Knapsack-Based Cutting Planes for the Max-Cut Problem

Adam N. Letchford[*] Konstantinos Kaparis[⋆]

[*] *Department of Management Science, Lancaster University*
Lancaster LA1 4YX, United Kingdom

[⋆] *School of Mathematics, Southampton University*
Highfield, Southampton SO17 1BJ, United Kingdom

Abstract

We present a new procedure for generating cutting planes for the max-cut problem. The procedure consists of three steps. First, we generate a violated (or near-violated) linear inequality that is valid for the semidefinite programming (SDP) relaxation of the max-cut problem. This can be done by computing the minimum eigenvalue of a certain matrix. Second, we use this linear inequality to construct a 'knapsack relaxation' of the given max-cut instance. Third, we generate cutting planes that are valid for the knapsack relaxation, using existing techniques from the literature on the knapsack problem. The procedure enables us to obtain upper bounds that are comparable with those obtained with SDP, but without using an SDP solver.

Keywords: *combinatorial optimisation, max-cut problem, semidefinite programming*

---

1 Introduction

Given an undirected graph with weights on the edges, the *max-cut* problem calls for the vertex set to be partitioned into two clusters, in such a way that the sum of the weights on the edges crossing from one cluster to the other is maximised. The max-cut problem is a well-known, strongly $\mathcal{NP}$-hard combinatorial optimisation problem. We refer the reader to [3, 5, 9] for surveys on the problem and its many applications.

There are several interesting heuristics and approximation algorithms for the max-cut problem, but here we are concerned with upper bounds computed with either linear programming (LP) or semidefinite programming (SDP). At present, SDP seems to be the clear winner [4, 10], although LP can work well for sparse instances [2, 7]. Our goal is to push the LP approach further, and try to make it work better for dense instances.

Our main idea is a new 'three-step' procedure for generating cutting planes for the max-cut problem. We begin by generating a violated (or near-violated) linear inequality that is valid for the SDP relaxation of the max-cut problem. Next, we use this linear inequality to construct a 'knapsack relaxation' of the given max-cut instance. Finally, we generate cutting planes that are valid for the knapsack relaxation, using existing techniques from the literature on the knapsack problem.

Theoretically speaking, our procedure can generate 'weak' cutting planes, since they do not in general define facets of the cut polytope. Nevertheless, our preliminary computational experiments indicate that the cutting planes are very useful in practice. Indeed, in many cases, the upper bounds that we obtain are comparable with those obtained with SDP.

2 Some Background

Since our procedure is non-conventional, we provide some background to motivate it. Assume without loss of generality that the graph is complete (adding dummy edges of weight zero if necessary). Let n denote the number, of vertices, and let w_{ij} denote the weight of edge $\{i, j\}$. It is well-known that the max-cut problem can be formulated as the following zero-one linear program:

$$\begin{aligned} \max \quad & \textstyle\sum_{1 \le i < j \le n} w_{ij} x_{ij} & & \\ \text{s.t.} \quad & x_{ij} + x_{ik} + x_{jk} \le 2 & (1 \le i < j < k \le n), & \quad (1) \\ & x_{ij} - x_{ik} - x_{jk} \le 0 & (1 \le i < j \le n; k \ne i, j) & \quad (2) \\ & x_{ij} \in \{0, 1\} & (1 \le i < j \le n). & \quad (3) \end{aligned}$$

Here, x_{ij} is a binary variable taking the value 1 if and only if vertices i and j are on opposite shores of the cut.

The convex hull in $\mathbb{R}^{\binom{n}{2}}$ of solutions to (1)-(3) is called the *cut polytope*. A wide variety of valid and facet-defining inequalities are known for this polytope (see Deza & Laurent [3]). For example, the triangle inequalities (1) and (2) themselves define facets, as shown by Barahona and Mahjoub [1].

Laurent and Poljak [6] showed that, for any vector $b \in \mathbb{R}^n$, the following inequality is valid for the cut polytope:

$$\sum_{1 \le i < j \le n} b_i b_j x_{ij} \le \left(\sum_{i=1}^{n} b_i \right)^2 / 4. \quad (4)$$

Laurent and Poljak also pointed out that the upper bound obtained by optimising over the convex set defined by these inequalities is equal to the one obtained by solving the standard SDP relaxation of max-cut.

