The Robust Network Loading Problem under Hose Demand Uncertainty: Formulation, Polyhedral Analysis, and Computations

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Abstract

We consider the Network Loading Problem (NLP) under a polyhedral uncertainty description of traffic demands for which we give a compact multi-commodity formulation and state a nice decomposition property obtained from projecting out the flow variables. This property considerably simplifies the resulting polyhedral analysis and computations by doing away with metric inequalities, an attendant feature of the most successful algorithms on NLP. Then we focus on the hose model of demand uncertainty description. We study the polyhedral aspects of NLP under hose demand uncertainty and use the results as the basis of an efficient Branch-and-Cut algorithm supported by a simple heuristic for generating upper bounds. We also provide the results of some computational experiments on well-known network design instances.

Keywords: *Network Loading Problem, polyhedral demand uncertainty, hose model, robust optimization, polyhedral analysis, branch-and-cut.*

1 Introduction

For a given undirected graph G, the Network Loading Problem (NLP) deals with the design of a least cost network by allocating discrete units of facilities with different capacities on the links of G so as to support *expected* pairwise demands between some endpoints of G. In this work, we relax the assumption of known demands and study *robust* NLP with polyhedral demands to obtain designs flexible to fluctuations in demand. Our aim is to design least cost networks which remain operational for *any feasible realization* in a prescribed demand polyhedron.

Since deterministic NLP is strongly NP-hard there have been various efforts for solving it as efficiently as possible through the use of alternative formulations, heuristics, and by a thorough polyhedral analysis ([6], [8]). Although the polyhedra of feasible flows for the single commodity problem with two facility types is fully characterized ([7]), the same problem with multi-commodity flows remains very hard, and its solution requires the use of metric inequalities to define the corresponding projection cone ([6], [7]).

Against this background, our main contribution to the rather limited literature on robust NLP is to consider a quite general definition of demand uncertainty and present an efficient single-stage approach to handle the problem. For the single-stage robust NLP under polyhedral demands, we are not aware of any attempt other than [5] where uncertainty was incorporated into the design of fiber optic networks with an emphasis on modeling rather than on a detailed polyhedral analysis and branch-and-cut.

An interesting feature of our MIP formulation for NLP with polyhedral uncertainty is that we avoid the use of metric inequalities due to a decomposition property obtained from a projection on the design components. [7] characterizes all extreme rays of the related projection cone for the deterministic single-commodity case using a similar projection. However, only necessary conditions were obtained for the deterministic multi-commodity variant, which is difficult since the coupling bundle constraints prevent the decomposition of the problem into single commodity subproblems. For our problem, we by-pass that difficulty using the observation that the existence of a multi-commodity flow can be certified by checking the existence of many single commodity flows. Although this result considerably simplifies the formulations, the problem still remains difficult. Hence we make a thorough polyhedral analysis based on which we present valid inequalities for robust NLP with arbitrary number of facilities and arbitrary capacity structures as the second main contribution of the present work. Finally, we develop a Branch-and-Cut algorithm and use it to solve several well-known network design instances.

In Section 2 we briefly summarize how we handle robust NLP with polyhedral demands (NLP_{pol}) and briefly mention our B&C algorithm in Section3. We give a sample of computational results in Section 4 and conclude with Section 5. The interested reader can refer to [2] for a more detailed discussion.

2 Problem Definition

For a given undirected graph $G = (V, E)$ with the set of nodes V and the set of edges E, let Q be the set of commodities where each commodity $q \in Q$ has the origin $s(q) \in V$ and destination $t(q) \in V$. The main concern of NLP_{pol} is to allocate discrete number of facilities with different capacities on the edges of G to design a least cost network, that is viable for any demand realization in a prescribed polyhedral set. Therefore, the capacity of each edge should be sufficient to support any one of the infinitely many feasible demands. As a result, we can model NLP_{pol} using the following semi-infinite MIP formulation

$$\min \sum_{\{h,k\}\in E} \sum_{l\in L} p^l_{hk} y^l_{hk} \tag{1}$$

$$\text{s.t} \quad \sum_{k:\{h,k\}\in E} (f^q_{hk} - f^q_{kh}) = \begin{cases} 1 & h = s(q) \\ -1 & h = t(q) \\ 0 & \text{otherwise} \end{cases} \qquad \forall h \in V, q \in Q \tag{2}$$

