The Multilayer Capacitated Survivable IP Network Design Problem: valid inequalities and Branch-and-Cut

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Abstract

Telecommunication networks can be seen as the stacking of several layers like, for instance, IP-over-Optical networks. This infrastructure has to be sufficiently survivable to restore the traffic in the event of a failure. Moreover, it should have adequate capacities so that the demands can be routed between the origin-destinations. In this paper we consider the Multilayer Capacitated Survivable IP Network Design problem. We study two variants of this problem with simple and multiple capacities. We give two multicommodity flow formulations for each variant of the problem and describe some valid inequalities. Using these we develop a Branch-and-Cut algorithm and a Branch-and-Cut-and-Price algorithm for each variant and present extensive computational results.

Keywords: *IP-over-optical network, survivability, capacities, Branch-and-Cut-and-Price algorithms.*

1 Introduction

In the past years, telecommunication networks have seen a big development with the advances in optical technologies and the explosive growth of the Internet. Telecommunication networks are now moving toward a model of high-speed routers interconnected by intelligent optical core networks. Moreover, there is a general consensus that the control plan of the optical networks should utilize IP-based protocols for dynamic provisioning and restoration of lightpaths.

The optical network consists of multiple switches interconnected by optical links. The IP and optical networks communicate through logical control interfaces. The optical network essentially provides point-to-point connectivity between routers in the form of fixed bandwidth lightpaths. These lightpaths define the topology of the IP network.

Each router in the IP network is connected to at least one of the optical switches. Moreover to each link between two routers in the IP network corresponds a routing path in the optical one between two switches corresponding to these routers. Figure 1 shows an IP-over-optical network. The IP network has four routers $R_1, \ldots, R_4$ and the optical network has seven switches $S_1, \ldots, S_7$. Only the optical switches $S_1, \ldots, S_4$ communicate with one router throught the optical-IP interface.

The introduction of this new infrastructure of telecommunication networks gives rise to survivability issues. For example consider the IP-over-optical network given in Figure 1. Suppose that the link $R_1 - R_2$ of the IP network, is routed on the optical path $S_1 - S_2$, and the link $R_1 - R_3$ corresponds to the path

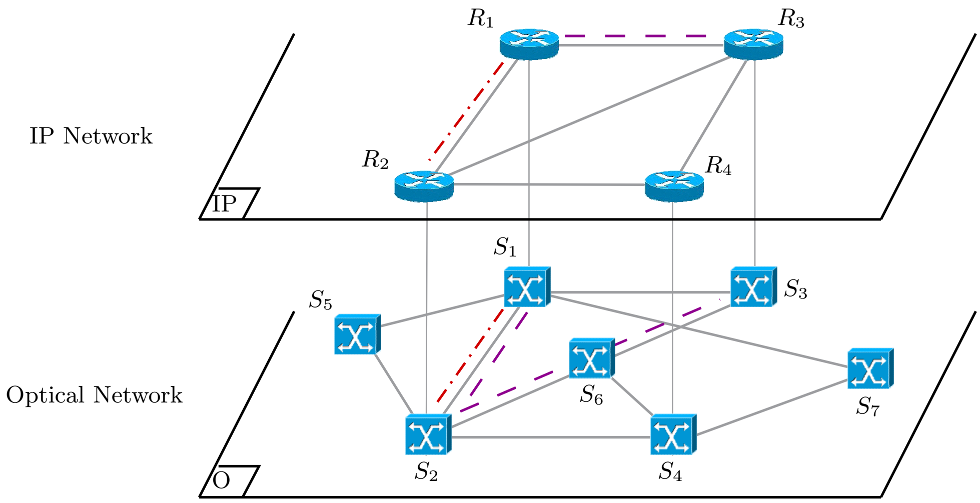


Figure 1: An IP-over-optical network

$S_1 - S_2 - S_6 - S_3$. Here, the network is not survivable to single link failure. For instance, if the optical link $S_1 - S_2$ fails, then the links in the IP network $R_1 - R_2$ and $R_1 - R_3$ are cut, and therefore the router R_1 is no more connected to the rest of the routers. As a consequence, survivability strategies have to be considered. If the transport network is fixed, one has to determine the suitable client network topology for the network to be survivable.