The separation problem for the inequalities (4) can be solved in polynomial time (to arbitrary precision) as follows:

1. Let $x^* \in [0, 1]^{\binom{n}{2}}$ be the point to be separated.
2. Construct a symmetric $n \times n$ matrix Y^*, in which $Y^*_{ii} = 1$ for $i = 1, \ldots, n$ and $Y^*_{ij} = 1 - 2x^*_{ij}$ for $1 \le i < j \le n$.
3. Compute the minimum eigenvalue of Y^* (to the desired precision).
4. If the eigenvalue is negative, the associated eigenvector provides a suitable vector b to obtain a violated inequality. Otherwise, no inequality is violated.

In theory, one could use the inequalities (4) in a standard LP-based cutting-plane algorithm. Such an algorithm would yield an upper bound that was at least as strong as the SDP bound. Unfortunately, the inequalities (4) have several undesirable features:

- They typically have 'nasty' (even irrational) coefficients, which can cause numerical problems in the simplex method.
- They are 'dense', typically involving every variable, which takes up a lot of memory and slows down the simplex method.
- Since the feasible region is convex but not polyhedral, a large number of iterations is often needed to obtain a solution of reasonable precision.
- As the optimum is approached, the inequalities tend to become near-parallel, which again causes numerical difficulties.

This led us to search for a more 'well-behaved' class of inequalities.

3 Our Solution

Our solution to the above dilemma is as follows: instead of adding the inequality (4) directly to the LP, we use it to construct a 'knapsack relaxation' of the max-cut problem, and then use existing separation procedures for the knapsack problem. The justification for this approach is that the knapsack-based cutting planes tend to have 'nice' integer coefficients and, in our experience, behave better numerically than the inequalities (4).

Here are the details. Once we have generated an inequality of the form (4), regardless of whether or not it is violated, we know that the cut polytope is contained in the following polytope:

$$\text{conv}\left\{x \in \{0,1\}^{\binom{n}{2}} : \text{ (4) holds}\right\}.$$

To convert this into a standard 0-1 knapsack polytope, we complement variables to make the coefficients non-negative, scale the resulting constraint so that the right-hand side is some convenient integer, and then round the coefficients down to integers. Finally, to generate cutting planes, we use some heuristic separation procedures that we have developed for the 0-1 knapsack polytope [8].

We are currently conducting computational experiments with random max-cut instances and the instances mentioned in [10]. The results that we have obtained so far are rather promising. For most of the instances that we have tried, the knapsack-based cutting planes close over 60% of the gap between the optimum and the upper bound obtained with triangle inequalities.

We remark that a similar procedure can be applied to zero-one quadratic programmes and zero-one quadratically-constrained programmes, for which similar SDP relaxations exist.

References

[1] F. Barahona and A.R. Mahjoub, *On the cut polytope*, Math. Program. **36** (1986) pp. 157–173.

[2] C. De Simone, M. Diehl, M. Jünger, P. Mutzel, G. Reinelt & G. Rinaldi, *Exact ground states of Ising spin glasses: new experimental results with a branch-and-cut algorithm*, J. Statistical Phys. **80** (1995) pp. 487-496.

[3] M.M. Deza and M. Laurent, *Geometry of Cuts and Metrics*, Springer-Verlag, Berlin, 1997.

[4] I. Fischer, G. Gruber, F. Rendl and R. Sotirov, *Computational experience with a bundle approach for semidefinite cutting plane relaxations of max-cut and equipartition*, Math. Program. **105** (2006) pp. 451–469.

[5] M. Laurent, *Max-cut problem*, in *Annotated Bibliographies in Combinatorial Optimization*, M. Dell'Amico, F. Maffioli and S. Martello, eds., Wiley, Chichester, 1997, pp. 241–259.

[6] M. Laurent and S. Poljak, *On a positive semidefinite relaxation of the cut polytope*, Lin. Alg. Appl. **223/224** (1995) pp. 439–461.

[7] M. Jünger, F. Liers, G. Reinelt and G. Rinaldi, *Computing exact ground states of hard Ising spin glass problems by branch-and-cut*, in *New Optimization Algorithms in Physics*, A.K. Hartmann and H. Rieger (eds.), Wiley-VCH, 2004, pp. 47–68.

[8] K. Kaparis and A.N. Letchford, *Separation algorithms for 0-1 knapsack polytopes*, Math. Program., to appear.

[9] S. Poljak and Zs. Tuza, *Maximum cuts and large bipartite subgraphs*, in *Combinatorial Optimization* (DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 20), W. Cook, L. Lovász and P. Seymour, eds., American Mathematical Society, 1995, pp. 181–244.

[10] F. Rendl, G. Rinaldi and A. Wiegele, *Solving max-cut to optimality by intersecting semidefinite and polyhedral relaxations*, Math. Program., to appear.