$$\max_{d\in D} \sum_{q\in Q} (f^q_{hk} + f^q_{kh}) d_q \le \sum_{l\in L} C^l y^l_{hk} \qquad \forall \{h,k\} \in E. \tag{3}$$

$$y^l_{hk} \ge 0 \text{ integer}, \; f^q_{hk}, f^q_{kh} \ge 0 \qquad \forall \{h,k\} \in E, \; l \in L, q \in Q \tag{4}$$

where $D = \{d \in \mathbb{R}^{|Q|} : \sum_{q\in Q} a^q_z d_q \le \alpha_z \; \forall z = 1, .., m, \; d_q \ge 0 \; \forall q \in Q\}$ is an arbitrary demand polyhedron and p^l_{hk} is the unit cost of installing type $l \in L$ facility with capacity C^l on edge $\{h, k\} \in E$. The variables of the model are y^l_{hk} for the number of type $l \in L$ facilities loaded on $\{h, k\} \in E$ for the flow on it in both directions and f^q_{hk} for the fraction of demand for commodity $q \in Q$ routed on $\{h, k\} \in E$ from h to k.

Firstly, we obtain an equivalent compact MIP model for NLP_{pol} by using a duality based transformation ([1], [3]). Then by projecting out the flow variables we provide the following formulation (NLP_{pro})

of the problem in a lower dimensional space

$$\min \sum_{e\in E}\sum_{l\in L} p_e^l y_e^l \tag{5}$$

$$\text{s.t} \quad \sum_{z=1}^{m} \alpha_z \lambda_z^e \le \sum_{l\in L} C^l y_e^l \qquad \forall e \in E \tag{6}$$

$$\sum_{z=1}^{m} a_z^q \lambda_z^e \ge 0 \qquad \forall e \in E,\ \forall q \in Q \tag{7}$$

$$\sum_{e\in\delta(S)}\sum_{z=1}^{m} a_z^q \lambda_z^e \ge 1 \qquad \forall q \in Q,\ S \subset V : s(q) \in S, t(q) \in V \setminus S \tag{8}$$

$$y_e^l \ge 0 \ \text{ integer},\ \lambda_z^e \ge 0 \quad \forall l \in L,\ z = 1,..,m,\ e \in E \tag{9}$$

where $\lambda \in \Re^{m|E|}$ are the dual variables used in duality transformation. [7] studied a similar projection both for single- and multi-commodity NLP with deterministic demands. Although he could characterize all extreme rays of the related projection cone for the single-commodity case, he could only give necessary conditions for the multi-commodity variant. The difficulty of the latter problem is due to the link capacity constraints, which prevent the decomposition of the problem into single commodity subproblems. The resulting projection inequalities are then the well-known metric inequalities. However, our duality transformation leads to a formulation without such bundle constraints. As a result, we could decompose the projection cone for the multi-commodity problem into several cones with one cone for each commodity and use results of [7], which yield the projection inequalities (7) and (8).

Subsequently, we focus on hose demand uncertainty where $D_{hose} = \{d \in \mathbb{R}^{|Q|} : \sum_{q\in Q:s(q)=i \text{ or } t(q)=i} d_q \le b_i \ \forall i \in W\}$ with $W \subseteq V$ as the set of nodes willing to communicate with other nodes given the bandwidth capacity b_i for each terminal node $i \in W$. Then for NLP_{hose}, (6) and (8) reduce to

$$\sum_{i\in W} b_i \lambda_i^e \le \sum_{l\in L} C^l y_e^l \qquad \forall e \in E \tag{10}$$

$$\sum_{e\in\delta(S)} (\lambda_{s(q)}^e + \lambda_{t(q)}^e) \ge 1 \quad \forall q \in Q, S \subset V : s(q) \in S, t(q) \in V \setminus S \tag{11}$$

whereas the corresponding constraints of type (7) are dominated by the nonnegativity constraints (9). We first observe that restricting λ variables as

