Multi-commodity Single Path Routing with Submodular Bandwidth Consumption[1]

Arie M.C.A. Koster* Manuel Kutschka*

*Centre for Discrete Mathematics and its Applications (DIMAP)
Warwick Business School, University of Warwick, Coventry, CV4 7AL, UK

Abstract

In this paper, we propose the use of submodular functions for robust network optimization. The approach is illustrated by a single path routing problem using results for the submodular knapsack problem.

Keywords: *robust network optimization, statistical multiplexing, submodular knapsack*

1 Introduction

Designing telecommunication networks that are robust against traffic fluctuations is an emerging research direction in network optimization. The virtual private network (VPN) design problem is the most prominent example in this context. Classically, a network has to be designed without knowledge of the actual traffic demands. A traffic matrix is usually based on estimations of the future demands, topped up by a safety factor. Hence, the more connection requests are routed across the same link, the more bandwidth is reserved for traffic fluctuations. It is however unlikely that all requests have high bandwidth requirements at the same time. This property of communication requests is known as *statistical multiplexing*. In [3, 4] the concept of statistical multiplexing is modeled for MPLS and multi-layer networks by the provision of two values for every demand q, the average bandwidth requirement d^q and the peak bandwidth requirement $p^q \geq d^q$. By modeling the demand requirements as independent stochastic processes with low probability of requiring p^q, we may assume that not all peak values occur at the same time. Hence, the network should be dimensioned for the sum of average values and the sum of the k highest peak values (where k is a parameter).

In this paper, we propose the modeling of robustness by submodular functions. Given a finite set N, a set function $f : 2^N \to \mathbb{R}_+$ is *submodular* if $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$ for all $S, T \subseteq N$. We suggest to model the bandwidth consumption on a link by the use of a submodular function, e. g., the above average and maximum peak value. We illustrate this concept by the single path routing problem with fixed link capacities where the objective is to admit demands such that the the weighted sum of them is maximized (with weights representing priorities, service classes, etc.). In this case the bandwidth capacity on a link is described by the *submodular knapsack polytope* which is defined as $K := \operatorname{conv}\{x \in \{0,1\}^N : f(X(x)) \leq b\}$ where $X(x) := \{i \in N : x_i = 1\}$. This polytope has recently been studied by Atamtürk and Narayanan [1].

In Section 2 we describe our model. In particular, we use a submodular function to describe the bandwidth consumption on a link. Moreover, we give some examples for submodular functions suitable to model statistical multiplexing. In Section 3 we state some polyhedral results for the submodular knapsack polytope. In addition, we generalize $(1, k)$-configurations to the submodular knapsack polytope. In Section 4 we describe our (preliminary) computational studies presenting results for two submodular functions. In Section 5 we deduce conclusions from the work carried out and discuss directions for further research.

[1]Research supported by the Centre for Discrete Mathematics and its Applications (DIMAP), University of Warwick, EPSRC award EP/D063191/1.

2 Model

We consider a directed telecommunication network with vertex set V, arc set A and capacities c_a for each arc $a \in A$. Given a vertex $i \in V$ the sets of incident arcs with startpoint (endpoint) i are denoted by $\delta^+(i)$ (resp. $\delta^-(i)$). Let Q be the set of commodities. For $q \in Q$ the source and target of this commodity are denoted by $s^q \in V$ and $t^q \in V$. In addition, a weight $d^q \in \mathbb{Q}_+$ (e. g., average demand, a Quality of Service (QoS) class, or priority) is given for each commodity. The binary variables x^q_{ij} determine a single path routing of the traffic, i. e., $x^q_{ij} = 1$ if the routing of commodity q uses arc $(i,j) \in A$, $x^q_{ij} = 0$ otherwise. The variable z^q describes if the commodity q could be realized ($z^q = 1$) or not ($z^q = 0$). The bandwidth consumption caused by routing traffic over an arc is given by the submodular set function $f : 2^Q \to \mathbb{R}_+$. Considering an arc $(i,j) \in A$ we define $X_{ij} := \{q \in Q : x^q_{ij} = 1\}$ as the (variable!) set of commodities routed via this arc. The objective of our model is to maximize the weighted number of realized/routed commodities subject to the arc capacities. Formulated as mathematical model, this reads:

$$\max \quad \sum_{q \in Q} d^q z^q \tag{1a}$$

$$s.t. \quad f(X_{ij}) \le c_{ij} \qquad \forall\, (i,j) \in A \tag{1b}$$

$$\sum_{j \in \delta^+(i)} x^q_{ij} - \sum_{j \in \delta^-(i)} x^q_{ji} = \begin{cases} z^q & \text{if } i = s^q \\ -z^q & \text{if } i = t^q \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, q \in Q,\ \forall\, i \in V \tag{1c}$$

$$x^q_{ij} \in \{0,1\},\ z^q \in \{0,1\} \qquad \forall\, q \in Q,\ \forall\, (i,j) \in A \tag{1d}$$

