On the Pickup and Delivery Traveling Salesman Problem with LIFO Loading

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Abstract

We provide a new 0-1 linear programming formulation for the Rear-loading TSP, a version of the TSP with pickup and deliveries where locations must be visited according to a last-in-first-out policy. The formulation, which combines typical TSP assignment variables and tournaments variables, has the following (somehow attractive) features: first, it is compact, because subtour elimination constraints are no longer needed to ensure the formulation correctness — although they still represent useful valid inequalities; secondly, the integrality of each of the two sets of variables is implied by the integrality of the other. The computational features of this formulation are currently under investigation.

Keywords: *Traveling Salesman Problem with restrictions.*

1 Introduction

Let V be a set of points or locations, and $c : V^2 \to I\!R_+$ a distance function. For any $A \subseteq V^2$, the cost of A is $c(A) = \sum_{uv \in A} c_{uv}$. The problem here addressed calls for a hamiltonian circuit on V, with pickup and deliveries specified by a set $C \subseteq V^2$ of ordered pairs of locations. Pickup and deliveries are special-constrained: locations must in fact be visited in a Last In First Out (LIFO) order, that is, for any ij, $hk \in C$, either $i \prec j \prec h \prec k$, or $h \prec k \prec i \prec j$, or $i \prec h \prec k \prec j$, or finally $h \prec i \prec j \prec k$. In other words, a solution to the problem is given by a set H of location pairs such that H is a hamiltonian circuit and $H \cup C$ admits a planar drawing with all the chords in C lying in the inner part of the circuit. Any such circuit will be said LIFO-feasible. Summarizing, we are interested in the following *Pickup and Delivery Traveling Salesman Problem with LIFO Loading* (TSP-PDLL)

Problem 1.1 *Find on V a LIFO-feasible hamiltonian circuit H of minimum cost $c(H)$.*

This problem has been considered by [3], by [4] (where is called Traveling Salesman Problem with Precedence and LIFO Constraints) and by [7]: the first reference provides an additive branch-and-bound algorithm, the second a heuristic algorithm, the third three different mathematical formulations and several families of valid inequalities which are used within a branch-and-cut algorithm. An exact algorithm is also proposed in [6], but to the best of our knowledge their results are still unpublished. In this paper we propose an alternative formulation which makes use of a further set of tournaments variables, and investigate some of its properties.

2 Formulation

Let $V = \{0, 1, \ldots, 2n\}$, where 0 denotes a special location called the depot, and $C = P \times D \subseteq V^2$, where $P = \{1, \ldots, n\}$ and $D = \{n + 1 \ldots, 2n\}$ respectively contain all the *pickup* and *delivery* locations. For distinct $i, j \in V$, define the binary variables

$$x_{ij} = \begin{cases} 1 & \text{if point } i \text{ is visited immediately before point } j \\ 0 & \text{otherwise,} \end{cases}$$

These are *assignment variables* defining a hamiltonian circuit. Note that $x_{i,n+j}$ is fixed to 0 for distinct $i, j \in P$. Similarly, x_{0k} is fixed to 0 for any $k \in D$. Our problem seeks for minimizing the total cost of a tour:

$$\min \sum_{ij \in V} c_{ij} x_{ij} \qquad (1)$$

with the x_{ij} constrained as follows.

Assignment inequalities
Each point must be visited exactly once:

$$\sum_{j \neq i} x_{ij} = \sum_{j \neq i} x_{ji} = 1 \qquad i \in V \qquad (2)$$

Subtour elimination inequalities
The tour must be connected:

$$\sum_{i,j \in S} x_{ij} \leq |S| - 1 \qquad S \subseteq V, |S| \geq 2 \qquad (3)$$

Other valid TSP inequalities
Obviously, all valid inequalities for the travelling salesman polytope are also valid for the Rear-loading TSP. Among these, let us recall:

- *Comb* and *simple comb* inequalities, [10] and [12];

- *2-Matching* inequalities, [11];

- D_h^+ and D_h^- inequalities, [11] and [5];

- *Predecessor* and *successor* inequalities, and *precedence cycle breaking* inequalities, [1];

- *Generalized order constraints*, [15].

Now define

$$y_{ij} = \begin{cases} 1 & \text{if point } i \text{ is visited (immediately or not) before point } j \\ 0 & \text{otherwise.} \end{cases}$$

for any distinct $i, j \in V$, $j \neq 0$. The y_{ij} define a total order among the elements of V, and are called *tournament variables*. Since a pickup would never follow the corresponding delivery, $y_{n+i,i}$ is clearly fixed to 0 for any $i \in P$. Also, the depot precedes all the locations, thus y_{0i} is fixed to 1 for any $i \in V - \{0\}$. Notice that the following (exponentially many) precedence inequalities, easily derived from [7], are valid for Problem 1.1 but not necessary for the correctness of the formulation addressed in this paper:

$$\sum_{i,j \in S} x_{ij} - \sum_{j \in S \cap D} x_{j0} \leq |S| - 2 \qquad (4)$$

for all $S \subset V$ such that $0 \in S$ and $\exists i \in P : i \notin S, n+i \in S$.

