Networks with unicyclic connected components

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Abstract

This paper focuses on the design of networks with unicyclic connected components. The size of each cycle should not be less than a given number. A polyhedral study is proposed. Many facets and valid inequalities are derived. Some of them can be exactly separated in polynomial time. Then the network design problem is solved by a cutting plane algorithm based on these inequalities and using a compact formulation issued from the transversality of the bicircular matroid.

Keywords: *Combinatorial Optimization, Network Design, polyhedra*

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1 Introduction

Given a weighted undirected graph $G = (V, E)$, we aim to compute a minimum weight subgraph $G' = (V, E')$ ($E' \subset E$) with unicyclic connected components. Moreover, the number of edges of each cycle of G' should not be less than $p + 1$. Said another way, the girth of G' will be at least $p + 1$.

This problem has some applications in telecommunications. We want to build a minimum cost network with a certain level of survivability provided by cycles. The traffic demands between nodes on the same cycle are fully protected against failures while the other demands can be disrupted. Since short cycles do not offer a high level of survivability, a lower bound of the girth is imposed. Some other technical constraints can also be considered (size of connected components, number of connected components, degree constraints, etc.). However, in this paper we only focus on the girth constraint in addition to the unicyclicity of the connected components. Work related to network survivability can be found in, for example, [2, 5, 8, 9, 10]. Problems related to cycle location in telecommunications (and transportation) networks are reviewed in [12].

This new problem has many connections to old problems. For example, if $p = |V| - 1$, then we get the traveling salesman problem. This clearly implies that our problem is also NP-hard [7].

If no girth constraint is imposed ($p = 2$), the problem becomes easy. In fact, it is already known that the set of edge subsets inducing a graph where each connected component contains at most one cycle is a matroid. This matroid is generally called the bicircular matroid (see, e.g., [13]). Then a simple greedy algorithm can be applied to compute a minimum cost network where each connected component contains exactly one cycle.

It is also easy to compute a minimum cost unicyclic network (the underlying graph should be connected and contains exactly one cycle). This comes from the fact that the set of edge subsets inducing a graph

containing at most one cycle is also a matroid. This matroid is obviously included in the bicircular matroid.

If the unicyclicity requirement is replaced by a cyclicity constraint (each connected component is exactly a cycle), we get the two-factors problem which is a well known easy problem (see, e.g., [14]). If a girth constraint is added (no cycles of length less than 5), the problem becomes difficult ([4]).

There are also some connections with the so-called ring star problem. Here we look for one cycle going through some nodes while the other nodes are connected to the closest node on the cycle. The ring star problem and some of its variations are studied in [1, 11].

If the unicyclicity constraint is relaxed, we get another known problem: find a minimum (or maximum) weight graph not containing cycles of length less than $p + 1$. This problem is equivalent to the covering problem where we look for a minimum (or maximum) weight set of edges intersecting each cycle whose length is not more than p. When $p = 3$, we know that the problem is easy if the graph is non-contractible to $K_5 \setminus e$ (the complete graph on 5 nodes minus one edge) [6]. However, the problem is NP-hard in the general case [15].

The remainder of this paper is organized as follows. Notation and a mathematical formulation are provided in next section. Some valid inequalities are described in Section 3. The faces induced by these valid inequalities are also studied. The separation of these inequalities is addressed in Section 4.

2 Notation and Mathematical Formulation

Let $\mathcal{H}_p$ be the set of graphs such that each connected component contains exactly one elementary cycle, and this cycle has at least $p + 1$ edges. Said another way, all connected components are unicyclic graphs and there is no any cycle of length less than or equal to p.

Given a graph $G = (V, E)$, we consider all the covering subgraphs $G' = (V, E')$ ($E' \subset E$) belonging to $\mathcal{H}_p$. A subgraph is represented by an incidence vector $(x_e)_{e \in E}$ where $x_e = 1$ if e is contained in the subgraph and 0 otherwise. The convex hull of the incidence vectors x of these covering subgraphs belonging to $\mathcal{H}_p$ is denoted by $P(G, p)$. Then computing a minimum weight covering subgraph belonging to $\mathcal{H}_p$ is equivalent to computing a minimum cost vector of $P(G, p)$.

We will assume throughout the paper that $G = (V, E)$ is a simple undirected connected graph (if G is not connected, one can independently consider each connected component). The number of vertices is denoted by $n = |V|$. Notice that we should at least assume that $n \geq 3$ and $p \leq n - 1$. A cost w_e is associated with each edge $e \in E$.