Our model consists of only two classes of constraints: capacity constraints (1b) and flow-conservation (or routing) constraints (1c). In the capacity constraints the submodular function f is used to describe the bandwidth consumption. This general approach is suitable to model many different behaviors.

For example, in the setting of statistical multiplexing an average bit-rate requirement d^q and a peak bandwidth requirement $p^q = d^q + v^q$ is given for every commodity q where v^q is the variation, i. e., the additional bandwidth required for the peak rate. The routing of the different demands has to satisfy the capacity limit of each arc such that all average and the worst, i. e., maximal, peak demand can be transported. This can be modeled by the following submodular function

$$f_{1MAX}(X_{ij}) := \sum_{q \in Q} d^q x^q_{ij} + \max_{q \in Q} \left\{ v^q x^q_{ij} \right\}$$

This idea can be extended to more complex statistical considerations, e. g. the routing must take into account not only the worst (maximal) but also the second worst peak value. Formulated as a submodular function this reads

$$f_{2MAX}(X_{ij}) := \sum_{q \in Q} d^q x^q_{ij} + \max_{\substack{q_1, q_2 \in Q \\ q_1 \neq q_2}} \left\{ v^{q_1} x^{q_1}_{ij} + v^{q_2} x^{q_2}_{ij} \right\}$$

Considering an arbitrary arc $a = (i,j) \in A$ we define the polytope related to the capacity constraint as $K^{sub}_{ij} = \operatorname{conv}\{x_{ij} \in \{0,1\}^Q : f(X_{ij}) \le c_{ij}\}$. This polytope is a relaxation of the convex hull of the integer solutions of our model and known as the *submodular knapsack* polytope. Therefore every facet of this polytope yields a valid inequality for the polytope described by our model. A better understanding of the structure of K^{sub}_{ij} may enable us to solve our model faster. This motivates the study of the polyhedral structure of the submodular knapsack polytope K^{sub}_{ij} in the following.

3 Submodular knapsack polytope

In this section we review some properties of the submodular knapsack polytope K recently derived by Atamtürk and Narayanan [1]. Their assumptions that f is nondecreasing and $0 < f(i) - f(\emptyset) < b$ for all $i \in N$ are realistic in the context of bandwidth consumption of multiple commodities. In addition, we generalize the class of $(1,k)$-*configuration inequalities* [10] to the submodular knapsack polytope.

Lemma 3.1 (Atamtürk and Narayanan [1]). *The set K is a full-dimensional polytope. The inequality $x_i \geq 0$, $i \in N$, is facet-defining for K. The inequality $x_i \leq 1$, $i \in N$, is facet-defining for K if and only if $f(\{i,j\}) \leq b$ for all $j \in N \setminus \{i\}$.*

The class of *cover inequalities* is a well-known class of facet-defining valid inequalities for the knapsack polytope [2, 5, 11]. In [1] the concept of a cover is generalized to the submodular knapsack.

Definition 3.2. A subset $C \subseteq N$ is called a *cover* if $f(C) > b$. A cover C is called *minimal* if $f(C \setminus \{i\}) \leq b$ for every $i \in C$.

Theorem 3.3 (Atamtürk and Narayanan [1]). *If $S \subseteq N$ is a cover for K, then the* cover inequality

$$\sum_{i \in S} x_i \leq |S| - 1 \tag{2}$$

is valid for K. Moreover, it defines a facet of $\{x \in K : x_i = 0$ for $i \in N \setminus S\}$ if and only if S is a minimal cover.

The concept of a $(1,k)$-*configuration* has been introduced by Padberg [10] for the knapsack. We generalize this concept to submodular knapsacks.

Definition 3.4. A subset $N' \subseteq N$ and an element $t \in N \setminus N'$ is called a $(1,k)$-*configuration* if $f(N') \leq b$ and $Q \cup \{t\}$ is a minimal cover for all $Q \subseteq N'$ with $|Q| = k$ and $2 \leq k \leq |N'|$.