In addition to the survivability aspect, we may need to install capacities on the IP network in order to route commodities between some routers. In this paper we shall be concerned by this problem which considers simultaneously both the survivability and the dimensionning of the IP network when the transport network is fixed. We shall discuss this problem from a polyhedral point of view. The multilayer networks have recently seen a particular attention [5, 10, 11].

The paper is organised as follows. We present first a multilayer survivable network design problem with capacity constraints, called the multilayer capacitated survivable IP network design problem. Then we describe two versions of this problem: either with at most one single equipment per edge, or with possibly multiple equipments on edges. We give two multicommodity flow formulations for each variant and describe some valid inequalities. Using these inequalities we develop a Branch-and-Cut algorithm and a Branch-and-Cut-and-Price algorithm for each variant and present extensive computational results.

In the rest of this section we give some definitions and notations. We denote a graph by $G = (V, E)$ where V is the *node set* and E the *edge set* of G. If $e \in E$ is an edge between two nodes u and v, then we also write $e = uv$ to denote e. We denote also by $D = (V, A)$ the bidirected graph associated with G such that each edge $e = uv \in E$ is replaced by two arcs (u, v) and (v, u), respectively from u to v and from v to u, in the *arc set* A of D. For an edge subset $F \subseteq E$ we denote by $\vec{F} \subseteq A$ the associated arc subset. For $F \subseteq E$ we let $G \backslash F$ denote the subgraph of G obtained by removing the edges of F and $D \backslash \vec{F}$ the associated subgraph obtain from D by removing the arcs of $\vec{F}$. Throughout the paper we will consider simple graphs. Let $G = (V, E)$ be an undirected graph. Given $W \subseteq V$, we denote by $\delta_G(W)$ the set of edges of G having exactly one node in W. The edge set $\delta_G(W)$ is called a *cut*. Given a vector $x \in \mathbb{R}^E$ and $F \subseteq E$, we let $x(F) = \sum_{e \in F} x(e)$.

Throughout the paper, given an IP-over-optical network, we suppose that to each router of the IP layer corresponds exactly one optical switch. We will represent an IP-over-optical network by two graphs $G^1 = (V^1, E^1)$ and $G^2 = (V^2, E^2)$, that represent the IP and optical networks, respectively. The nodes of G^1 (resp. G^2) correspond to the routers of the IP layer (resp. the optical switches), and the edges represent the possible links between the routers (resp. switches). Each node $v_i \in V^1$ is associated with a node $w_i \in V^2$. For an edge $f \in E^1$, we denote by P_f the path in G^2 corresponding to f.

2 The Multilayer Capacitated Survivable IP Network Design Problem

The first major survivability requirement used in telecommunication networks is the 2-connectivity [1, 6, 7, 13]. That is there must exist at least two edge-disjoint paths between every pair of nodes in the network. This assumption, that only one edge may fail at a time, is based on the naive idea that the links in the network are independent and no equipment can be commonly used by two distinct links. However, this is not the case, for instance, for the IP-over-optical networks, when the optical layer is taken into account in the management of the IP network.

In fact, any edge of the client network is supported by a path in the optical network (lightpaths). That is the traffic of an edge in the client network is routed in the optical network along the path corresponding to that edge. Therefore an edge of the optical network may appear in several paths supporting distinct edges. In consequence, the failure of an edge in the optical network may affect several optical paths, and hence the edges of the client network corresponding to these paths. As a result, several edges may fail at the same time in the IP layer.

The multilayer survivable IP network design problem (MSIPND problem) introduced by Borne et al [3], consists in finding the set of links to be installed in the IP network so that if a failure occurs on an optical link, the IP subnetwork obtained by removing the corresponding edges is connected.

In our problem, we can install capacities of 2.5 Gbits or 10 Gbits on any link of the IP network. Since the capacities are symmetric, we consider that, each time a capacity is installed between a router R_1 and a router R_2, one has to install the same capacity from R_2 to R_1.

