

Partial Path Column Generation for the Vehicle Routing Problem with Time Windows

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Abstract

This paper presents a column generation algorithm for the Vehicle Routing Problem with Time Windows (VRPTW). Traditionally, column generation models of the VRPTW have consisted of a Set Partitioning master problem with each column representing a route, i.e., a resource feasible path starting and ending at the depot. Elementary routes (no customer visited more than once) have shown superior results on difficult instances (less restrictive capacity and time windows). However, the pricing problems do not scale well when the number of feasible routes increases, i.e., when a route may contain a large number of customers. We suggest to relax that ‘each column is a route’ into ‘each column is a part of the giant tour’; a so-called partial path, i.e., not necessarily starting and ending in the depot. This way, the length of the partial path can be bounded and a better control of the size of the solution space for the pricing problem can be obtained.

Keywords: *Vehicle Routing Problem, Column Generation, Elementary Shortest Path Problem with Resource Constraints*

1 Introduction

The VRPTW can be described as follows: A set of customers, each with a demand, needs to be serviced by a number of vehicles all starting and ending at a central depot. Each customer must be visited exactly once within a given time window and the capacity of the vehicles may not be exceeded. The objective is to service all customers traveling the least possible distance. In this paper we consider a homogeneous fleet, i.e., all vehicles are identical.

The standard Dantzig-Wolfe decomposition of the arc flow formulation of the VRPTW is to split the problem into a master problem (a Set Partitioning Problem) and a pricing problem (an Elementary Shortest Path Problem with Resource Constraints (ESPPRC), where capacity and time are the constrained resources). A restricted master problem can be solved with delayed column generation and embedded in a branch-and-bound framework to ensure integrality. Applying cutting planes either in the master or the pricing problem leads to a Branch-and-Cut-and-Price algorithm (BCP). Dror [5] showed that the ESPPRC (with time and capacity) is strongly \mathcal{NP} -hard.

We propose a decomposition approach based on the generation of partial paths and the concatenation of these. In the bounded partial path decomposition approach the main idea is to limit the solution space of the pricing problem by bounding some resource, e.g., the number of nodes on a path. The master problem combines a known number of these bounded partial paths to ensure all customers are visited.

The paper is organized as follows: In Section 2 an overview of the Dantzig-Wolfe decomposition of the VRPTW is given and it is described how to calculate the reduced cost of columns when column generation is used. Section 5 concludes on the model.

2 The Vehicle Routing Problem with Time Windows

The VRPTW can formally be stated as: Given a graph $G(V,A)$ with nodes V and arcs A , a set R of resources ($R = \{\text{load, time}\}$) where each resource $r \in R$ has a lower bound a_i^r and an upper bound b_i^r for all $i \in V$ and a positive consumption τ_{ij}^r when using arc $(i, j) \in A$, find a set of routes starting and ending at the depot node $0 \in V$ satisfying all resource limits, such that the cost is minimized and all customers $C = V \setminus \{0\}$ are visited.

2-index formulation of the VRPTW In the following let c_{ij} be the cost of arc $(i, j) \in A$, x_{ij} be the binary variable indicating the use of arc $(i, j) \in A$, and T_{ij}^r be the consumption of resource $r \in R$ at the beginning of arc $(i, j) \in A$. Let $\delta^+(i)$ and $\delta^-(i)$ be the set of outgoing respectively ingoing arcs of node $i \in V$. The mathematical model of VRPTW adapted from Bard et al. [2] and Ascheuer et al. [1]:

$$\min \sum_{(i,j) \in A} c_{ij} x_{ij} \quad (1)$$

$$\text{s.t.} \quad \sum_{(i,j) \in \delta^+(i)} x_{ij} = 1 \quad \forall i \in C \quad (2)$$

$$\sum_{(j,i) \in \delta^-(i)} x_{ji} = \sum_{(i,j) \in \delta^+(i)} x_{ij} \quad \forall i \in V \quad (3)$$

$$\sum_{(j,i) \in \delta^-(i)} (T_{ji}^r + \tau_{ji}^r x_{ji}) \leq \sum_{(i,j) \in \delta^+(i)} T_{ij}^r \quad \forall r \in R, \forall i \in C \quad (4)$$