For $k = |N'|$ a $(1,k)$-configuration is a minimal cover. Padberg [10] has shown that $(1,k)$-configurations yield facet-defining inequalities for the knapsack polytope. This result can be extended for the submodular knapsack polytope as follows.

Theorem 3.5. *Given a submodular knapsack polytope K with elements $N = \{1, \ldots, n\}$. Let $t \in N$ and $N' \subseteq N \setminus \{t\}$ be a $(1,k)$-configuration. The inequalities*

$$(r - k + 1)x_t + \sum_{j \in T} x_j \leq r \tag{3}$$

where $T \subseteq N'$ with $|T| = r$ and $k \leq r \leq |N'|$ are valid for the submodular knapsack polytope. If $N' = N \setminus \{t\}$, these inequalities are facet-defining.

Proof. First, we show the validity of the inequalities (3). Given a $(1,k)$-configuration and a related inequality (3) we consider the two possible values of x_t: If $x_t = 0$, then the inequality reads as $\sum_{j \in T} x_j \leq r$. This is valid for K by definition of T. If $x_t = 1$, then inequality (3) reduces to $\sum_{j \in T} x_j \leq k - 1$ which is valid for K since $Q \cup \{t\}$ is a minimal cover for all $Q \subseteq N'$ with $|Q| = k$ and so it holds that $f(Q \cup \{t\}) > b$ for each k-element subset Q of T.

Second, we show that the face $\{x \in K : x$ satisfies (3) with equality$\}$ contains n affinely independent vectors. W. l. o. g. we assume $T = \{1, \ldots, r\}$ and $t = n$. We define for $i = 1, \ldots, r$ the sets

$$U_i := \{((i-1) \bmod r) + 1,\ (i \bmod r) + 1,\ ((i+1) \bmod r) + 1,\ \ldots,\ ((i+k-3) \bmod r) + 1\}.$$

Since $|U_i| = k - 1$, $f(U_i \cup \{t\}) \leq b$ by definition. Further, we define sets $W_j := \{1, \ldots, k-2, j, n\}$ for $j = r+1, \ldots, n-1$. For these sets, $f(W_j) \leq b$ holds because $t \in W_j$ and $|W_j \setminus \{t\}| = k - 1$. Finally, we consider the set T for which $f(T) \leq b$ holds by definition. The n characteristic vectors of $U_i \cup \{t\}$ ($1 \leq i \leq r$), W_j ($r \leq j \leq n-1$) and T are clearly affinely independent and on the face defined by inequality (3). □

4 Preliminary computational results

In this section we describe some preliminary computational results for the studied network optimization problem. On the one hand, we study the contribution of the minimal cover inequalities (2) to improve the performance of integer programming solvers. On the other hand, we study the impact on the network capacity of a higher peak value in the statistical multiplexing model with one or two maximum values taken into account.

We implemented a linearized version of our model in C++ using ILOG CPLEX 11.1 [6] as integer programming solver. In addition, we added separators for the min-cover inequalities for the submodular knapsack polytopes defined by the capacity constraints in our model. The computations are carried out on a computer with linux OS, 2.5 GHz CPU and 3 GB RAM. A time limit of 30min is set for solving each problem instance. The min-cover inequalities are only added in the root of the branch-and-bound tree (like all default separators of CPLEX).

Name	SNDlib model	$\|V\|$	$\|A\|$	$\|Q\|$
ATLANTA	D-B-N-N-S-A-N-N	15	44	210
FRANCE	D-B-M-N-C-A-N-N	25	90	300
NOBEL-GERMANY	D-B-E-N-C-A-N-N	17	52	121
PDH	D-B-E-N-C-A-N-N	11	68	24
POLSKA	D-B-M-N-C-A-N-N	12	36	66

Table 1: Problem instances

To achieve meaningful results we took instances from the SNDlib [9] using arc capacities from known solutions of network design problems for these instances (provided by SNDlib as well). Table 1 shows the instances, their number of vertices $|V|$, number of arcs $|A|$ and demands $|Q|$. The column 'SNDlib model' states the model (resp. solution) in SNDlib notation we used as base for assigning capacities to the arcs. Our computational studies contain two specific submodular functions: f_{1MAX} and f_{2MAX} as introduced in Section 2. Each of them is linearized by replacing it by a set of linear capacity constraints where all average demand values but only one peak value (resp. one pair of peak values) at a time contribute to the bandwidth consumption.

