On the x-and-y-axes travelling salesman problem

Eranda Cela[*] Vladimir Deineko[⋆] Gerhard Woeginger[◇]

[*] *Institute of Mathematics, Graz University of Technology*
Steyergasse 30, A-8010 Graz, Austria

[⋆] *Warwick Business School, the University of Warwick*
Coventry, CV4 9JA, United Kingdom

[◇] *Department of Mathematics and Computer Science, TU Eindhoven*
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Abstract

This paper deals with a special case of the Euclidean Travelling Salesman Problem known as the *x-and-y-axes* TSP. In this special case, all cities are situated on the x-axis and on the y-axis of an orthogonal coordinate system for the Euclidean plane. We show that the problem can be solved in $O(n^2)$ time.

Keywords: *Euclidean travelling salesman problem, Combinatorial optimization, Geometry, Polynomial algorithms.*

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1 Introduction

The *travelling salesman problem* (TSP) is defined as follows. Given an $n \times n$ distance matrix $C = (c_{ij})$, find a permutation $\pi \in S_n$ that minimizes the sum $\sum_{i=1}^{n-1} c_{\pi(i)\pi(i+1)} + c_{\pi(n)\pi(1)}$. Set S_n here is the set of all $n!$ permutations. In other words: the salesman must visit the cities 1 to n in arbitrary order and wants to do this while minimizing his total travel length. This problem is one of the fundamental problems in combinatorial optimization and well known to be NP hard. For more detailed information, the reader is referred to the book by Lawler, Lenstra, Rinnooy Kan and Shmoys [11].

Several special cases of the TSP are solvable in polynomial time, due to special combinatorial structures in the distance matrix (see surveys [3, 10]). In this paper we will deal with a special case of the so-called *Euclidean* TSP. In the Euclidean TSP cities are points in the two-dimensional plane and their distances are measured according to the Euclidean metric. It is easy to see that in this case, the shortest TSP tour does not intersect itself (cf. Flood [8]) and hence, geometry makes the problem somewhat easier. Nevertheless, this special case is still NP-hard (see e.g. Papadimitriou [12] or chapter 3 in the TSP book [11]). Although S. Arora in his well-known paper [2] suggested a polynomial-time approximation scheme (approximation with ratio arbitrarily close to 1) for fixed-dimension Euclidean TSP, his algorithms have high complexity and therefore are primarily of theoretical interest. Identifying polynomially solvable cases of the Euclidean TSP is still a popular and challenging topic in combinatorial optimisation. A wide range of previously known families of Euclidean TSP instances [1, 4, 5, 6, 7, 9, 14] have been generalised in an excellent paper of Rubinstein, Thomas, and Wormald [15], were the so-called Constrained TSP (CTSP) was introduced and fully investigated. In the CTSP a finite set G of smooth compact curves in plane is given, and all n points in the corresponding TSP must lie on this set G. Each curve in G has finite length. Moreover we assume that G contains a finite number of intersections or even self-intersections and at any intersection the different branches of the curves approach from different directions. (We refer the reader to [15] for formal definitions.) It is shown in [15] that in this case the problem can be solved in polynomial time, where the polynomial depends on G and on the number of points n. The CTSP as described above is a quite general problem and hence it is not surprising that the degree of the corresponding polynomial is huge, and the result, similarly to Arora's result [2], is mainly of theoretical interest. It is therefore an interesting and non-trivial problem to further investigate special cases related to the CTSP and design low degree polynomial time algorithms.

In this paper we deal with a special case related to the CTSP known as *x-and-y-axes* traveling salesman problem (XY-TSP). In this special case, all cities are situated on the x-axis and on the y-axis of an orthogonal coordinate system for the Euclidean plane. If the coordinates of all cities in the XY-TSP are bounded by a constant then the XY-TSP is a special case of the CTSP. Clearly, the XY-TSP is a quite simple case of the CTSP; there are only two curves (which are actually line segments) and there is only one intersection at 90^0. Nevertheless the problem remained unsolved at least since 1980, when it was formulated in [4] as an open problem. We exploit the main idea of [15] and show that the XY-TSP can be solved in $O(n^2)$ time.