$$a_i x_{ij} \leq T_{ij}^r \leq (b_j - \tau_{ij}^r) x_{ij} \quad \forall r \in R, \forall (i, j) \in A \quad (5)$$

$$T_{ij}^r \geq 0 \quad \forall r \in R, \forall (i, j) \in A \quad (6)$$

$$x_{ij} \in \{0, 1\} \quad \forall (i, j) \in A \quad (7)$$

The objective (1) sums up the cost of the used arcs. Constraints (2) ensure that each customer is visited exactly once, and (3) are the flow conservation constraints. Constraints (4) and (5) ensure the time windows are satisfied. It is assumed that the bounds on the depot are always satisfied. Note, no sub-tours can be present since only one time stamp per arc exists and the travel times are positive.

3 Bounded partial paths

A solution to the VRPTW: $v_0 \rightarrow c_1^1 \rightarrow \dots \rightarrow c_{k_1}^1 \rightarrow v_0, v_0 \rightarrow c_1^2 \rightarrow \dots \rightarrow c_{k_2}^2 \rightarrow v_0, v_0 \rightarrow c_1^n \rightarrow \dots \rightarrow c_{k_n}^n \rightarrow v_0$ can be represented by the giant-tour representation of Christofides and Eilon [3]:

$$v_0 \rightarrow c_1^1 \rightarrow \dots \rightarrow c_{k_1}^1 \rightarrow v_0 \rightarrow c_1^2 \rightarrow \dots \rightarrow c_{k_2}^2 \rightarrow v_0 \rightarrow c_1^n \rightarrow \dots \rightarrow c_{k_n}^n \rightarrow v_0$$

which is one long path visiting all customers. The consumption of resources is reset each time the depot node is encountered.

The idea is to partition the problem so that the solution space of each part is smaller than the original problem. This is done by splitting the giant-tour into smaller segments by imposing an upper limit on some resource, e.g., bounding the path length in the number of nodes. In the following the number of visited customers is considered the bounding resource, i.e., the number of visits to the non-depot node set C . Each segment represents a partial path of the giant-tour. With a fixed number of customers on each partial path, say L , a fixed number of partial paths, say K , is needed to ensure that all customers are visited, i.e., $L \cdot K \geq |C|$. The partial paths can start and end in any node in V and can visit the depot several times. Example of a partial path:

$$c_1 \rightarrow c_2 \rightarrow v_0 \rightarrow c_3 \rightarrow v_0 \rightarrow c_4$$

Consider the graph $G'(V', A')$ consisting of a set of layers $K = \{1, \dots, |K|\}$, each one representing G for a partial path. Let G^k be the sub graph of G' representing layer k with node set $V^k = \{(i, k) : i \in V\}$ for all $k \in K$ and arc set $A^k = \{(i, j, k) : (i, j) \in A\}$ for all $k \in K$. Let $A^* = \{(i, i, k) : (i, k) \in V^k \wedge (i, k+1) \in V^{k+1} \wedge k \in K\}$ be

the set of interconnecting arcs, i.e., the arcs connecting a layer k with the layer above k namely layer $k + 1$ for all $k \in K$ and all nodes $i \in V$ (layer $|K| + 1$ is defined to be layer 1 $\in K$ and layer 0 is defined to be layer $|K| \in K$). Let $V' = \bigcup_{k \in K} V^k$ and let $A' = \bigcup_{k \in K} A^k \cup A^*$. An illustration of G' can be seen on Figure 1. Note, that arc (i, i, k) does not exist in A^k and that arc (i, j, k) with $i \neq j$ does exist in A^* , so all arcs $(i, j, k) \in A'$ can be uniquely indexed. With the length of a path defined as the number of customers on it, the problem is now to find partial paths of length at most L in $|K|$ layers with $L \cdot |K| \geq |C| > L \cdot (|K| - 1)$, so that each partial path p ending in node $i \in V$ is met by another partial path p' starting in i . All partial paths are combined while not visiting any customers more than once and satisfying all resource windows. A customer $c \in C$ is considered to be on a partial path p if c is visited on p and is not the end node of p .