An edge $e \in E$ is sometimes denoted by (i, j) where i and j are the extremities of the edge. Then given any vector $(\alpha_e)_{e \in E}$, α_e can also be denoted by α_{ij}. If A is a subset of vertices, then $\delta(A)$ denotes the set of edges having exactly one extremity in A. If $A = \{v\}$ where $v \in V$, then we write $\delta(v)$ instead of $\delta(\{v\})$.

Given two disjoint subsets of vertices A and B, $\delta(A) \setminus \delta(B)$ denotes the set of edges having one extremity in A and the other extremity in $V \setminus \{A \cup B\}$. Let $(z_e)_{e \in E}$ be any vector indexed by E. Then we use the following notation: $z(A, B) = \sum\limits_{(i,j) \in E, i \in A, j \in B} z_{ij}$. When $B = \{v\}$, we write $z(A, v)$ instead of $z(A, \{v\})$.

We also use $z(A)$ to denote $\sum\limits_{(i,j) \in E, i \in A, j \in A} z_{ij}$. If C is a set of edges $C \subset E$, then $z(C)$ naturally denotes $\sum\limits_{e \in C} z_e$.

Let us now focus on the mathematical formulation of the network design problem.

First, it is already known that the bicircular matroid is a transversal matroid [13]. Said another way, the independents of the bicircular matroid can be represented by matching. More precisely, let H be the bipartite graph where the set of nodes is given by $V \cup E$ and there is an edge between $v \in V$ and $e \in E$ if and only if v is an extremity of e. If we consider any matching of H, the set of edges of E included in this matching is obviously an independent of the bicircular matroid. The bases of the matroid are given by matchings of size n (in fact our problem does not have a solution if the rank of the bicircular matroid associated with G is less than n).

These simple (known) observations lead to a simple formulation of the unicyclicity constraint. For each edge $e = (i, j) \in E$ we have a variable x_e and two variables y_{ie} and y_{je} where y_{ie} is equal to 1 if vertex i is matched with the edge (i, j). x_e is of course given by $y_{ie} + y_{je}$. Using the classical polyhedral characterization of bipartite matching (see, e.g., [14]), we deduce that our problem can be formulated as follows (when we relax the girth constraint):

$$
\begin{aligned}
\min \sum_{e \in E} w_e x_e & \\
x_e = y_{ie} + y_{je}, \forall e = (i, j) \in E & \\
\sum_{e=(i,j) \in E} y_{ie} = 1, \forall i \in V & \\
0 \le y_{ie}, \forall i \in V, e = (i, j) \in E & \\
x_e \le 1, \forall e \in E. &
\end{aligned}
\tag{1}
$$

Girth constraints can also be integrated in a simple way. We only have to write that the number of edges inside any set of vertices A of cardinality at most p cannot be more than $|A| - 1$.

$$
x(A) \le |A| - 1, A \subset V, 2 \le |A| \le p
\tag{2}
$$

Then our problem is well formulated by (1), (2) and the integrality constraints (3):

$$
x_e \in \{0, 1\}, \forall e \in E.
\tag{3}
$$

While the extremal solutions of (1) are 0-1 vectors, those of (1)+(2) can be fractional. Moreover, the separation of inequalities (2) is not easy. We will then try to reinforce the system (1)+(2) by adding some of the inequalities that are valid to $P(G, p)$.

3   A polyhedral study

We will present some valid inequalities. Due to size limitation, no proofs are given in this section. They can be found in [3].

Proposition 3.1 *Let A and B be two disjoint subsets of V such that $|A| \le p - 1$. Then the following inequality is valid for $P(G, p)$:*

$$
2x(A) + x(A, B) \le 2|A| + |B| - 2.
\tag{4}
$$

Proposition 3.2 *Let A and B be two disjoint subsets of V. Then the following inequality is valid for $P(G, p)$:*

$$
\sum_{v \in A} (2 - x(\delta(v))) + \sum_{v \in B} (x(\delta(v) \setminus \delta(A)) - 2) \le n - (p + 1).
\tag{5}
$$

We will see later that inequalities (5) can be separated in polynomial time.
It is interesting to notice that when $A = \emptyset$ and $B = \{v\}$, inequalities (5) reduce to

$$
x(\delta(v)) \le n - (p - 1).
$$

We study the dimension of $P(G, p)$ and the valid inequalities defined above. To simplify this study, we will assume in the rest of this section that $G = (V, E)$ is a complete graph. Let $m = n(n-1)/2$.

Proposition 3.3 *The dimension of $P(G, p)$ is $m - 1$ if $2 \le p \le n - 2$, and $m - n$ if $p = n - 1$.*

Proposition 3.4 *Inequality $x_e \ge 0$ defines a facet of $P(G, p)$ if and only if $n \ge 5$.*

Proposition 3.5 *Inequality (2) defines a facet of $P(G,p)$ if and only if $n \geq 4$ and $2 \leq |A| \leq min(n - 2, p)$.*

Notice that inequalities $x_e \leq 1$ are a special case of inequalities (2).