Generally speaking, the definition of tournament asks for the following constraints.

Total order inequalities
$$y_{ij} + y_{ji} = 1 \qquad \text{(symmetry)} \tag{5}$$
for any $i, j \in V, j \neq 0$, and
$$y_{ij} + y_{jk} + y_{ki} \leq 2 \qquad \text{(transitivity)} \tag{6}$$
for any three distinct $i, j, k \in V$, $i, j, k \neq 0$. (Putting together (5) and (6) one gets $y_{ij} + y_{jk} - y_{ik} \leq 1$.)

Assignment and tournament variables are jointly constrained as follows.

Crossover inequalities
First, every x_{ij} with $j \neq 0$ implies the corresponding y_{ij}:
$$x_{ij} - y_{ij} \leq 0 \qquad \text{for any distinct } i, j \in V, j \neq 0 \tag{7}$$

Moreover, if point i directly precedes point k, then no j separates it from k:
$$\begin{aligned} x_{ik} + y_{ij} + y_{jk} &\leq& 2 \qquad &\text{for any distinct } i, j, k \in V, j, k \neq 0 \\ x_{i0} + y_{ij} &\leq& 1 \qquad &\text{for any distinct } i, j \in V, j \neq 0 \end{aligned} \tag{8}$$

Proposition 2.1 *Through inequalities (6) and (7) one gets rid of subtour elimination inequalities (3).*

Proof.. In fact should $\mathbf{x}$ define a subtour H' not touching 0, (7) would imply $y_{ij} = 1$ for all $ij \in H'$, but then (6) would be violated for any triple u, v, w of points touched by H'. Notice that this argument does not require $\mathbf{y}$ integer. $\qquad \square$

Still, (3) are valid inequalities to be possibly used in order to improve the lower bound obtained by linear relaxation.

Another set of crossover inequalities is obtained by considering that every y_{st} asserts or denies the existence of an (s, t)-path π_{st} belonging to the hamiltonian circuit. Thus, every such path is defective of at least one edge whenever $y_{st} = 0$, that is
$$\sum_{ij \in \pi_{st}} x_{ij} \leq |\pi_{st}| - y_{ts} \qquad \text{for any distinct } s, t \in V \tag{9}$$

Observe, however, that the above inequalities are still valid replacing x_{ij} by y_{ij}. The resulting inequalities are in turn dominated by (6).

Tournament variables make it easy to express last-in first-out constraints.

LIFO inequalities
Consider two distinct pickup-delivery pairs $i, n+i$ and $j, n+j$, for any $i, j \in P$ and $n+i, n+j \in D$, and any hamiltonian circuit H. Suppose H LIFO-feasible. Then $n+j$ preceding i implies $n+i$ not preceding j:
$$y_{n+j,i} + y_{n+i,j} \leq 1 \qquad\qquad i, j \in P, i \neq j \tag{10}$$
Moreover, a hamiltonian circuit H containing a path which joins in the order i, j, $n+i$ and $n+j$ is LIFO-infeasible: hence
$$y_{ij} + y_{j,n+i} + y_{n+i,n+j} \leq 2 \qquad\qquad i, j \in P, i \neq j \tag{11}$$

Other valid LIFO inequalities
In [7], the Authors introduce the following three new families of valid inequalities for the special case of the TSP with LIFO constraints:

3

- *Incompatible predecessor and successor* inequalities,

- *Hamburger* inequalities, and

- *Incompatible paths* inequalities. Notice that summing up inequalities (9) for two incompatible paths gives rise to an inequality non-dominated by the corresponding incompatible paths inequality.

Other valid acyclic tournament inequalities

Inequalities (5) plus the nonnegativity constraints give a complete and non-redundant description of the *tournament polytope*, as proved in [9]. On the other hand, polyhedral results for the *linear ordering polytope* on a complete digraph, i.e. for the convex hull of the incidence vectors of all acyclic tournaments in the digraph, are also valid for the TSPPDL polytope. In particular, Grötschel et al. [9] showed that inequalities (6) induce facets of the linear ordering polytope, and they proved again that the following two classes of inequalities

- *Simple k-fence* inequalities, and

- *Simple Möbius ladder* inequalities

define facets of this polytope. Observe that, since the linear ordering polytope is not full dimensional (due to the fact that in a tournament every pair of nodes has to be connected by exaclty one arc) all the facets are express in the *normal form*, where an inequality $\mathbf{ax} \leq b$ is in normal form if all coefficients a_{ij} are nonnegative integers and $a_{ij}a_{ji} = 0$.

Some variants of the simple k-fence inequalities have been introduced by [13] and [2]. Furthermore, *Paley digraph* inequalities, and k-*wheels* and Z_k-*digraphs* inequalities are described in [8] and [14], respectively.