$$\lambda_s^e \le 1 \quad \forall e \in E, s \in W \tag{12}$$

does not violate the validity of the model. Let $\Lambda = \{\lambda \in \Re^{|E||Q|} : (12)\}$. Next, we make an extensive polyhedral analysis of $F = \{(\lambda, y) \in \Re_+^{|W||E|} \times Z_+^{|E||L|} : (10) - (11)\}$ and $F' = F \cap \Lambda$. To this end, we firstly study $F_\lambda = \{\lambda^e \in \Re_+^{|W|} : (10)\}$ $(F'_\lambda = F_\lambda \cap \Lambda)$, which is the projection of $F(F')$ into the subspace of λ variables and show that the facet defining inequalities for $P = conv(F)$ (resp. for $P' = conv(F')$) and F_λ (F'_λ) are the same. Then we consider the single edge restriction of NLP_{hose}, i.e., $F_e = \{(\lambda^e, y_e) \in \Re_+^{|W| \times Z_+^{|L|}} : (10)\}$ $(F'_e = F_e \cap \{(\lambda^e, y_e) \in \Re_+^{|W|} \times Z_+^{|L|} : (12)\})$ and prove the following theorem to display the relationship between the facet defining inequalities of $P_e = conv(F_e)$ $(P'_e = conv(F'_e))$ and P (P').

Theorem 1. *Let $e \in E$ be such that $\delta(S) \setminus \{e\} \ne \emptyset$ for every $S \subset V$ such that there exists $q \in Q$ with $s(q) \in S$ and $t(q) \in V \setminus S$. Inequality $\alpha\lambda^e + \beta y_e \ge \gamma$ is facet defining for P_e (P'_e) if and only if it is facet defining for P (P').*

Finally, we discuss the cut restriction of NLP_{hose} for $S \subset V$ such that the subgraphs induced by S and $V \setminus S$ are connected. Let $y_{\delta(S)}$ be the restriction of the vector y to $e \in \delta(S)$, $F(S) = \{y_{\delta(S)} \in Z_+^{|\delta(S)||L|} :$

$\sum_{l\in L}\sum_{e\in\delta(S)} C^l y_e^l \geq B(S)\}$ and $P(S) = conv(F(S))$ where $B(S) = \min\{\sum_{i\in S\cap W} b_i, \sum_{i\in(V\setminus S)\cap W} b_i\}$ is the maximum total traffic that can be routed on the edges in $\delta(S)$. We first show that $F(S) = Proj_{y(\delta(S))}(F) = Proj_{y(\delta(S))}(F^{'})$ when $B(S) > 0$ and then state the following theorem.

Theorem 2. *Let $S \subset V$ be such that the subgraphs induced by S and $V \setminus S$ are both connected and $B(S) > 0$. If $\sum_{l\in L}\sum_{e\in\delta(S)} \beta_e^l y_e^l \geq \beta_0$ is facet defining for $conv(F(S))$ and for each $e^{'} \in \delta(S)$ there exists a vector $y_{\delta(S)} \in F(S)$ such that $\sum_{l\in L}\sum_{e\in\delta(S)} \beta_e^l y_e^l \geq \beta_0$ and $\sum_{l\in L} C^l y_{e^{'}}^l > B(S)$, then the inequality is facet defining for P and $P^{'}$.*

Moreover, using these results, we modify two well-known families of valid inequalities for NLP to render them valid for NLP_{hose} and prove that they are facet defining under several mild conditions. For $S \subset V$ and $l \in L$, let $Y^l(S) = \sum_{e\in\delta(S)} y_e^l$, $b(S) = \sum_{i\in S\cap W} b_i$, $r^l(S) = b(S) \mod C^l$, and $R^l(S) = \lceil B(S)\rceil \mod C^l$. Also for l_1 and l_2 in L, let $g(l_1, l_2) = C^{l_1} \mod C^{l_2}$.

Proposition 1. *For $S \subset V$ and $l^* \in L$ such that $R^{l^*}(S) > 0$, the cutset inequality*

$$\sum_{l\in L: C^l < B(S)} \left(R^{l^*}(S) \left\lfloor \frac{C^l}{C^{l^*}} \right\rfloor + \min\{g(l,l^*), R^{l^*}(S)\} \right) Y^l(S) + \sum_{l\in L: C^l \geq B(S)} R^{l^*}(S) \left\lceil \frac{B(S)}{C^{l^*}} \right\rceil Y^{l^*}(S) \geq R^{l^*}(S) \left\lceil \frac{B(S)}{C^{l^*}} \right\rceil \tag{13}$$

is valid for P and $P^{'}$. Let C^1 be the capacity of the smallest facility. If the subgraphs induced by S and $V \setminus S$ are both connected and $C^1 = 1$, then it is also facet defining for P .