Table 2 states the results of our computational studies on the effect of adding cuts for the submodular knapsack polytope. For each problem instance and each considered variation (stated as ratio v^q/d^q) the columns 'time' state the solving times and 'gap%' the remaining integrality gaps for the following scenarios: CPLEX without additional separators and CPLEX with min-cover inequality separator. In the presence of a separator the table also shows the number of added cuts in the column '#cuts'. Further details are omitted due to paper length restrictions.

For f_{1MAX} we observe that oftentimes by adding violated min-cover inequalities the remaining integrality gap can be reduced, despite a reduction in the number of branch&bound nodes processed. Nevertheless, the time limit is reached for many values for v^q/d^q in both scenarios. Only for the instances NOBEL-GERMANY (variation of 25%) and POLSKA (variation of 50%) a significant acceleration of the solving time could be achieved by adding only a reasonable number of violated min-cover cuts.

For f_{2MAX} only two instances could be tested. In this case we cannot observe a trend for the reduction of the remaining integrality gap as the instance PDH gets solved to optimality very easily and POLSKA does have decreases for some settings and increases for others. As PDH gets solved very fast the introduction of a separation algorithm results in a high overhead with an increase of the solving time.

Figures 1 and 2 show the objective values with respect to the variation and the submodular function used, normalized for the case without variation. Due to memory consumption of the linearizations of the submodular function only the instances PDH and POLSKA can be tested with f_{2MAX} as submodular function.

We observe that the impact of higher peak values on the objective function depends on the network topology and traffic matrix, i. e., the objective of PDH decreases significantly faster than the objective of the other networks. The results also show that the impact of second worst peak value for this network is less severe than for POLSKA.

5 Conclusion

This paper reports on some first results on the use of submodular functions for robust network optimization. By statistical multiplexing of commodities on a link, the bandwidth consumption follows a submodular function. We illustrate this concept for a single path routing problem where the capacity consumption on a link can be modeled as a submodular knapsack, a problem recently studied by Atamtürk and Narayanan [1]. We contributed a small

		f_{1MAX}					f_{2MAX}				
		CPLEX		min-cover			CPLEX		min-cover		
Name	v^q/d^q	gap%	time	gap%	time	#cuts	gap%	time	gap%	time	#cuts
ATLANTA	0.00	0.00	30.49	0.00	144.52	2007	-	-	-	-	-
ATLANTA	0.10	0.14	1834.59	0.15	1825.67	285	-	-	-	-	-
ATLANTA	0.25	0.52	1835.82	0.50	1826.31	276	-	-	-	-	-
ATLANTA	0.50	1.36	1827.21	1.06	1823.19	577	-	-	-	-	-
ATLANTA	0.75	2.03	1826.53	1.21	1823.06	841	-	-	-	-	-
ATLANTA	1.00	2.36	1826.87	1.61	1823.14	772	-	-	-	-	-
FRANCE	0.00	0.23	1813.39	1.35	2297.90	3263	-	-	-	-	-
FRANCE	0.10	2.05	1823.15	1.74	1905.47	216	-	-	-	-	-
FRANCE	0.25	1.87	1825.34	0.49	1913.79	289	-	-	-	-	-
FRANCE	0.50	3.56	1821.43	4.18	2002.34	449	-	-	-	-	-
FRANCE	0.75	5.04	1824.18	4.65	1979.98	483	-	-	-	-	-
FRANCE	1.00	5.76	1819.59	6.19	1990.86	409	-	-	-	-	-
NOBEL-GERMANY	0.00	0.00	4.71	0.00	3.41	33	-	-	-	-	-
NOBEL-GERMANY	0.10	0.00	128.71	0.00	83.80	15	-	-	-	-	-
NOBEL-GERMANY	0.25	0.34	1840.94	0.00	84.46	22	-	-	-	-	-
NOBEL-GERMANY	0.50	0.49	1837.88	0.49	1820.80	79	-	-	-	-	-
NOBEL-GERMANY	0.75	1.00	1835.67	1.01	1820.34	87	-	-	-	-	-
NOBEL-GERMANY	1.00	0.00	1322.81	0.37	1824.15	82	-	-	-	-	-
PDH	0.00	0.00	0.07	0.00	0.46	26	0.00	0.18	0.00	0.72	26
PDH	0.10	0.00	0.14	0.00	0.56	25	0.00	3.77	0.00	41.29	23
PDH	0.25	0.00	0.13	0.00	0.44	25	0.00	1.17	0.00	1.84	7
PDH	0.50	0.00	0.06	0.00	0.15	10	0.00	0.83	0.00	4.39	13
PDH	0.75	0.00	0.10	0.00	0.31	29	0.00	0.55	0.00	0.98	13
PDH	1.00	0.00	0.05	0.00	0.10	8	0.00	0.33	0.00	0.49	8
POLSKA	0.00	0.00	62.03	0.00	58.85	80	0.00	64.41	0.00	61.34	80
POLSKA	0.10	0.50	1820.10	0.48	1809.32	60	1.43	1825.64	1.43	1821.89	45
POLSKA	0.25	0.00	277.73	0.00	267.64	51	2.23	1831.46	2.18	1824.92	36
POLSKA	0.50	0.00	1616.43	0.00	1199.29	34	6.87	1828.94	9.40	1826.49	30
POLSKA	0.75	1.22	1821.37	1.09	1811.53	40	12.97	1827.35	13.05	1825.20	23
POLSKA	1.00	2.08	1819.52	2.34	1809.86	31	13.76	1826.97	13.04	1825.85	22

