On the Vehicle Routing Problem with lower bound capacities

Luis Gouveia* Jorge Riera⋆ Juan-José Salazar⋆

*DEIO–CIO, University of Lisbon
Bloco C/2 - Campo Grande, 1749-016 Lisbon, Portugal

⋆DEIOC, University of La Laguna
Av. Astrofísico Francisco Sánchez, 38271 La Laguna, Tenerife, Spain

Abstract

In this paper we show and discuss a family of inequalities for solving a variant of the classical vehicle routing problem where also a lower bound is considered. The inequalities are related to the projected inequalities from a single commodity flow formulation. Other inequalities are based on rounding procedures. We also show computational experiments proving the utility of the new inequalities.

Keywords: *Vehicle Routing Problems, Multistar and Reverse-Mulstar Inequalities, Branch-and-cut.*

1 Introduction

This paper discusses inequalities to address a variant of the classical vehicle routing problem where also lower bound capacities are taken into account. This problem is of interest when the decision maker desires to implement a solution where all routes may have a similar work load.

Routing problems are usually modelled through a directed graph $G = (V, A)$. A special node in $V = \{1, 2, \ldots, n\}$, node 1, represents a depot. Nodes in $V \setminus \{1\}$ represent clients. Each node i has a demand d_i such that $\sum_{i \in V} d_i = 0$. Demands can be negative. An arc is represented by one index a, or by two indices ij when its head j and its tail i are convenient for the notation. Each arc $a \in A$ is associated with a lower capacity $\underline{q}_a$ and an upper capacity $\overline{q}_a$ such that $\underline{q}_a \leq \overline{q}_a$, meaning that if arc a is in the solution (that is, used by a vehicle) then the vehicle load when traversing this arc cannot be greater than $\overline{q}_a$ and not smaller than $\underline{q}_a$. Also, each arc a is associated with a value c_a representing the cost of using the arc by a vehicle. A generic single-commodity flow (SCF) model uses two variables for each arc a:

(i) a 0-1 variable x_a indicating whether arc a is used by a vehicle and

(ii) a continuous variable f_a representing the load (flow) of a vehicle traversing the arc.

To simplify notation, if $S, T \subset V$ then we write $x(S : T)$ instead of $\sum_{(i,j) \in A, i \in S, j \in T} x_{ij}$. Given $S \subseteq V \setminus \{1\}$, we denote $V \setminus (S \cup \{1\})$ by S'. For brevity of notation, we also write i instead of $\{i\}$ for any $i \in V$. In addition, $\delta^+(S)$ stands for $\{(i,j) \in A : i \in S, j \notin S\}$ and $\delta^-(S)$ stands for $\{(i,j) \in A : i \notin S, j \in S\}$.

Then, the model minimizes a cost function

$$\min \sum_{a \in A} c_a x_a$$

subject to two sets of constraints.

One set involves only the design variables, and imposes the in-degree and out-degree constraints on each client node:

$$x(i : V \setminus \{i\}) = x(V \setminus \{i\} : i) = 1 \qquad \text{for all } i \in V \setminus \{1\} \tag{1}$$

$$x_a \in \{0,1\} \qquad \text{for all } a \in A \tag{2}$$

The other set of constraints impose the connectivity and the capacity constraints, and involve the use of the *flow* variables:

$$\sum_{a \in \delta^-(i)} f_a - \sum_{a \in \delta^+(i)} f_a = d_i \qquad \text{for all } i \in V \tag{3}$$

$$\underline{q}_a x_a \leq f_a \leq \overline{q}_a x_a \qquad \text{for all } a \in A. \tag{4}$$

As one specific example, a model for the unit-demand Capacitated Vehicle Routing Problem (1CVRP) with vehicle capacity $Q \geq 2$ can be defined by setting d_i to 1 if $i \in V \setminus \{1\}$ and to $1 - |V|$ if $i = 1$, and

$$\underline{q}_a = \overline{q}_a = 0 \qquad \text{for all } a \in \delta^-(1)$$

$$\underline{q}_a = 1 \text{ and } \overline{q}_a = Q \qquad \text{for all } a \in \delta^+(1)$$

$$\underline{q}_a = 1 \text{ and } \overline{q}_a = Q - 1 \qquad \text{for all } a \notin \delta^+(1) \cup \delta^-(1).$$

Different variations of vehicle routing problems can be formulated by changing some of the parameters given in the previous generic SCF formulation or setting new ones (see, e.g., Toth and Vigo [9]).

