whose height is smaller than or equal to the strip's height). For this extension, we consider $H^* = \{h_1^*, ..., h_{m_h}^*\} \subseteq \{h_1, \ldots, h_m\}$ as the set of m_h different heights (without repetitions) ordered by their increasing values, a graph $G^0 = (V^0, A^0)$, with $V^0 = \{0, 1, ..., H\}$ and $A^0 = \{(a,b) : 0 \le a < b \le H \quad \text{and} \quad b - a = h_i^*, \forall i \in \{1, ..., m_h\}\}$, for the first stage, and a set of graphs $G^s = (V^s, A^s)$, with $V^s = \{0, 1, ..., W\}$ and $A^s = \{(d,e) : 0 \le d < e \le W \quad \text{and} \quad e - d = w_i, \forall i \in \{1, ..., m\} : h_i \le h_s\}$, for each of the m_h sets of patterns of the second stage. As in [8], every set A^s, $\forall s \in \{0, 1, ..., m_h\}$, includes unit arcs that represent the stock sheet's waste.

$$\min \quad z^0 \tag{5}$$

$$\text{s.t.} \quad \sum_{(a,b)\in A^0} x^0_{ab} - \sum_{(b,c)\in A^0} x^0_{bc} = \begin{cases} -z^0 & \text{, if } b = 0 \\ 0 & \text{, if } b = 1, 2, ..., H-1 \\ z^0 & \text{, if } b = H \end{cases} \tag{6}$$

$$\sum_{(c,c+h_s^*)\in A^0} x^0_{c,c+h_s^*} - z^s = 0, \quad \forall s \in \{1, ..., m_h\} \tag{7}$$

$$\sum_{\substack{(d,e)\,\in\, A^s \\ h^*\,\in\, H^*}} x^s_{deh^*} - \sum_{\substack{(e,f)\,\in\, A^s \\ h^*\,\in\, H^*}} x^s_{efh^*} = \begin{cases} -z^s & \text{, if } e = 0 \\ 0 & \text{, if } e = 1, 2, ..., W-1 \\ z^s & \text{, if } e = W \end{cases} \quad , \forall s \in \{1, ..., m_h\} \tag{8}$$

$$\sum_{s=1}^{m_h} \sum_{(f,f+w_i)\in A^s} x^s_{f,f+w_i,h_i} \ge b_i, \quad \forall i \in \{1, ..., m\} \tag{9}$$

$$x^0_{ab} \ge 0 \text{ and integer}, \quad \forall (a,b) \in A^0 \tag{10}$$

$$x^s_{deh^*} \ge 0 \text{ and integer}, \quad \forall (d,e) \in A^s, \quad \forall s \in \{1, ..., m_h\}, \quad \forall h^* \in H^* \tag{11}$$

In this formulation, variable z^0 represents the number of used stock sheets and variables z^s, $\forall s \in \{1, ..., m_h\}$, denote the number of strips of height h_s^* cut in the first stage. Variables x^0_{ab} represent an arc's (a,b) flow, on graph G^0, and variables $x^s_{cdh^*}$ represent the flow, on graph G^s, which goes through an arc (c,d) corresponding to items of width $(d-c)$ and height $h^* \in H^*$. The third index h^* differentiates items with the same width but different heights, within a same graph G^s. Constraints (7) make the connection between the two stages of the problem: the number of strips of height h_i^*, $\forall i \in \{1, ..., m_h\}$, cut in the first stage must be equal to the number of strips of height h_i^* to be cut into the demanded items in the second stage. Constraints (6) and (8) are related with flow conservation of the first and second stages, respectively, and constraints (9) ensure that all the demands are fulfilled.

In order to strengthen the model, we derived a new family of cutting planes. They rely on the fact that all items with heights greater than or equal to h_j must be cut out of strips of height greater than or equal to h_j. Considering the trivial lower bound, with $I_j = \{i \in \{1, ..., m\} : h_i \ge h_j\}$, we can say that

$$\sum_{l=j}^{m_h} z^l \geqslant \left\lceil \frac{\sum_{i\in I_j} w_i b_i}{W} \right\rceil \qquad \forall j \in \{1, \ldots, m_h\} \tag{12}$$

The model has a pseudo-polynomial number of variables and constraints. Variables represent arcs in the arc-flow model, and reducing the number of arcs reduces the size of the model, increasing its efficiency. Three reduction criteria were applied, none of them eliminating cutting patterns that cannot be replaced by equivalent ones, in what concerns finding the minimum number of used stock sheets.