Proposition 2. *Let $e \in E$, $l^* \in L$, and $S \subseteq W$ be such that $r^{l^*}(S) > 0$. The residual capacity inequality*

$$\sum_{l\in L} \left(r^{l^*(S)} \left\lfloor \frac{C^l}{C^{l^*}} \right\rfloor + \min\{g(l,l^*), r^{l^*}(S)\} \right) y_e^l + \sum_{i\in S} b_i(1-\lambda_i^e) \geq r^{l^*}(S) \left\lceil \frac{b(S)}{C^{l^*}} \right\rceil \tag{14}$$

is valid for $P^{'}$.

Corollary 1. *Let $e \in E$ be such that $\delta(S^{'}) \setminus \{e\} \neq \emptyset$ for every $S^{'} \subset V$ such that there exists $q \in Q$ with $s(q) \in S^{'}$ and $t(q) \in V \setminus S^{'}$. Suppose that $|L| = 1$ and let $S \subseteq W$ be such that $r^1(S) > 0$. The residual capacity inequalities (14) define a facet of $P^{'}$ if $\left\lceil \frac{b(S)}{C^1} \right\rceil \geq 2$ or if $\left\lceil \frac{b(S)}{C^1} \right\rceil = 1$ and $|S| = 1$.*

3 Branch-and-Cut Algorithm

Our B&C algorithm starts with a larger feasible set $\{(\lambda, y) \in \Re_+^{|W||L|} \times Z_+^{|E||L|} : (10)\}$ and adds violated valid inequalities (11), (13), and (14) at each iteration. We solve minimum cut problems to separate (11) and use this cut information to separate (13). Finally, we implement a polynomial time algorithm to separate a relaxation of (14) for the residual capacity inequalities. We also use a simple rounding heuristic to get upper bounds on the optimal solution, which we observe to be quite efficient for our problem.

4 Experimental Results

In this section we report a sample of our computational test results for NLP_{hose} with single facility and two facilities. An extensive set of additional test results together with a comprehensive discussion is provided in Altın et al. [2]. Currently, we compare the performance of the B&C algorithm with that of Cplex on some well-known instances from the SNDLIB web site ([10]) as well as some others used in [1] for a VPN design problem. We use AMPL to model NLP_{hose} and Cplex 9.1 MIP solver to solve them. The B&C algorithm is implemented in C using MINTO ([9]) and Cplex 9.1 as LPsolver.

4.1 Single Facility NLP_{hose}

Firstly, we compare our B&C algorithm with solving the single facility NLP_{hose} using Cplex. We could solve 7 out of 18 instances to optimality in 2 hours using both Cplex and B&C. The first chart in Figure 1 shows how much the solution times are reduced by using our B&C algorithm rather than Cplex for these 7 instances. B&C is much faster in all cases and the gap grows as large as 99.7% for *bhvdc*.

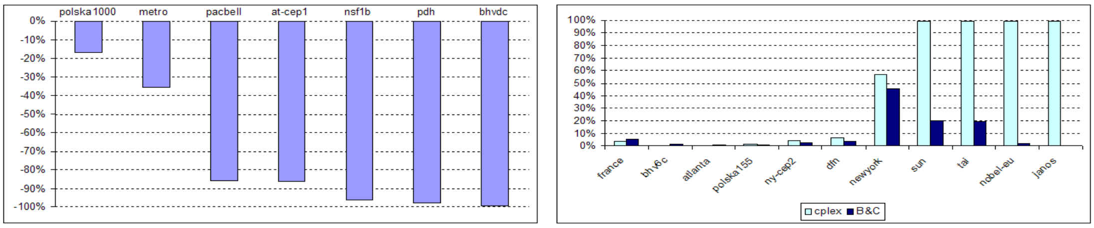

Figure 1: Change in solution times and termination gaps if we use B&C rather than Cplex.