2 Structure of optimal tours in the XY-TSP

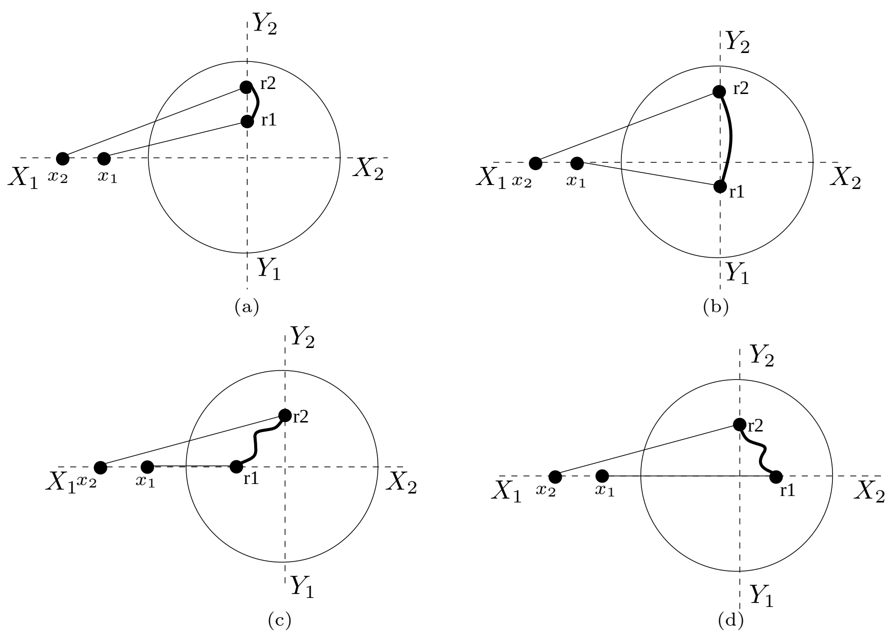

Figure 1: *Possible configurations to be considered in Theorem 2.1.*

In our investigation we exploit the main idea from [15] which can be formulated as follows in the case of the XY-TSP. Let $\mathcal{R}$ be a circle around the origin $(0,0)$ of an orthogonal coordinate system in the Euclidean plane. It is intuitively clear that the number of distinct edges of an optimal tour that leave the circle is bounded by a small constant. For the general case of the CTSP considered in [15] it was not possible to implicitly calculate this constant. Fortunately it can easily be done for the XY-TSP. The first result will follow from the next theorem.

Through the rest of the paper the notation $[a, b]$ will denote an edge connecting vertices a and b in a TSP tour, and in the case where a and b lie on the same axis, it will also denote a path connecting vertices a and b where all the intermediate vertices of the path lie on the same axis as a and b. In the later case we will refer to such a path $[a, b]$ as an edge in the tour.

Theorem 2.1 *Consider an instance of the XY-TSP and let $\mathcal{R}$ be a circle around the origin $(0, 0)$ of an orthogonal coordinate system in the Euclidean plane. Let $[x_1, r_1]$ and $[x_2, r_2]$ be two edges in an optimal tour for the given XY-TSP instance such that the endpoints r_1 and r_2 lie within $\mathcal{R}$ and the endpoints x_1 and x_2 lie outside of $\mathcal{R}$ and, moreover, assume that they lie on one of the half-axes, say on a half-axes of X. The optimal tour must then contain the edge $[x_1, x_2]$.*