In this paper, we consider a more realistic model in which one has to decide a minimum cost set capacities to install on the client IP layer, so that all the demands can be routed as a multicommodity flow, and this, for any single link failure in the transport network. More precisely we consider the overlay model where each one of the IP and the optical networks have their own control and routing mechanisms. We suppose that the topology and the routing of the optical network are fixed and satisfy some survivability requirements. We also suppose that a set of IP routers (resp. optical switches) is given as well as the possible links between the routers (resp. switches). As the routing of the optical network is known, one can determine for each optical link e, the set of edges of the IP network that may be affected if e is cut. If a certain cost is associated with each type of capacity on each edge of the IP network, the *Multilayer Capacitated Survivable IP Network Design problem* (MCSIPND problem) is to find the set of links to be installed in the IP network and facilities to be loaded on theses links so that if a failure occurs on an optical link, the IP subnetwork obtained by removing the corresponding edges allows a multicommodity flow which satisfies the capacities.

We consider two variants of the problem: the multiple MCSIPND which allows multiple equipments and the simple MCSIPND where only one single equipment can be loaded between two routers. The orientation of the links between the routers is omitted because the capacities are symmetric.

3 Node-arc formulation

For an edge e of graph $G^2 = (V^2, E^2)$ corresponding to the optical network, let F_e be the set of edges of the IP network that may be affected by a failure of e, that is $F_e = \{f \in E^1 \,|\, e \in P_f\}$. We let $\mathcal{F} = \{F_e,\, e \in E^2\}$. Also we denote by D^1 the directed graph associated with G^1 and $\vec{\mathcal{F}} = \{\vec{F_e} \,|\, F_e \in \mathcal{F}\}$. We denote by K the set of commodities. For each $k \in K$, we know the origin o_k, the destination d_k and the amount ω_k of the demand k.

Let $\mu^1 = 2.5$ Gbit/s and $\mu^2 = 10$ Gbit/s be the possible facilities. For each $ij \in E^1$, let c^l_{ij} be the cost of installing a capacity μ^l on ij for $l = 1, 2$. Then, the MCSIPND$_m$ problem consists in finding a minimum cost subgraph H of G^1 such that for every edge $e \in E^2$, the graph obtained from H by removing the edges of F_e has enough capacity to route the commodities of K with respect to the capacity of the remaining edges.

In order to give a node-arc formulation for the MCSIPND problem, let us denote by $f_{uv}^{k,e}$ for an arc $(u,v) \in A^1$, an edge $e \in E^2$ and a commodity $k \in K$, the flow of k on (u,v) from u to v in case of failure of e (*i.e.* when the arcs of $\vec{F}_e$ are removed in D^1). For an edge $uv \in E^1$ let x_{uv}^l be the number of facilities μ^l installed on uv, for $l = 1, 2$. Hence the multiple MCSIPND problem is equivalent to the following integer programming problem.

$$\text{Minimize} \sum_{l=1,2} \sum_{uv \in E^1} c_{uv}^l x_{uv}^l$$

$$\sum_{u:(u,v) \in A^1 \setminus \vec{F}_e} f_{uv}^{k,e} - \sum_{u:(v,u) \in A^1 \setminus \vec{F}_e} f_{vu}^{k,e} = \begin{cases} -\omega_k & \text{if } v = o_k, \\ 0 & \text{if } v \neq o_k, d_k, \\ \omega_k & \text{si } v = d_k, \end{cases} \qquad \forall v \in V^1, \forall k \in K, \forall e \in E^2, \quad (1)$$

$$\sum_{k \in K} f_{uv}^{k,e} \leq \sum_{l=1,2} \mu^l x_{uv}^l \qquad \forall uv \in E^1, \forall e \in E^2, \quad (2)$$

$$\sum_{k \in K} f_{vu}^{k,e} \leq \sum_{l=1,2} \mu^l x_{uv}^l \qquad \forall uv \in E^1, \forall e \in E^2, \quad (3)$$

$$x_{uv}^l \geq 0 \text{ and integer} \qquad \forall uv \in E^1,\ l = 1, 2, \quad (4)$$

$$f_{uv}^{k,e}, f_{vu}^{k,e} \geq 0 \qquad \forall uv \in E^1, \forall k \in K, \forall e \in E^2. \quad (5)$$

By adding the following inequalities

$$x_{uv}^1 + x_{uv}^2 \leq 1 \qquad \forall uv \in E^1, \quad (6)$$

we obtain a valid formulation for the simple MCSIPND problem.