The SCF model is an example of compact model. We can create a natural model involving only the x_a variables and with a LP relaxation bound equal to the LP relaxation bound of the SCF model, by using the tool of projection. The procedure to project out the flow variables from the LP relaxation of the SCF model is based on the following result:

Theorem 1.1 (Hoffman 1960 [6]) *There is a solution f_a of the linear system (3)–(4) if and only if*

$$\sum_{a \in \delta^-(S)} \overline{q}_a x_a \geq \sum_{a \in \delta^+(S)} \underline{q}_a x_a + \sum_{i \in S} d_i \qquad \textit{for all } S \subset V. \tag{5}$$

We can give some intuition on how to generate these inequalities from the SCF model. Suppose we add the flow conservation constraints (3) for all nodes i in set S and cancel equal terms, leading to:

$$\sum_{a \in \delta^-(S)} f_a = \sum_{a \in \delta^+(S)} f_a + \sum_{i \in S} d_i.$$

Then, by using the upper bounding part in constraints (4) on the term in the left-hand side of the previous equality and by using the lower bounding part of constraints (4) on the right-hand side, we obtain (5). In fact, by reversing the bounding procedure just suggested (i.e., using the lower bounding part in (4) on the term in the left-hand side, and using the upper bounding part in (4) on the term in the right-hand side) we obtain the following alternative necessary-and-sufficient condition for the theorem

$$\sum_{a \in \delta^-(S)} \underline{q}_a x_a \leq \sum_{a \in \delta^+(S)} \overline{q}_a x_a + \sum_{i \in S} d_i \qquad \text{for all } S \subset V. \tag{6}$$

One can easily see that the inequality (6) associated with S coincides with the inequality (5) associated with the set $V \setminus S$. This follows from the fact that $\delta^+(S) = \delta^-(V \setminus S)$, $\delta^-(S) = \delta^+(V \setminus S)$ and that $\sum_{i \in S} d_i + \sum_{i \in V \setminus S} d_i = 0$. Thus, the two families of inequalities (5) and (6) are equivalent. However, the two families of inequalities (5) and (6) (together with the way we suggested above for generating these inequalities) permit us to cast Hoffman's theorem in an alternative form. We can divide each family of inequalities in two groups: one group is associated with the sets S containing node 1 and the other group with the sets S not containing node 1. Then we use one group in both families of inequalities to generate a complete projection, in particular the groups of inequalities (5) and (6) defined by sets S not containing node 1.

The reason for this option is that it is easier to enhance the different modelling properties of the inequalities from each group. First, for most of the standard vehicle routing problems, the first group (given by (5) for sets S not containing node 1) of projected inequalities is quite interesting while the second group (given by (6) for sets S not containing node 1) is redundant. Second, expression (6) permits us to detect quite easily less standard variations of the problem where these inequalities might be useful.

We use the name *Multistar* (MS) inequalities for constraints (5) with $1 \notin S$, while the name *Reverse Multistar* (RMS) inequalities is used for constraints (6) with $1 \notin S$. An intuition for the designation “reverse” will be given later on.

As noted before, it is well known (see, for instance, Gouveia [5], Letchford, Eglese and Lysgaard [7], Letchford and Salazar [8]) that for “capacitated” routing problems, that is, in routing problems with an upper bound on the number of clients on each route (or more generally, an upper bound on the sum of the demands of the clients on each route) the resulting MS inequalities turn out to be rather interesting inequalities. However, for these problems, the RMS inequalities are not of interest since they are implied by other inequalities in the model.

To exemplify this, consider the 1CVRP. The resulting MS inequalities are as follows

$$Qx(1:S) + (Q-1)x(S':S) \geq x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}. \tag{7}$$

They are the directed version of known inequalities that define facets of the associated undirected polytope (see Araque et al. [1]).

The resulting RMS inequalities for the 1CVRP are $x(1:S) + x(S':S) \leq (Q-1)x(S:S') + |S|$ for all $S \subseteq V \setminus \{1\}$. It is easy to see that for a given set S the corresponding RMS inequality is implied by the equality obtained by adding the in-degree constraints (1) for nodes $j \in S$. Thus, for the 1CVRP, the RMS inequalities are not of interest.

The next section presents and discusses a variant of the 1CVRP where the RMS inequalities are not redundant, and they have a specific and intuitive interpretation. To model this problem we need new families of inequalities. We investigate the separation problem of these inequalities and implement a cutting-plane approach to solve it. Due to the limitation of this abstract, we do not give here details on these aspects.

2 VRP with lower capacities

As we have noted before, expression (6) permits us to “guess” situations where the RMS inequalities might be of interest. The first situation that comes to mind is one where the demand summation on the left-hand side of (6) is negative. This may happen, for instance, in situations where some of the client demands are negative. This corresponds to pick-up and delivery variations of the 1CVRP (see, e.g., [4]) where typically the clients with positive demands correspond to clients which receive some commodity from the depot and where clients with negative demands send some commodity to the depot. However, it is also the case that a RMS inequality corresponds to a MS inequality in the symmetrized version of the problem (we exchange the sign of all demand numbers) and from a structural point of view, a RMS inequality is similar to a MS inequality.