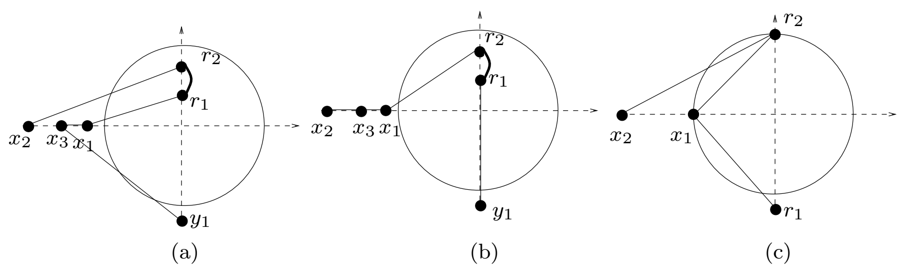

Figure 2: *Illustration to Theorem 2.1, Case 1(i): (a) forbidden configuration; (b) forbidden configuration transformed; (c) extreme case for the forbidden configuration.*

An immediate corollary from Theorem 2.1 is that for every circly $\mathcal{R}$ around the origin of a coordinate system in the euclidean plane there are no more than eight distinct edges of an optimal tour that connect a vertex within circle to a vertex outside circle. A straightforward idea would be to consider n circles (a circle for each vertex) and find the shortest length configuration amon all possible configurations for each each circle. The dynamic programming approach would find an optimal TSP tour in this case in $O(n^8)$ time. We will show that the complexity can be reduced to $O(n^2)$. Although there is a couple of obvious speed up tricks that can improve the multiplicative constant in the complexity of the algorithm, we will concentrate in this paper only on the order of complexity.

Theorem 2.2 *Let $\mathcal{R}$ be a circle around the origin of an orthogonal coordinate system in the Euclidean plane. There are no more than six distinct edges of an optimal XY-TSP tour that leave the circle.*

It follows from this theorem that the naive dynamic programming algorithm mentioned above would find an optimal tour in $O(n^6)$ time. To further improve the complexity we analyse possible positions of the points where the edges from a circle could go. We need the following definition.

Given a circle $\mathcal{R}$, we define the neighborhood of the circle, $\mathcal{L}(\mathcal{R})$, to consist of vertices lying outside the circle, such that from each half-axis $\mathcal{L}(\mathcal{R})$ contains the closest vertex to the origin among all vertices of that half-axis lying outside the circle, if any. So $\mathcal{L}(\mathcal{R})$ contains at most 4 vertices, at most one vertex from each half-axis.

Lemma 2.3 *Given an optimal XY-TSP tour and a circle $\mathcal{R}$. Let $[x_1, y_1]$ be an edge of the tour leaving the circle $\mathcal{R}$, i.e. vertex x_1 lies within (or on) the circle and vertex y_1 lies outside of the circle, say on half-axis T. If the edge $[x_1, y_1]$ is the unique edge leading from a vertex inside (or on) the circle to a vertex on half-axis T, then point y_1 belongs to the neighbourhood $\mathcal{L}(\mathcal{R})$: $y_1 \in \mathcal{L}(\mathcal{R})$. If there are two edges of the optimal tour, $[x_1, y_1]$ and $[x_2, y_2]$, that leave the circle, such that the vertices y_1 and y_2 lie outside the circle on the same half-axis, then one of these endpoints belongs to the neighborhood $\mathcal{L}(\mathcal{R})$, and the interval $[l_1, l_2]$ belongs to the optimal tour.*

Based on this lemma and on Theorem 2.2 we can classify possible configurations of an optimal tour with respect to some circle $\mathcal{R}$. This leads to an $O(n^2)$ algorithm as stated in the following theorem.

Theorem 2.4 *An optimal XY-TSP tour can be founsd in $O(n^2)$ time.*

3 Conclusion

This paper deals with a special case of the Euclidean Travelling Salesman Problem known as *x-and-y-axes* TSP. In this special case, all cities are situated on the x-axis and on the y-axis of an orthogonal coordinate system for the Euclidean plane. We have shown that the problem can be solved in $O(n^2)$ time. Although there is a couple of obvious speed up tricks that can improve the multiplicative constant in the time complexity of the algorithm, we have concentrated only on the order of the complexity. It is not clear whether it is possible to reduce the achieved $O(n^2)$ time complexity. It would also be interested to see low complexity algorithms for the other special cases related to the CTSP.

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