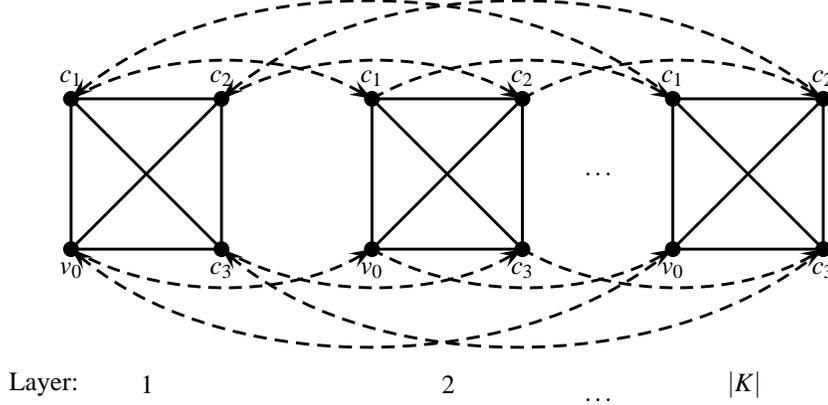


Figure 1: Illustration of G' with $|C| = 3$, $|K| = 3$, and $|L| = 1$. Edges (full-drawn) represent two arcs; one in each direction. Dashed lines are the interconnecting arcs A^* .

Let L be the upper bound on the length of each partial path, and let $|C|$ be the length of the combined path (the giant-tour). Now, exactly $|K| = \lceil |C|/L \rceil$ partial paths are needed to make the combined path, since $L \lceil |C|/L \rceil \geq |C| > L(\lceil |C|/L \rceil - 1)$. Note that given a $|K|$, L can be reduced to $L = \lceil |C|/|K| \rceil$.

3-index formulation of the VRPTW Let x_{ij}^k be the variable indicating the use of arc $(i, j, k) \in A'$. Problem (1)–(7) is rewritten:

$$\min \sum_{k \in K} \sum_{(i,j) \in A} c_{ij} x_{ij}^k \quad (8)$$

$$\sum_{k \in K} \sum_{(i,j) \in \delta^+(i)} x_{ij}^k = 1 \quad \forall i \in C \quad (9)$$

$$\sum_{(i,j) \in \delta^+(i)} x_{ij}^k \leq 1 \quad \forall k \in K, \forall i \in C \quad (10)$$

$$\sum_{k \in K} \left(x_{ii}^{k-1} + \sum_{(j,i) \in \delta^-(i)} x_{ji}^k \right) = \sum_{k \in K} \left(x_{ii}^k + \sum_{(i,j) \in \delta^+(i)} x_{ij}^k \right) \quad \forall i \in V \quad (11)$$

$$x_{ii}^{k-1} + \sum_{(j,i) \in \delta^-(i)} x_{ji}^k = x_{ii}^k + \sum_{(i,j) \in \delta^+(i)} x_{ij}^k \quad \forall k \in K, \forall i \in V \quad (12)$$

$$\sum_{k \in K} \sum_{i \in V} x_{ii}^k = K \quad (13)$$

$$\sum_{i \in C} \sum_{(i,j) \in \delta^+(i)} x_{ij}^k \leq L \quad \forall k \in K \quad (14)$$

$$\sum_{k \in K} \sum_{(j,i) \in \delta^-(i)} \left(T_{ji}^r + \tau_{ji}^r x_{ji}^k \right) \leq \sum_{k \in K} \sum_{(i,j) \in \delta^+(i)} T_{ij}^r \quad \forall r \in R, \forall i \in C \quad (15)$$

$$\sum_{(j,i) \in \delta^-(i)} \left(T_{ji}^r + \tau_{ji}^r x_{ji}^k \right) \leq \sum_{(i,j) \in \delta^+(i)} T_{ij}^r \quad \forall r \in R, \forall k \in K, \forall i \in C \quad (16)$$

$$a_i \sum_{k \in K} x_{ij}^k \leq T_{ij}^r \leq b_i \sum_{k \in K} x_{ij}^k \quad \forall r \in R, \forall (i,j) \in A \quad (17)$$

$$a_i x_{ij}^k \leq T_{ij}^r \leq b_i x_{ij}^k \quad \forall r \in R, \forall k \in K, \forall (i,j) \in A \quad (18)$$

$$x_{ij}^k \in \{0, 1\} \quad \forall k \in K, \forall (i,j) \in A \quad (19)$$

$$T_{ij}^r \geq 0 \quad \forall r \in R, \forall (i,j) \in A \quad (20)$$

The objective (8) sums up the cost of the used edges. Constraints (9) ensure that all customers are visited exactly once, while the redundant constraints (10) ensure that no customer is visited more than once. Constraints (11) maintain flow conservation between the original nodes V . Constraints (11) can be rewritten as

