On the Unit Demand Vehicle Routing Problem: Flow Based Inequalities implied by a Time-dependent Formulation

Maria Teresa Godinho*, Luís Gouveia+, Thomas L. Magnanti□ , Pierre Pesneau ◊, José Pires#

**CIO and Dep. of Mathematics, Instituto Politécnico de Beja,*
Rua Afonso III, nº 1 e 3, 7800-800 Beja, Portugal

+*CIO and Dep.of Statistics and Operations Research,, Faculdade de Ciências da Universidade de Lisboa*
Bloco C/6 Piso 4, Campo Grande, 1749-016 Lisboa, Portugal

□*Dep. of Electrical Engineering and Computer Science and Sloan School of Management, MIT,*
77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA

◊*MAB, Université de Bordeaux 1,*
351, Cours de la Libération F33405 Talence Cedex France, France

#*CIO and Dep. of Mathematics,Instituto Superior de Contabilidade e Administração de Lisboa*
Rua Miguel Bombarda, nº 20, 1069-035 Lisboa, Portugal

Abstract

In this paper we study the relationship between the linear programming relaxation of a pure time-dependent formulation (that can be seen as modified version of the well-known Picard and Queyranne formulation for the TSP [11]) and the linear programming relaxation of a well known single-commodity flow model due to Gavish and Graves [6]. In particular, we show that the time-dependent formulation implies a large class of upper bounding and lower bounding flow constraints and that some of them are facet defining for the CVRP.

Keywords: Vehicle Routing, Hop-indexed Network Flow Models

---

1. Introduction

The unit-demand Capacitated Vehicle Routing Problem (CVRP) is defined on a given directed graph G = (V,A) with a node set V = {1,…,n} and an arc set A with an integer weight (cost) c_a associated with each arc a of A, as well as a given natural number Q. The problem seeks a minimum cost set of routes originating and terminating at the depot (we assume that node 1 is the depot) with each node in V\{1} visited exactly once and each route containing at most Q nodes (plus the depot). [12] provides surveys on the problem; [2, 5, 10] discuss the most successful algorithms for solving this problem as well as general demand cases; [1, 3, 4, 9] have studied the polyhedral structure of the CVRP, and recently, while [6] presents and compares the linear programming relaxation of several so-called multicommodity flow time-dependent formulations.

In this paper we study the relationship between the linear programming relaxation of a pure time-dependent formulation (that can be seen as modified version of the well-known Picard and Queyranne formulation for the TSP [11]) and the linear programming relaxation of a well known single-commodity flow model due to Gavish and Graves [6]. In particular, we show that the time-dependent formulation implies a new large class of upper bounding and lower bounding flow constraints and that some of them are facet defining for the CVRP.

2. A Generic Formulation and the Single-Commodity Formulation for the CVRP

Throughout this paper, for any formulation P, F(P) denotes its set of feasible solutions, v(P) the value of an optimal solution, and P_L its linear programming relaxation. If g is any quantity defined on the arcs (i,j), we let $g(A,B)$ denote $\sum_{i\in A}\sum_{j\in B} g_{ij}$. When A = B, we will simply use the notation $g(A)$ and when A (or B) is a single node set, for instance A = {i}, we will use $g(i,B)$. In particular, we will use this notation when referring to binary arc variables x_{ij} associated with arc inclusion.

Consider the following generic formulation for the CVRP:

$$\begin{aligned} &\textit{minimize} \quad \sum_{(i,j)\in A} c_{ij}x_{ij} \\ &\textit{subject to} \quad x_{ij} \in \textit{Assign} \quad (i,j)\in A \\ &\qquad \{(i,j):\ x_{ij}=1 \textit{ does not contain routes with more than } Q \textit{ nodes}\} \end{aligned}$$

with Assign denoting the feasible set of the well-known assignment relaxation arising in formulations for the problem:

$$\begin{aligned} &x(V,j)=1 \quad j\in V\setminus\{1\} \\ &x(i,V)=1 \quad i\in V\setminus\{1\} \\ &x_{ij}\in\{0,1\} \quad (i,j)\in A. \end{aligned}$$