Inequalities (1) are called *flow conservation constraints*. Inequalities (2) and (3) express the fact that the sum of the flows of all commodities $k \in K$ on an edge has to be less than or equal to the capacity of this edge. They will be called *capacity constraints*. Inequalities (4) and (5) are called *trivial inequalities*. Inequalities (6) express the fact that only one link can be used between two given nodes. Then we have only one type of capacity on an edge.

4 Path formulation

As in the node-arc formulation, for an edge $uv \in E^1$ we denote by x_{uv}^l the number of facilities μ^l installed on uv for $l = 1, 2$. For an edge $e \in E^2$ and a commodity k we denote by $\mathcal{P}_k^e$ the set of paths from o_k to d_k in the graph $D^1 \setminus \vec{F}_e$ (*i.e.* when the edge $e \in E^2$ fails). For a path P of $\mathcal{P}_k^e$, let $y_k^e(P)$ be the amount of flow of commodity k on P in case of faillure of e. The following mixed integer programming formulation is valid for the multiple MCSIPND problem.

$$\text{Minimiser} \sum_{l=1,2} \sum_{uv \in E^1} c_{uv}^l x_{uv}^l$$

$$\sum_{P \in \mathcal{P}_k^e} y_k^e(P) = \omega_k \qquad \forall k \in K,\ \forall e \in E^2, \quad (7)$$

$$\sum_{k \in K} \sum_{P \in \mathcal{P}_k^e \,|\, (u,v) \in P} y_k^e(P) \leq \sum_{l=1,2} \mu^l x_{uv}^l \qquad \forall uv \in E^1,\ \forall in E^2, \quad (8)$$

$$\sum_{k \in K} \sum_{P \in \mathcal{P}_k^e \,|\, (v,u) \in P} y_k^e(P) \leq \sum_{l=1,2} \mu^l x_{uv}^l \qquad \forall uv \in E^1,\ \forall e \in E^2, \quad (9)$$

$$y_k^e(P) \geq 0 \qquad \forall e \in E^2,\ \forall k \in K, \forall P \in \mathcal{P}_k^e, \quad (10)$$

$$x_{uv}^l \geq 0 \text{ and integer} \qquad \forall uv \in E^1,\ l = 1, 2. \quad (11)$$

This formulation has a collection of $|K|$ demand constraints (7) that represent the flow of each path P in $\mathcal{P}^e_k$, $k \in K$ for each failure $e \in E^2$ and ($\sum_{P \in \mathcal{P}^e_k} y^e_k(P)$ represents the amount of flow of commodity k passing through the set of paths from o_k to d_k). This flow has to be equal to the amount ω_k between o_k and d_k. Inequalities (8) and (9) are called *capacity constraints*. The flow through the edge uv has to be less than the capacity of this edge from u to v (constraints (8)) and from v to u (constraints (9)). Inequalities (10) and (11) are the *trivial constraints*.

By adding inequalities (6), we obtain a valid formulation for the simple MCSIPND problem.

The linear relaxation of the path formulation contains a moderate number of constraints but a huge number of variables. An appropriate method to solve this would be the column generation approach.

5 Column generation

Column generation will be used to solve the linear relaxation of the path based formulation of the MCSIPND problem (called the *master problem*). This approach has been extensively used for modeling and solving large versions of the linear multicommodity flow problem [2, 8]. The general idea of column generation is to solve a restricted linear program with a small number of columns (variables) in order to determine an optimal solution for the master problem. In fact a limited number of variables may induce an optimal basis solution for the master problem. So the column generation algorithm solves the linear relaxation of the master problem by solving the linear relaxations of several restricted master problems. After determining the solution of the linear relaxation of a restricted master problem, we use the *pricing problem* which consists in finding whether there are any column not yet in the restricted master problem with negative reduced cost. If none can be found, the current solution is the optimal for the linear relaxation of the master problem. However, if one or more such columns do exist, then they are added to the restricted master problem and the process is repeated. When defining the initial restricted master problem, it is necessary to ensure the existence of a feasible solution. To this end, we use an ε formulation which consists in minimizing an additional capacity ε which permits to carry the flow. Indeed this permits to determine the initial dual variables which will be passed to the pricing problem. Here the pricing problem can then be reduced to the search of several shortest path problems with non-negative costs. In the next section, we describe some valid inequalities which may strengthen the linear relaxation.