Inequalities (2), (5)-(8), (10), (11) plus nonnegativity and integrality clauses are sufficient to describe the feasible region of Problem 1.1. The above statement can be strenghtened as follows:

Proposition 2.2 *Problem* $\min_{\mathbf{x,y}}\{\mathbf{cx}\}$ *subject to (2), (5)-(8), (10), (11) and*

$$\mathbf{x} \geq \mathbf{0} \qquad \mathbf{y} \geq \mathbf{0} \qquad \mathbf{y} \text{ integer}$$

admits an optimal integer solution, i.e., a minimum cost rear-loading hamiltonian circuit can be found even by dropping integrality clauses on the x variables.

Proof. For any integer valued y-vector, the resulting constraint set has a total unimodular matrix and an integer right-hand side. □

Proposition 2.3 *Problem* $\min_{\mathbf{x,y}}\{\mathbf{cx}\}$ *subject to (2), (5)-(8), (10), (11) and*

$$\mathbf{x} \geq \mathbf{0} \qquad \mathbf{y} \geq \mathbf{0} \qquad \mathbf{x} \text{ integer}$$

admits an optimal integer solution, i.e., a minimum cost rear-loading hamiltonian circuit can be found even by dropping integrality clauses on the y variables.

Proof. Let $\mathbf{x}$ be an integer x-solution to (2), (5)-(8), (10), (11): as such, it defines an assignment and therefore for any $i \in V$, $x_{ij} = 1$ for some $j \neq i$ and $x_{ik} = 0$ for all $k \neq j$. In particular, by Proposition 2.1 — that as observed holds also for y-variables not constrained to integrality — we know that $\mathbf{x}$ defines a hamiltonian circuit. Let $n + i$ be such that $x_{n+i,0} = 1$, that is, $n + i$ is the last (delivery) node of the circuit that contains the depot. Thus, $x_{n+i,j} = 0$ for all $j \neq 0$. Using (7) and the second of (8) we then get $y_{n+i,j} = 0$ for any $j \neq 0$, and by symmetry $y_{j,n+i} = 1$.

Let now $s(k) \neq 0$ be the successor of node $k \neq n + i$, that is, $x_{k,s(k)} = 1$ and, in particular, $x_{k0} = 0$. Again, using (7) and (8) we obtain $y_{k,s(k)} = 1$ and $y_{s(k),k} = 0$.

By definition of $s(k)$, clearly $x_{kj} = 0$ for all $j \neq s(k)$. Also, by Proposition 2.1, for any node pair $k = i_1$, $j = i_p$ there is a (k, j)-path $(i_1, i_2, \ldots, i_p)$ such that $x_{i_t i_{t+1}} = 1$ for $t = 1, \ldots, p - 1$, and then clearly $y_{i_1 i_2} = y_{i_{p-1} i_p} = 1$. Suppose inductively $y_{i_1 i_{p-1}} = 1$: then (6) and (5) imply $y_{i_1 i_p} = 1$, that is $y_{kj} = 1$ and $y_{jk} = 0$. □

4

Name	Nodes	Optimal integer value	Lower bound	GAP	time (sec.)
a280.11	11	207,64378	207,64378	0,00%	0,03
a280.15	15	2529,84598	2529,84598	0,00%	0,09
brd14051.13	13	4136,79644	3767,69073	8,92%	0,05
brd14051.15	15	4046,91183	3800,54352	6,09%	0,08
brd14051.17	17	4353,90904	4083,18743	6,22%	0,14
nrw1379.11	11	2418,09993	2191,15147	9,39%	0,03
nrw1379.19	19	2555,91128	2324,14267	9,07%	0,25
pr1002.15	15	11721,31988	11652,22133	0,59%	0,09
pr1002.21	21	13774,60517	13535,44178	1,74%	0,37

Table 1: Computational results

3 Preliminary computational experiments

This section refers on a few, very much preliminary computational experiments carried out with formulation (2), (5)-(8), (10), (11). CPLEX 9.0 was used for solving both the integer problem and the linear relaxation on a laptop with standard configuration. Problem instances were generated by considerably reducing the number of nodes of some instances taken from the TSP Library [16] out of four distinct classes: for example, a280.11 indicates the instance obtained by truncating a280.tsp after the first 11 nodes. Results are illustrated in Table 1, the rows of which provide the following data.

1. Problem identifier (column 1);

2. Number of nodes of the problem (column 2);

3. Optimal value of the integer LP (column 3);

4. Optimal value of the linear relaxation (column 4);

5. Integrality gap (column 5);

6. Computation time in seconds (column 6).

These small problems were solved in fraction of seconds with an integrality gap ranging from 0 to less than 10%. The test indicates that that some of the valid inequalities discussed above can help reducing the gap. For example, the solution of the linear relaxation of brd14051.13, brd14051.15 and nrw1379.19 violates precedence inequalities (4). Ongoing experiments will hopefully help understanding which inequalities are best suitable to improve the lower bounds obtained.

4 Conclusions

We presented an alternative 0-1 linear programming formulation of the Pickup and Delivery Traveling Salesman Problem with LIFO Loading (TSP-PDLL). Unlike previous ones, this formulation requires polynomially many variables and constraints. Some properties of the formulation have been discussed. Further research is needed in order to evaluate the potential of this formulation in terms of computational features.

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