Moreover, we compare termination gaps with Cplex and our B&C algorithm for the remaining 11 instances in the second chart in Figure 1. Even though Cplex gives better upper bounds in *dfn, ny-cep2*, and *atlanta*, the gaps at termination are better for the B&C algorithm in *dfn, ny-cep2*. On the other hand, B&C is clearly superior for *newyork, tai, janos, nobel-eu,* and *sun*. Furthermore, the performance of Cplex degrades significantly when compared with the B&C algorithm for larger networks like *tai, janos, nobel-eu,* and *sun*. Except *tai*, $|Q| = |V|(|V| - 1)$ in these instances, and we observe that among such cases only in *dfn* and *atlanta* Cplex has performed slightly better than B&C. Actually the upper bound of Cplex is just 0.07% and 0.2% tighter than the one of B&C in *dfn* and *atlanta*, respectively. On the other hand the upper bounds we obtain with B&C are 100% better than the bounds with Cplex in *tai, janos, nobel-eu,* and *sun*. Finally, the gaps at termination for B&C algorithm is clearly superior in 8 of the 11 instances with much lower gaps for *tai*, *nobel-eu*, and *sun* in addition to the zero gap for *janos*.

Next, we compare the total design cost for the hose case (c_{hose}) with the cost of the design for the average (c_{ave}) and the worst-case (c_{worst}) demand expectations on several instances. We observe that c_{hose} is 17.62% higher than c_{ave} on the average whereas c_{worst} is 6 to 25 times larger than c_{hose}. Hence, even though we need to pay for the additional flexibility that the hose model provides, we still make significant savings by exploiting the hose information in the design process.

4.2 Two Facility NLP_{hose}

Initially, we consider the five instances, which we could solve to optimality both with Cplex and the B&C algorithm. The first chart on the left in Figure 2 shows the percent reduction in solution times we have obtained by using B&C rather than Cplex.

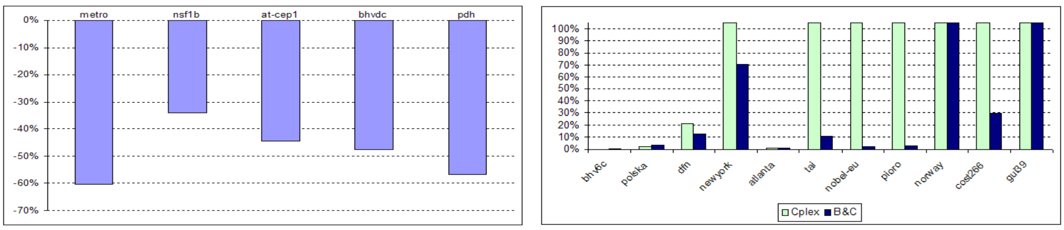

Figure 2: Change in solution times and termination gaps if we use B&C rather than Cplex.

Moreover, we provide a comparison of termination gaps with Cplex and our B&C algorithm for the remaining 11 instances in the second chart in Figure 2. Just like the single facility case, we see that our B&C algorithm is superior especially for larger instances where $|Q| = |V|(|V| - 1)$. This is quite obvious especially for *tai, nobel-eu, pioro*, and *cost266* since the MIP solver could not even find a feasible solution by solving the two-facility NLP_{hose} problem in 2 hours whereas the B&C algorithm successfully produced some upper bounds. Finally, c_{hose} is 18.86% larger than c_{ave} for the two facility case.

5 Conclusion

In this paper we study the Network Loading Problem under a polyhedral definition of traffic demands. We discuss the design of a network which is capable to support infinitely many non-simultaneous demand realizations. Based on a compact formulation and a decomposition property we briefly mention our detailed polyhedral analysis for the hose demand model. The polyhedral analysis formed the basis of an efficient Branch-and-Cut algorithm equipped with a heuristic for computing upper bounds. Our computational results revealed that projecting out the flow variables and using our Branch-and-Cut algorithm is quite effective for both single and two-facility problem types. We also work on the problem with another demand polyhedron and we hope to discuss our results in the conference as well.

Acknowledgments

The authors would like to thank A. Ridha Mahjoub and Hervé Kerivin for helpful discussions.

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