Table 2: Computational results: f_{1MAX} & f_{2MAX}

extension to their work by generalizing the class of $(1, k)$-configuration inequalities to submodular knapsacks. Preliminary computational results for the model have been presented for some realistic instances.

Many directions for further research are to be pursued. First of all, the separation of the $(1, k)$-configuration inequalities has to be implemented and the computations have to be extended to larger instances that are currently unsolvable due to memory consumption of the linearizations of the submodular functions. Submodular functions relevant to robust network optimization have to be studied and further facet-defining inequalities for the submodular knapsack polytope and their lifting and separation have to be investigated (cf. [7]). Also, extending the results to the case where capacity decisions have to be taken is on our agenda. Finally, a comparison with the robust knapsack problem as introduced in [8] is considered.

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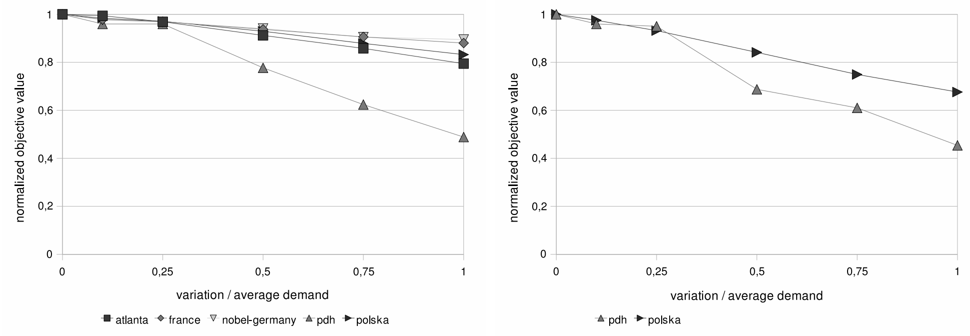
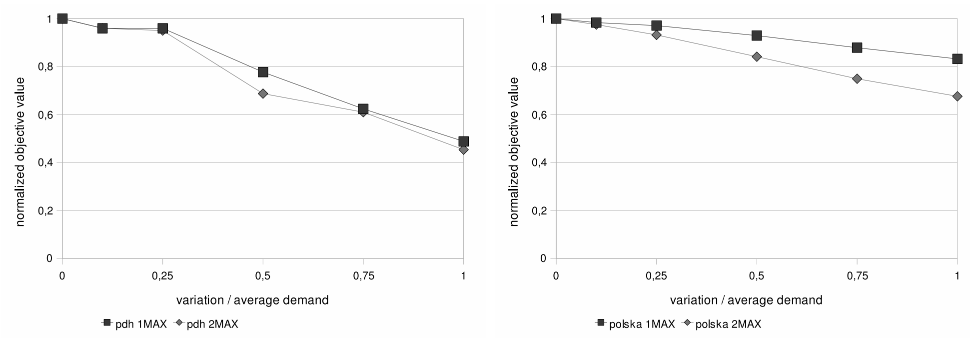


(a) f_{1MAX} (b) f_{2MAX}

Figure 1: Comparison of problem instances: degree of variation vs. objective value, normalized for the case without variation



(a) PDH (b) POLSKA

Figure 2: Comparison of f_{1MAX} and f_{2MAX}: degree of variation vs. objective value, normalized for the case without variation

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