$$\sum_{k \in K} \sum_{(j,i) \in \delta^-(i)} x_{ji}^k = \sum_{k \in K} \sum_{(i,j) \in \delta^+(i)} x_{ij}^k \quad \forall i \in V$$

since $\sum_{k \in K} x_{ii}^{k-1} = \sum_{k \in K} x_{ii}^k$. Constraints (12) maintain flow conservation within a layer. Constraint (13) ensures that K partial paths are selected and constraints (14) that the length of the partial path in each layer is at most L . Constraints (15) connect the resource variables on a global level and constraints (16) connect the resource variables within each single layer, note that since there is no (15) and (16) for the depot it is not constrained by resources. Constraints (17) globally enforce the resource windows and the redundant constraints (18) enforce the resource windows within each layer.

4 Dantzig-Wolfe decomposition

The 3-index formulation of the VRPTW (8)–(20) is Dantzig-Wolfe decomposed whereby a master and a pricing problem is obtained.

Master problem: Let λ_p be the variable indicating the use of partial path p . Using Dantzig-Wolfe decomposition where the constraints (9), (11), (13), (15), and (17) are kept in the master problem the following master problem is obtained:

$$\min \sum_{p \in P} c_p \lambda_p \quad (21)$$

$$\sum_{p \in P} \sum_{(i,j) \in \delta^+(i)} \alpha_{ij}^p \lambda_p = 1 \quad \forall i \in C \quad (22)$$

$$\sum_{p \in P: e^p = i} \lambda_p = \sum_{p \in P: s^p = i} \lambda_p \quad \forall i \in V \quad (23)$$

$$\sum_{p \in P} \lambda_p = K \quad (24)$$

$$\sum_{(j,i) \in \delta^-(i)} \left(T_{ji}^r + \sum_{p \in P} \tau_{ji}^r \alpha_{ji}^p \lambda_p \right) \leq \sum_{(i,j) \in \delta^+(i)} T_{ij}^r \quad \forall r \in R, \forall i \in C \quad (25)$$

$$a_i \sum_{p \in P} \alpha_{ij}^p \lambda_p \leq T_{ij}^r \leq b_i \sum_{p \in P} \alpha_{ij}^p \lambda_p \quad \forall r \in R, \forall (i,j) \in A \quad (26)$$

$$T_{ij}^r \geq 0 \quad \forall r \in R, \forall (i,j) \in A \quad (27)$$

$$\lambda_p \in \{0, 1\} \quad \forall p \in P \quad (28)$$

Where α_{ij}^p is the number of times arc $(i,j) \in A$ is used on path $p \in P$ and s^p and e^p indicates the start respectively the end node of partial path $p \in P$. Constraints (22) ensure that each customer is visited exactly once. Constraints (23) link the partial paths together by flow conservation. Constraint (24) is the convexity constraint ensuring that K partial paths are selected. Constraints (25) and (26) enforce the resource windows.

Pricing problem: The $|K|$ pricing problems corresponding to the master problem (21)–(28) contains constraints (10), (12), (14), (16), and (18) and can be formulated as a single ESPPRC where the depot is allowed to be visited more than once. Let s and e be a super source respectively a super target node. Arcs (s, i) and (i, e) for all $i \in V$ are added to G .

$$\min \sum_{(i,j) \in A} \bar{c}_{ij} x_{ij} \quad (29)$$

$$\text{s.t.} \quad \sum_{(s,i) \in \delta^+(s)} x_{si} = 1 \quad (30)$$

$$\sum_{(i,e) \in \delta^-(e)} x_{ie} = 1 \quad (31)$$

$$\sum_{(i,j) \in A} x_{ij} \leq 1 \quad \forall i \in C \quad (32)$$

$$\sum_{(j,i) \in \delta^-(i)} x_{ji} = \sum_{(i,j) \in \delta^+(i)} x_{ij} \quad \forall i \in V \quad (33)$$