There are several ways to model the implicit route constraints. One modelling approach is to use extra variables such as in [6] that considers additional flow variables f_{ij} indicating the amount of flow on arc (i,j) (assuming that the depot, node 1, sends one unit of flow to every other node) and the additional set of inequalities:

$$\begin{aligned} &f(V,j)-f(j,V\setminus\{1\})=1 && j\in V\setminus\{1\} \\ &f(1,V\setminus\{1\}=n-1 && \\ &f_{ij}\le (Q-1)x_{ij} && (i,j)\in A,\ i,j\ne 1 \\ &f_{1j}\le Qx_{1j} && j\in V\setminus\{1\} \\ &f_{ij}\ge x_{ij} && (i,j)\in A,\ j\ne 1 \\ &f_{ij}\ge 0 && (i,j)\in A. \end{aligned}$$

The reader is referred to [6] for the validity of this formulation.

3. The Modified Picard and Queyranne Formulation

The well-known Picard and Queyranne formulation for the TSP (see [11]) is easily modified for the CVRP. The main modifications between the model for the TSP and ours are specified in [8]. The Modified Picard and Queyranne formulation, MPQ, is obtained by replacing the implicit part of the generic formulation with the following system:

$$
\begin{array}{ll}
z_{1j}^{1} = \sum_{i \in V \setminus \{1\}} z_{ji}^{2} & j = 2,\dots,n \\
\sum_{i \in V} z_{ij}^{h} = \sum_{i \in V \setminus \{1\}} z_{ji}^{h+1} & j = 2,\dots,n \text{ and } h = 2,\dots,Q-1 \\
\sum_{i \in V} z_{ij}^{Q} = z_{j1}^{Q+1} & j = 2,\dots,n \\
x_{1j} = \sum_{h=1,\dots,Q} z_{1j}^{h} & j = 2,\dots.,n \\
x_{ij} = \sum_{h=2,\dots,Q} z_{ij}^{h} & i,j = 2,\dots.,n \\
x_{j1} = z_{j1}^{Q+1} & j = 2,\dots,n \\
z_{ij}^{h} \in \{0,1\} & (1,j) \in A \text{ and } h = 1,\dots.,Q \\
 & \text{or } (i,j) \in A,\ i,j \neq 1 \text{ and } h = 2,\dots,Q \\
 & \text{or } (i,1) \in A \text{ and } h = Q+1.
\end{array}
$$

In this model, the variable z_{ij}^{h} indicates that arc (i,j) is in a path that contains Q-h+1 nodes after the arc (including node j, but not node 1). The equality constraints imposed upon the z variables simply define a network flow system in this layered graph whose solution (in integer variables) are paths (corresponding to routes in the original graph) from the source to the destination versions of node 1. These constraints permit each path to visit several copies of the same original node. However, in the overall problem, the constraints linking the z_{ij}^{h} with the x_{ij} variables and the assignment constraints in the x_{ij} variables rule out that situation.

The following result is proved in [8]:

Proposition 3.1:

$$v(MPQ_L) \geq v(SCF_L).$$

Is the previous inequality strict for some cases? The reference [8] provides some computational results showing that the MPQ formulation provides, in general, better linear programming bounds than the SCF formulation. These results suggest that it would be worth investigating what inequalities are implied by the linear programming relaxation of MPQ that are not redundant in the linear programming relaxation of SCF. This will be the topic of the next section.

4. Inequalities Implied by MPQ in the x_{ij} and f_{ij} Space

The set of linking constraints included in the SCF formulation models the fact that the flow in any "internal" arc with $x_{ij} = 1$ cannot exceed Q-1 (since the first node in any route absorbs one unit of flow) and cannot be less than 1 (since only the arc returning to the depot would have flow equal to zero). We can generalize this concept by introducing a variation of the stronger constraints that reflects the maximum or minimum flow in arcs in positions two away (after or before) node 1. The validity of the following constraints is easy to establish:

$$\begin{aligned} f_{ij} &\le (Q-2)x_{ij} + x_{1i} \qquad (i,j) \in A,\ i,j \ne 1 \\ f_{ij} &\ge 2x_{ij} - x_{j1} \qquad (i,j) \in A,\ j \ne 1. \end{aligned} \tag{4.1a/b}$$

Note that these constraints do not imply the previous set. However, we can add them to the single commodity flow model to tighten the linear programming relaxation. In the conference we will exhibit a feasible solution for the linear programming relaxation of SCF that violates the new inequalities, thus proving that they are not redundant in the linear programming relaxation of SCF.