The second situation, which is also motivated by a simple analysis of expression (6), is to consider variations of the 1CVRP where lower bound values on the arc flows are bigger than one. To illustrate this statement, consider the so-called *Balanced Vehicle Routing Problem* (BVRP), where a minimum number of clients $\underline{Q}$ and a maximum number of clients Q are required to each route in a feasible solution. In general, the designation “balanced” applies to a variant of the problem when $Q - \underline{Q}$ is small. Here we relax this designation since our aim is to consider situations with a lower bound $\underline{Q}$ and we may even allow examples where $Q = |V| - 1$ (i.e., no upper capacity on the vehicles). The lower limited capacity can be easily modelled through a SCF model by setting the lower bound value $\underline{q}_a$ on the arcs leaving the depot as being equal to $\underline{Q}$. More precisely, a SCF model for the BVRP can be obtained by setting the parameters as follows:

$$\begin{aligned} \underline{q}_a = \overline{q}_a = 0 & \qquad \text{for all } a \in \delta^-(1) \\ \underline{q}_a = \underline{Q} \text{ and } \overline{q}_a = Q & \qquad \text{for all } a \in \delta^+(1) \\ \underline{q}_a = 1 \text{ and } \overline{q}_a = Q - 1 & \qquad \text{for all } a \notin \delta^+(1) \cup \delta^-(1). \end{aligned}$$

As seen above, it is quite easy to include lower bound information in a SCF formulation. On the other hand, finding from scratch, inequalities only involving the x_a variables and that guarantee the minimum required number of clients in each route seems to be far from easy. Fortunately, the projection result permits us to obtain one of such set of inequalities. The MS inequalities are exactly the ones given in (7). Note that they do not depend on the lower bound value $\underline{Q}$. The RMS inequalities, instead, take into account the upper bound capacity as well as the new lower bound capacity:

$$\underline{Q}x(1:S) + x(S':S) \leq (Q-1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}. \tag{8}$$

The MS inequalities are known to define facets, under mild conditions, of the undirected polytope associated to the 1CVRP (see, e.g., [7]). Although no similar study has been done for the BVRP, we have no reason to suspect that the MS inequalities can be strengthened when non-trivial flow lower bounds are imposed. In contrast, the RMS inequalities can be strengthened by decreasing the coefficient of the variables in the right hand-side term leading to inequalities only involving the lower bound capacity, the following *Enhanced RMS* (ERMS) inequalities:

$$\underline{Q}x(1:S) + x(S':S) \leq (\underline{Q}-1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}. \tag{9}$$

Proposition 2.1 *The ERMS inequalities (9) are valid for the BVRP polytope.*

We end this section noting that for the special case of the BVRP where $Q = \underline{Q}$ the ERMS inequalities (9) do not provide new information since they can be shown to be equivalent to the MS inequalities (7). We refer to the case where $Q = \underline{Q}$. We shall show that the MS inequality (7) for a given set S is equivalent to the ERMS inequality (9) for the complement set S'. For the proof we use of the fact that, when $Q = \underline{Q}$, all feasible solutions have a fixed number of vehicles given by $(|V|-1)/Q$, leading to the equality $x(1 : V \setminus 1) = (|V|-1)/Q$.

Proposition 2.2 *When $Q = \underline{Q}$, the ERMS inequality (9) for set S is equivalent to the MS inequality (7) for the set S', and vice-versa.*

3 Rounded Inequalities

It is well known that "projected" inequalities can be used to produce (by adequate division and rounding) other interesting sets of inequalities (see, e.g., [8]). As an example, consider the MS inequalities (7) for the 1CVRP. Dividing by Q these inequalities and rounding we obtain the following *rounded MS inequalities*

$$x(1:S) + \left\lceil \frac{Q-1}{Q} \right\rceil x(S':S) \geq \left\lfloor \frac{1}{Q} \right\rfloor x(S:S') + \left\lceil \frac{|S|}{Q} \right\rceil \tag{10}$$

that correspond to the well-known *generalized cut constraints* $x(V \setminus S : S) \geq \left\lceil \frac{|S|}{Q} \right\rceil$ for all $S \subseteq V \setminus \{1\}$. These inequalities are the directed version of facet-defining inequalities for the undirected 1CVRP polytope (see, e.g. Campos et al. [2], Cornuejols and Harche [3] and Araque et al. [1]) and are by far the most relevant inequalities in cutting-plane approaches for solving the 1CVRP and related problems.

As it has been done with the original RMS inequalities, the ERMS inequalities can be rewritten as follows:

$$(\underline{Q}-1)|S| \leq (\underline{Q}-1)x(S:S') + (\underline{Q}-1)x(S':S) + \underline{Q}x(S:S).$$

Then, if we divide this inequality by $\underline{Q}$, and then apply rounding as before, we obtain the new *rounded ERMS inequalities*:

$$|S| - \left\lfloor \frac{|S|}{\underline{Q}} \right\rfloor \leq \left\lceil \frac{\underline{Q}-1}{\underline{Q}} \right\rceil x(S:S') + \left\lceil \frac{\underline{Q}-1}{\underline{Q}} \right\rceil x(S':S) + x(S:S) \tag{11}$$

Proposition 3.1 *When $Q = \underline{Q}$, the rounded ERMS inequality (11) for set S is equivalent to the rounded MS inequality (10) for the set $\overline{S'}$.*

We are currently working on a branch-and-cut algorithm making use of the inequalities introduced in this abstract to solve the BVRP. Computational results will be presented during the INOC2009.

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