$$\sum_{i \in C} \sum_{(i,j) \in \delta^+(i)} x_{ij} \leq L \quad (34)$$

$$\sum_{(j,i) \in \delta^-(i)} (T_{ji}^r + \tau_{ji}^r x_{ji}) \leq \sum_{(i,j) \in \delta^+(i)} T_{ij}^r \quad \forall r \in R, \forall i \in C \quad (35)$$

$$a_i x_{ij} \leq T_{ij}^r \leq b_i x_{ij} \quad \forall r \in R, \forall (i, j) \in A \quad (36)$$

$$x_{ij} \in \{0, 1\} \quad \forall (i, j) \in A \quad (37)$$

The objective (29) minimizes the reduced cost of a column. Constraints (30) and (31) ensure that the path starts in s respectively ends in e . Constraints (32) dictates that no node is visited more than once, thereby ensuring elementarity, and constraints (33) conserve the flow. Constraint (35) and (36) ensure the resource windows are satisfied for all customers. Note, since the depot is missing in (35) each time a path leaves the depot a resource is only restricted by its lower limit a_0^r for all $r \in R$.

Let π ($\pi_i \geq 0 : \forall i \in C$) be the duals of (22), let $\pi_0 = 0$, let μ be the duals of (23), let $\beta \leq 0$ be the dual of (24), let ν ($\nu \leq 0 : \forall i \in C$) be the duals of (25), let $\nu_0 = 0$, and let $\underline{\omega} \leq 0$ and $\bar{\omega} \geq 0$ be the dual of (26). The cost of the arcs in this ESPPRC are then given as:

$$\bar{c}_{ij} = -\beta + \begin{cases} c_{ij} - \pi_i - \tau_{ij} \nu_j - a_i \underline{\omega}_i + b_i \bar{\omega}_i & \forall (i, j) \in A \setminus (\delta^+(s) \cup \delta^-(e)) \\ \mu_j & \forall (s, j) \in \delta^+(s) \\ \mu_i & \forall (i, e) \in \delta^-(e) \end{cases}$$

and the pricing problem becomes finding the shortest path from s to e .

Solving the pricing problem: ESPPRCs can be solved by a labeling algorithm. For details regarding labeling algorithms we refer to Desaulniers et al. [4], Irnich [6], Irnich and Desaulniers [7], and Righini and Salani [10].

Branching: Integrality can be obtained by branching on the original variables, which can be accomplished by cuts in the master problem (see Vanderbeck [11]), e.g., let X_{ij} be the set of partial paths that utilize arc (i, j) then the branch rule $x_{ij} = 0 \vee x_{ij} = 1$ can be expressed by:

$$\sum_{p \in X_{ij}} \lambda_p = 0 \vee \sum_{p \in X_{ij}} \lambda_p = 1.$$

Bounds: The following theorem justifies the approach presented in this paper.

Theorem 1. Let z_{lp} be an LP-relaxed solution to (1)–(7) and let z_{pp} be an LP-relaxed solution to (21)–(28) then $Z_{lp} \leq Z_{pp}$ for all instances of VRPTW and $Z_{lp} < Z_{pp}$ for some instances of VRPTW.

Proof. $Z_{lp} \leq Z_{pp}$ since all solutions to (21)–(28) map to solutions to (1)–(7). An instance with $Z_{lp} < Z_{pp}$ is obtained with four customers each with a demand of resource r of half the global maximum b_r of r , the distance from the customers to the depot larger than the distance between the customers, and $L = 4$. The solution to (21)–(28) would use the expensive edges four times, whereas the solution to (1)–(7) only would use them twice. \square

5 Conclusion

A new decomposition model of the VRPTW has been presented with ESPPRCs as the pricing problems. The model facilitates control of the running time of the pricing problems. Due to the aggregation of the model, LP relaxed bounds of (21)–(28) are better than the direct model (1)–(7). Since (21)–(28) is a relaxation of the traditional Dantzig-Wolfe decomposition model with elementary routes as columns, the LP relaxed bounds may be weaker yielding a larger branch-and-bound tree. The difference in bound quality can be decreased with the use of special purpose cutting planes, which this paper does not leave room for. Furthermore, effective cuts such as Subset Row-inequalities by Jepsen et al. [8] and Chvátal-Gomory Rank-1 cuts (see Petersen et al. [9]) can be applied to the Set Partition master problem to strengthen the bound. Future experimental results will conclude on the effectiveness of this approach.

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