6 Valid inequalities and facets

For $W \subseteq V$, we denote by $\gamma^+(W)$ (resp. $\gamma^-(W)$) the set of demands which have their origin (resp. destination) in W and their destination (resp. origin) in $V \backslash W$. We denote also by $\gamma(W)$ the set $\gamma^+(W) \cup \gamma^-(W)$. Given a set of nodes $W \subseteq V$, $\emptyset \neq W \neq V$, let

$$D_W = \left\lceil \max \left\{ \frac{\sum_{k \in \gamma^+(W)} \omega_k}{2.5}, \frac{\sum_{k \in \gamma^-(W)} \omega_k}{2.5} \right\} \right\rceil .$$

Let $F_e \in \mathcal{F}$ be an edge subset of E and $W \subseteq V$, $\emptyset \neq W \neq V$, the following inequalities, called *capacity demand cut inequalities*, are valid for the multiple and the simple MCSIPND problems:

$$x^1(\delta_{G \backslash F_e}(W)) + x^2(\delta_{G \backslash F_e}(W)) \geq \left\lceil \frac{D_W}{4} \right\rceil \tag{12}$$

$$x^1(\delta_{G \backslash F_e}(W)) + 2x^2(\delta_{G \backslash F_e}(W)) \geq \begin{cases} \lceil \frac{D_W}{2} \rceil + 1 & \text{if } D_W \bmod 4 = 2, \\ \lceil \frac{D_W}{2} \rceil & \text{otherwise ,} \end{cases} \tag{13}$$

$$x^1(\delta_{G\setminus F_e}(W)) + 3x^2(\delta_{G\setminus F_e}(W)) \geq \left\lceil \frac{3D_W}{4} \right\rceil, \tag{14}$$

$$x^1(\delta_{G\setminus F_e}(W)) + 4x^2(\delta_{G\setminus F_e}(W)) \geq D_W. \tag{15}$$

One may generate further cut based valid inequalities by combining inequalities of type (15), (12), (14), (13) and trivial inequalities. However all inequalities obtained this way are redundant with respect to the capacity demand cut inequalities (see [4]).

In the following, we give necessary conditions and sufficient conditions for inequality (15) to be facet defining for the multiple MCSIPND problem. As in [3], a subgraph $H = (W, F)$ of $G = (V, E)$ is said to be $\mathcal{F}$-*connected* with respect to $\mathcal{F} = \{F_e, e \in E^2\}$ if for all $e \in E^2$, the graph $H\setminus F_e$ is connected.

Theorem 6.1 *Let $F_e \in \mathcal{F}$ be an edge subset of E and $W \subseteq V$, $\emptyset \neq W \neq V$, inequality (15) defines a facet of the multiple MCSIPND polytope only if*

1. *$G\setminus F_e(W)$ and $G\setminus F_e(\overline{W})$ are connected,*
2. *there is no $F_g \in \mathcal{F}\setminus\{F_e\}$ such that $F_e \cap \delta_G(W) \subset F_g \cap \delta_G(W)$,*
3. *$G(W)$ and $G(\overline{W})$ are $\mathcal{F}$-connected, if $\delta_G(W) \cap F_e = \emptyset$,*
4. *$D_W > max \left\{ \frac{\sum_{k\in\gamma^+(W)} \omega_k}{2.5}, \frac{\sum_{k\in\gamma^-(W)} \omega_k}{2.5} \right\}$,*
5. *$D_W \geq 4$.*

Theorem 6.2 *Inequality (15) defines a facet of the multiple MCSIPND polytope if*

1. *condition 1), 2), 4, 5) of Theorem 6.1 are satisfied,*
2. *$G(W)$ and $G(\overline{W})$ are $\mathcal{F}$-connected.*

In [9], Magnanti et al. introduce cutset inequalities valid for the Two-Facility Capacitated Network Loading Problem (TFLP). These can be easily extended to the MCSIPND problems. The extended ones are specials cases of the capacity demand cut inequalities.