As noted in [8], we can generalize the constraints (4.1a/b) by (i) bounding the flow in arcs that are more than 2 arcs away from the depot, and (ii) considering constraints for arc sets instead of a single arc (i,j). For the moment we list some facet defining results of the inequalities presented so far.

Let P_n denote the Capacitated Vehicle Routing with Flows Polytope, defined as the convex hull of all the incidence vectors $x = (x_{ij}, f_{ij}) \in R^{2|A|}$ of the feasible routes and flows in a given graph G=(V,A)

Proposition 4.1: If $Q \ge 3$ and $n \ge Q+4$, the inequality $f_{ij} \ge x_{ij}$ is facet defining for P_n.

Proposition 4.2: If $Q \ge 3$ and $n \ge Q+4$ inequality $f_{ij} \le (Q-1)x_{ij}$ is facet defining for P_n.

Proposition 4.3: If $Q \ge 4$ and $n \ge Q+7$ inequality $f_{ij} \le (Q-2)x_{ij} + 2x_{1i}$ is facet defining for P_n.

In the talk we will also discuss the more general inequalities and will be present results assessing the quality of the projected inequalities.

References

[1]J.R Araque, L. Hall, T.L. Magnanti, ''Capacitated trees, capacitated routing and associated polyedra,'' Discussion Paper 90-61, CORE, University of Louvin La Neuve, Belgium, 1990.

[2]J. R. Araque, G. Kudva, T. L. Morin and J. F. Pekny, ''A Branch-and-Cut Algorithm for Vehicle Routing Problems'', Annals of Operations Research, 50, 37-59,1994.

[3]P. Augerat, ''Approche Polyèdrale du Problème de Tournées de Véhicules,'' PhD Thesis, Institut National Polytechnique de Grenoble, France, 1995.

[4]V. Campos, A. Corberan, and E. Mota, "Polyhedral Results for a Vehicle Routing Problem," European Journal of Operational Research , 52, 75-85, 1991.

[5]R. Fukasawa, H. Longo, J. Lysgaard, M. Poggi de Aragão, M. Reis, E. Uchoa and R. F. Werneck, ‘‘Robust branch-and-cut-and-price for the capacitated vehicle routing problem,” Mathematical Programming, 106 (3), 49-511, 2006.

[6] B. Gavish and S. Graves, “The Travelling Salesman Problem and Related Problems,” Working Paper, Graduate School of Management, University of Rochester, 1978

[7]M.T. Godinho, L. Gouveia, T.L. Magnanti, “Combined Route Capacity and Route Length Models for Unit Demand Vehicle Routing Problems,” Discrete Optimization 5, 350-372, 2008.

[8]M.T. Godinho, L. Gouveia, T.L. Magnanti, P. Pesneau, J.M. Pires “On a Time-Dependent Formulation for the Vehicle Routing Problem” Working Paper nº 11 / 200/, Centro de Investigação Operacional, Lisbon.

[9]A.N. Letchford and J.J. Salazar-Gonzalez, ‘‘Vehicle Routing Inequalities obtained by Projection of Flow Variables,’’ Mathematical Programming B (105), 251-274, 2006.

[10]J. Lysgaard, A.N. Letchford and R.W. Eglese, ‘‘A new branch-and-cut algorithm for the capacitated vehicle routing problem,’’ Mathematical Programming, 100 (2), 423-445, 2004.

[11] J.C. Picard, J.C. and M. Queyranne, “The time-dependent travelling salesman problem and its application to the tardiness in one-machine scheduling” Operations Research 26, 86–110, 1978.

[12]P. Toth, D. Vigo (eds.), ‘‘The Vehicle Routing Problem,’’ SIAM Monographs on Discrete Mathematics and Applications, Philadelphia, 2002.