4 Extensions to the Model

The first extension to the model is for the *Non-Oriented case*. If the stock sheets are made of materials that have any kind of pattern (for example, the wood grain), the rotation of the items is not allowed. This means that an item of width w and height h is considered to be different from an item of width h and height w, as they are meant to have a different orientation of the original pattern (the *oriented* case, as mentioned above). When this does not happen, and rotation is allowed, the number of feasible cutting patterns increases. Our model does not consider rotation. However, it can be easily considered by defining a different item i' for each item i, such that $h_{i'} = w_i$, $w_{i'} = h_i$ and $b_{i'} = 0$.

Another extension is the *orientation of the first stage's cuts.* So far, we considered that, in the first stage, cuts are horizontal (generating horizontal strips) and that, in the second stage, cuts are vertical. Clearly, the opposite can also happen: first-stage vertical cuts and second-stage horizontal cuts. This obviously generates different cutting patterns, which may be attractive. Therefore, in order to eventually reduce the number of used stock sheets, it may be convenient to allow the first cut to be either horizontal or vertical, making it possible to have, in a same cutting plan, some stock sheets with horizontal strips and others with vertical ones. Although this can potentially improve the optimal solution, as more cutting patterns are allowed, it increases considerably the model's size, as we have to consider two sets of graphs for the second stage, one for the horizontal strips and another one for the vertical strips.

5 Computational Results

The model was tested with two sets of real instances from the furniture industry, sets A and B. The set A instances have an average demand of 12.7, while in the set B instances the average demand is 249.8. The average total number of items of the two sets of instances are, respectively, 28.7 and 8.9.

Tables 1 and 2 describe the computational results for the set A and set B instances, respectively, for the original version of the arc-flow model without the cutting planes, $Arc-Flow$, for the branch-and-price algorithm described in Alves et al. [9], *G&G*, and for the ILP model proposed in [10], *Lodi*. Z_{RL} and Z stand for the linear relaxation and integer solution, respectively, and $t(s)$ represents the total computational time, in seconds. An asterisk (*) means that the algorithm did not find the optimal solution of the instance within the time limit of 7200 seconds.

Table 1: Set A instances results.

		$Arc-Flow$		G&G		Lodi				$Arc-Flow$		G&G		Lodi	
Name	Z	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	**Name**	Z	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$
A-1	4	3.300	0.203	3.300	0.020	2.221	0.063	A-23	14	12.922	0.765	12.922	2093.860	11.801	*
A-2	36	36.000	0.329	36.000	0.020	22.415	0.016	A-24	35	34.321	1012.610	34.356	1739.190	30.936	*
A-3	8	8.000	0.156	8.000	0.050	4.861	0.016	A-25	18	17.015	2595.828	17.327	*	16.261	*
A-4	3	2.667	0.187	2.667	0.000	2.185	0.016	A-26	8	7.049	19.984	7.080	24.380	6.436	0.828
A-5	13	12.525	0.751	12.525	0.750	11.158	32.000	A-27	20	19.037	3965.547	19.200	*	17.979	*
A-6	2	1.889	0.094	1.889	0.020	1.179	0.016	A-28	12	10.813	118.875	11.167	985.020	10.253	18.109
A-7	14	13.125	0.641	13.125	0.030	9.388	*	A-29	28	27.281	3155.829	27.287	*	25.324	*
A-8	2	1.067	0.078	1.067	0.010	1.000	0.047	A-30	4	3.593	15.500	3.677	14.310	3.408	0.219
A-9	61	60.671	27.343	60.671	*	54.742	*	A-31	8	7.470	23.390	7.487	81.970	6.837	2.328
A-10	3	1.969	0.344	2.000	0.060	1.844	48.125	A-32	27	26.849	83.907	26.897	2257.090	24.324	*
A-11	46	45.758	7.718	45.758	54.800	43.331	*	A-33	35	34.634	4793.702	34.670	*	32.705	*
A-12	14	14.000	0.329	14.000	0.010	10.861	0.141	A-34	6	5.195	3.125	5.221	6.990	4.634	0.203
A-13	14	13.533	0.906	13.533	0.750	11.864	*	A-35	17	16.360	44.719	16.401	358.700	15.170	*
A-14	67	66.824	183.531	66.824	*	63.081	*	A-36	9	8.875	1.719	8.875	0.700	6.546	0.375
A-15	39	38.942	1.110	38.942	3.030	35.056	*	A-37	5	4.631	0.859	4.631	*	4.169	16.484
A-16	83	82.256	12.031	82.256	15.610	77.936	*	A-38	23	22.107	11.828	22.123	54.730	20.205	*
A-17	5	4.704	0.344	4.704	*	4.458	13.828	A-39	4	3.017	1.000	3.058	0.380	2.782	0.047
A-18	65	64.389	2.969	64.681	*	59.339	*	A-40	17	15.813	79.203	15.896	*	14.390	*
A-19	58	57.234	17.328	57.234	15.580	50.929	*	A-41	19	18.500	4.719	18.500	11.970	14.183	0.281
A-20	27	26.098	1.219	26.098	1.280	22.415	*	A-42	8	7.167	0.954	7.167	0.080	4.533	0.000
A-21	28	27.235	57.625	27.235	10.080	25.329	*	A-43	7	6.375	0.999	6.417	5.030	4.797	0.031
A-22	3	2.400	0.235	2.400	0.010	1.934	0.047								