We have introduced further classes of valid inequalities which are extensions of valid inequalities introduced in [3] for the problem without capacities. These are the *cut-cycle inequalities* and the *star-partition inequalities*. We have also introduced a class of inequalities called the *saturation inequalities* which are only valid for the simple MCSIPND problem.

7 Branch-and-cut and branch-and-cut-and-price algorithms

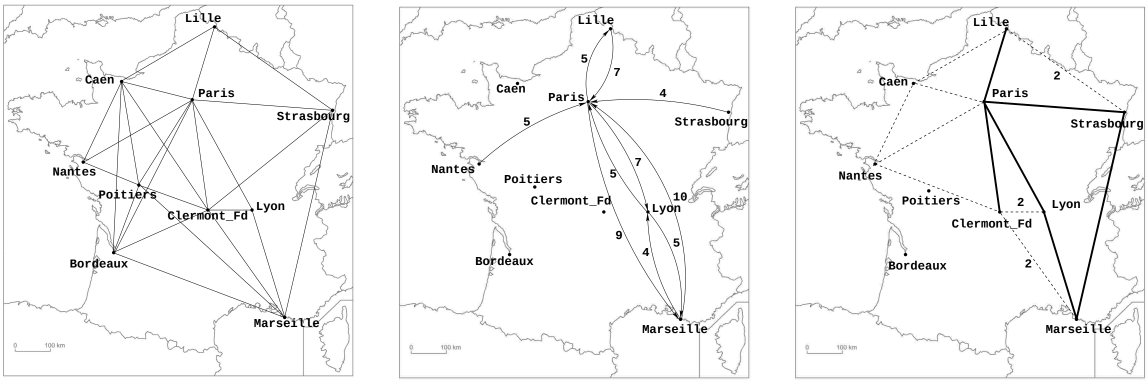
For each variant (simple or multiple MCSIPND) we have developed a Branch-and-Cut algorithm based on the node-arc formulation and a Branch-and-Cut-and-Price algorithm based on the path formulation. Our aim is to address the algorithmic applications of the previous results.

Usually, the optimal solution of the linear relaxation is not feasible, and thus, in each iteration of the Branch-and-Cut and the Branch-and-Cut-and-Price algorithms, it is necessary to generate further inequalities that are valid for the MCSIPND problem but violated by the current solution. For this, one has to solve the so-called *separation problem*. This consists, given a class of inequalities, in deciding whether the current solution satisfies the inequalities, and if not, in finding an inequality that is violated by this solution. The inequalities given just above are all valid for the two variants of the MCSIPND problem except the saturation inequalities which are valid only for the simple one. We have used all theses classes of inequalities in our algorithms.

8 Computational results

Extensive computational results will be presented. We have tested our four algorithms (two for each variant) on real-life instances and instances obtained from problems of the TSP Library ([12]). These instances were generated with 6, 8 or 10 nodes, $|\mathcal{F}| = 10, 20$ and $|\mathcal{K}| = 5, 10, 20$. The real instances have been provided by the french telecommunications operator France Télécom. These instances have 6 to 18 nodes and $\mathcal{F}$ with 11 to 32 edge sets. The number of commodities is between 5 and 20.

Our algorithms have given good results. The most part of theses instances have been solved to optimality in the time limit. We can see in Figure 2 a little real instance with an optimal solution for the multiple MCSIPND problem.



Optical network | Commodities | IP network solution

Figure 2: A real french instance with 10 nodes, $|\mathcal{F}| = 25$ and 10 commodities

9 Conclusion

In this paper we have considered the multilayer survivable network design problem which has applications to the design of reliable IP-over-optical network. We have considered the capacity dimensioning of the network. We have proposed two integer programming formulations for each of the two variants of the problem (simple and multiple). We have identified some valid inequalities, and we have also described necessary conditions and sufficent conditions for a class of inequalities to define facets. Using this, we have developed Branch-and-Cut and Branch-and-Cut-and-Price algorithms for the problems and presented extensive computational results. Other variants of the multilayer network design problem are of interest for telecommunication operators and merit to be investigated. In particular those which IP and optical layers should be treated simultaneously.

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