Table 2: Set B instances results.

		$Arc-Flow$		$G\&G$		$Lodi$				$Arc-Flow$		$G\&G$		$Lodi$	
Name	Z	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	**Name**	Z	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$	Z_{LR}	$t(s)$
B-1	76	74.953	0.593	74.953	8.980	68.159	*	B-62	18	17.548	0.532	17.548	0.060	16.299	*
B-2	171	170.792	1.078	170.792	0.220	149.817	*	B-63	38	37.755	1.062	37.755	0.310	33.980	*
B-3	110	108.913	0.235	108.912	0.110	87.949	*	B-64	20	19.276	1.156	19.276	2.780	18.383	*
B-4	21	20.700	0.984	20.700	0.110	17.824	334.578	B-65	172	171.559	3.171	171.559	10.170	*	*
B-5	11	10.528	0.454	10.528	0.140	8.935	8.422	B-66	41	39.997	1.172	39.997	1.520	35.999	*
B-6	10	9.667	0.375	9.667	0.010	8.893	0.078	B-67	262	260.960	21.859	260.960	3049.280	*	*
B-7	537	536.250	0.343	536.250	0.010	*	*	B-68	49	48.937	2.281	48.937	*	46.135	*
B-8	334	333.889	0.297	333.889	0.130	*	*	B-69	20	19.938	0.219	19.938	0.160	18.265	*
B-9	85	84.189	63.141	84.189	*	*	*	B-70	117	116.156	2.016	116.156	*	108.342	*
B-10	78	77.744	388.453	77.746	*	*	*	B-71	66	65.081	1.344	65.081	17.500	60.677	*
B-11	271	270.049	20.109	*	*	*	*	B-72	34	33.780	0.546	33.780	1.580	32.316	*
B-12	465	464.963	0.344	464.963	0.080	*	*	B-73	19	18.258	0.641	18.258	0.480	17.003	*
B-13	230	229.109	1.297	257.950	*	*	*	B-74	309	308.594	1.548	308.594	*	*	*
B-14	35	34.333	0.781	34.333	0.980	29.598	*	B-75	126	125.859	0.703	125.859	1.360	*	*
B-15	16	15.444	1.422	15.444	0.310	12.059	15.922	B-76	221	220.511	0.328	220.511	0.160	*	*
B-16	24	23.254	0.845	23.254	0.940	21.708	*	B-77	208	207.001	1.609	207.001	67.670	*	*
B-17	12	11.125	0.171	11.125	0.080	7.592	0.250	B-78	73	72.533	0.281	72.533	0.130	*	*
B-18	14	13.104	0.266	13.104	*	11.950	10.797	B-79	50	49.534	0.500	49.534	2.810	44.858	*
B-19	47	46.214	1.094	46.214	2.170	43.371	*	B-80	57	56.911	0.500	56.911	1.240	*	*
B-20	95	95.000	0.844	95.000	0.030	84.370	*	B-81	425	424.042	1.828	424.042	62.450	*	*
B-21	229	228.333	0.422	228.333	0.110	*	*	B-82	105	104.828	1.187	104.828	4.440	91.376	*
B-22	17	16.497	0.610	16.497	0.230	15.075	*	B-83	97	96.548	1.641	96.548	9.700	89.383	*
B-23	85	84.332	1.109	84.332	3.110	80.823	*	B-84	103	102.167	1.265	102.167	*	94.452	*
B-24	9	8.827	0.203	8.827	0.110	8.273	*	B-85	106	105.048	0.578	105.048	0.530	*	*
B-25	47	46.750	0.344	46.750	0.030	40.111	*	B-86	106	105.048	0.563	105.048	0.530	*	*
B-26	128	127.705	0.657	127.705	*	*	*	B-87	128	127.731	0.672	127.731	0.940	*	*
B-27	221	220.511	0.343	220.511	0.160	*	*	B-88	226	225.615	0.328	225.615	0.610	*	*
B-28	38	36.981	0.297	36.981	0.160	35.986	*	B-89	353	352.436	0.656	352.436	4.140	*	*
B-29	19	18.243	0.391	18.243	0.410	16.970	*	B-90	841	840.424	0.500	840.424	0.130	*	*
B-30	57	56.768	16.515	56.768	5.690	54.929	*	B-91	317	316.501	0.812	316.501	*	*	*
B-31	30	29.333	0.609	29.333	0.050	25.644	*	B-92	176	175.238	0.344	175.238	0.250	*	*
B-32	192	191.068	3.563	191.068	*	*	*	B-93	201	200.066	0.781	200.066	4.700	*	*
B-33	52	50.889	0.750	50.889	0.780	44.801	*	B-94	12	11.083	0.438	11.083	0.050	10.304	*
B-34	83	82.667	1.610	82.667	1.640	67.846	*	B-95	60	59.603	0.344	59.603	*	52.306	*
B-35	96	95.833	0.391	95.833	0.080	85.315	226.531	B-96	124	123.750	0.625	123.750	*	*	*
B-36	96	95.833	0.406	95.833	0.030	87.387	*	B-97	60	60.000	0.203	60.000	0.010	*	*
B-37	53	52.222	0.391	52.222	0.030	44.228	*	B-98	68	67.017	1.266	67.017	3.280	57.340	*
B-38	35	34.583	0.172	34.583	0.000	26.891	705.750	B-99	1095	1095.000	0.781	1095.000	0.010	*	*
B-39	35	35.000	0.188	35.000	0.000	26.963	35.875	B-100	59	58.571	15.859	58.571	13.640	56.814	*
B-40	19	18.639	2.469	18.639	0.060	17.268	*	B-101	59	58.339	3.813	58.339	*	51.903	*
B-41	17	16.673	4.219	16.673	0.660	14.546	*	B-102	46	45.218	2.125	45.218	*	43.111	*
B-42	16	15.389	0.281	15.389	0.090	14.501	*	B-103	41	39.860	3.266	39.860	*	36.522	*
B-43	6	5.333	0.687	5.333	0.550	4.468	0.563	B-104	3	2.333	0.141	2.333	0.020	1.958	0.047
B-44	24	23.315	12.375	23.315	*	22.023	*	B-105	34	33.333	0.109	33.333	0.000	29.922	73.406
B-45	11	10.167	2.875	10.167	0.110	9.723	*	B-106	71	70.178	2.047	70.178	*	64.672	*
B-46	28	26.986	1.062	26.986	0.030	22.120	*	B-107	300	298.996	586.718	298.996	*	*	*
B-47	8	7.917	0.125	7.917	0.060	7.430	4.063	B-108	185	183.716	*	183.716	*	169.267	*
B-48	14	13.446	10.407	13.446	1.310	13.216	*	B-109	26	25.136	4.609	25.136	9.230	21.087	*
B-49	21	20.071	0.437	20.071	0.030	15.958	1.313	B-110	4	3.200	0.125	3.200	0.000	2.319	0.078
B-50	71	70.357	0.109	70.357	0.000	61.279	5293.469	B-111	56	55.170	79.595	55.171	186.770	51.114	*
B-51	63	62.143	1.047	62.143	0.190	52.234	*	B-112	9	8.103	1.172	8.103	1.380	6.991	*
B-52	120	119.500	0.141	119.500	0.000	*	*	B-113	231	230.436	0.422	230.436	0.660	*	*
B-53	3	2.800	0.469	2.800	0.030	2.354	6.703	B-114	-	279.984	*	279.985	*	*	*
B-54	13	12.200	0.265	12.200	4.720	10.916	*	B-115	15	14.188	9.031	14.188	0.010	9.966	116.969
B-55	51	49.938	0.860	49.938	26.330	47.064	*	B-116	192	191.272	1765.781	191.272	*	*	*
B-56	214	213.831	4.047	213.831	16.630	*	*	B-117	6	5.300	0.656	5.300	0.000	4.787	0.219
B-57	44	43.856	34.547	43.857	*	41.751	*	B-118	36	35.250	4.188	35.250	0.020	32.296	*
B-58	20	19.119	0.360	19.119	0.140	17.972	*	B-119	144	143.318	8.140	143.318	*	*	*
B-59	11	10.393	0.219	10.393	0.010	9.123	1648.484	B-120	5	4.652	0.609	4.652	*	4.172	10.156
B-60	13	12.165	0.531	12.165	0.090	11.251	*	B-121	3	2.100	0.266	2.100	0.270	1.529	*
B-61	6	5.333	0.235	5.333	0.010	4.688	*								

Furthermore, the arc-flow model of the first extension, the second extension and both extensions, as described in Section 4, solved exactly within the time limit about 70%, 77% and 58% of the set A instances, and 93%, 98% and 92% of the set B instances, respectively.

Although we did not use the family of cutting planes defined in Section 3, we compared the linear relaxations of the original version of the arc-flow model without and with the cutting planes for the set A and set B instances, respectively. The cutting planes improved the linear relaxation in about 35% of the set A instances, and about 17% of the set B instances. In three instances of set A and one instance of set B, the lower bound provided by the linear relaxation increased one unit.

6 Conclusions

We presented an exact arc-flow model for the two-dimensional two-stage Cutting Stock Problem with guillotine constraints. Starting with all variables and constraints, we applied reduction criteria, to reduce the size and symmetry of the model, and solved it with CPLEX considering a new family of cutting planes. We also explored some extensions of the problem: rotation of items, and horizontal or vertical orientation for the first cut. Finally, we presented computational results for real instances from the furniture industry. For the instances tested, it proved to be more efficient than other previous methods.

Acknowledgement

This work was partially supported by FCT through the doctoral grant SFRH/BD/39607/2007 for Rita Macedo and by the Project SCOOP (Sheet cutting and process optimization for furniture enterprises) (COOP-CT-2006-032998), funded by the EC, 6th FP on RTD & D, specific actions for SMEs.

References

[1] K. A. Dowsland and W. B. Dowsland, *Packing Problems*, European Journal of Operational Research **56** (1992) pp. 2–14.

[2] H. Dyckhoff and U. Finke, *Cutting and Packing in Production and Distribution*, Physical Verlag, Heidelberg, 1990.

[3] A. Lodi, S. Martello and M. Monaci, *Two-dimensional packing problems: a survey*, European Journal of Operational Research **141** (2002) pp. 241–252.

[4] A. Lodi, S. Martello and D. Vigo, *Recent Advances on Two-dimensional Bin Packing Problems*, Discrete Applied Mathematics **123** (2002) pp. 379–396.

[5] H. Dyckhoff, G. Scheithauer and J. Terno, *Cutting and packing: An annotated bibliography*, in *Annotated Bibliographies in Combinatorial Optimization*, M. Dell ' Amico, F. Maffioli and S. Martello, eds., Wiley, Chichester, 1997, pp. 393–412.

[6] H. Dyckhoff, *A typology of cutting and packing problems*, European Journal of Operational Research **44** (1990) pp. 145–159.

[7] G. Wäscher, H. Haußner and H. Schumann, *An Improved Typology of Cutting and Packing Problems*, European Journal of Operational Research **183** (2007) pp. 1109–1130.

[8] J. M. Valério de Carvalho, *Exact solution of bin-packing problems using column generation and branch-and-bound*, Annals of Operations Research **86** (1999) pp. 629–659.

[9] C. Alves, R. Macedo, M. Mrad, J. M. Valério de Carvalho, F. Alvelos, T. M. Chan and E. Silva, *An Exact Branch-and-Price Algorithm for the Two-Dimensional Cutting Stock Problem*, submitted to International Transactions in Operational Research (2008).

[10] A. Lodi, S. Martello and D. Vigo, *Models and Bounds for Two-Dimensional Level Packing Problems*, Journal of Combinatorial Optimization **8** (2004) pp. 363–379.