Let us now study inequalities (4). It is clear that if we take $|A| = 1$, then inequality (4) is a combination of the trivial inequalities $x_e \leq 1$. The case $|B| = 0$ is also included in inequalities (2). Then, we will assume that $|A| \geq 2$ and $|B| \geq 1$. This directly implies that $p \geq 3$ (since $|A| \leq p - 1$) and $n \geq 4$.

Proposition 3.6 *Inequality (4) defines a facet of $P(G,p)$ if and only if $|A| = p - 1$, $3 \leq p \leq n - 2$ and $|B| \geq 3$.*

Proposition 3.7 *Let A and B be two disjoint subsets of V and let $C = V \setminus \{A \cup B\}$. Inequality (5) defines a facet of $P(G,p)$ if and only if $3 \leq p \leq n - 2$ and one of the following conditions holds:*
- *$|B| = 0$, $1 \leq |C| \leq p - 1$*
- *$1 \leq |B| \leq p - 4$, $|C| \geq 3$*
- *$|B| = p - 3$, $|C| \geq 4$, $p \geq 4$*
- *$|B| = p - 2$, $|A| \leq 1$, $|C| \geq 4$*
- *$|B| = p - 1$, $|A| = 0$.*

4 A Cutting Plane Algorithm

A cutting plane algorithm is implemented based on the inequalities of the previous section. We will now give some details about the separation of these inequalities.

In fact, we prove in [3] that the separation problems of inequalities (2) and (4) are NP-hard. Then heuristics are implemented to separate inequalities (2) and (4) (they are described in [3]).

We will focus on the separation of inequalities (5). We prove below that this can be done in polynomial time by solving a minimum cut problem in an appropriate graph.

Given a set A, it is clear that a set B maximizing $\sum_{v \in A} (2 - x(\delta(v))) + \sum_{v \in B} (x(\delta(v) \setminus \delta(A)) - 2)$ will only contain the vertices v such that $x(\delta(v) \setminus \delta(A)) - 2 \geq 0$. Said another way, maximizing $\sum_{v \in A} (2 - x(\delta(v))) + \sum_{v \in B} (x(\delta(v) \setminus \delta(A)) - 2)$ is equivalent to minimizing $\sum_{v \in A} (x(\delta(v)) - 2) + \sum_{v \in V \setminus A} \min (2 - x(\delta(v) \setminus \delta(A)), 0)$.

Let us then define a function $f : 2^V \to \mathbb{R}$ by

$$f(A) = \sum_{v \in A} (x(\delta(v)) - 2) + \sum_{v \in V \setminus A} \min (2 - x(\delta(v) \setminus \delta(A)), 0).$$

Inequalities (5) can surprisingly be separated by a maximum flow algorithm. Given a graph $G(V, E)$ and a possibly fractional vector x, we build a network (a capacitated directed graph) as follows. Each vertex u_i of V is replaced by two vertices v_i and w_i. Each edge $e = (u_i, u_j) \in E$ is represented by two directed arcs (w_i, v_j) and (w_j, v_i). The capacity of each arc is x_e. We also consider two new vertices s and t. There are arcs from s to v_i and w_i with a capacity of 2. There are also arcs from v_i and w_i to t where the capacity is $x(\delta(u_i))$. By computing a maximum flow from s to t we get a minimum cut separating s and t. The vertices u_i such that both v_i and w_i are in the same side than s will constitute the set A. Similarly, B contains all the vertices u_i such that both v_i and w_i are in the same side than t. The other vertices u_i (those where w_i and v_i are not in the same side) constitute $C = V \setminus \{A \cup B\}$.

First, it is clear that all the vertices $u_i \in C$ are such that v_i is in the same side as s and w_i is in the other side. Said another way, by switching v_i and w_i the capacity of the cut cannot decrease. We can also see that if $u_i \in C$, then the capacity of the edges related to either v_i or w_i and belonging to the cut is exactly $2 + x(\delta(u_i))$. Moreover, if $u_i \in B$, then the capacity of the edges related to either v_i or w_i and belonging to the cut is $x(u_i, A) + 4$.

The capacity of the cut is then given by $\sum_{u_i \in A} 2x(\delta(u_i)) + \sum_{u_i \in C} 2 + x(\delta(u_i)) + \sum_{u_i \in B} 4 + x(u_i, A)$. Since the capacity of the cut is minimum, if u_i belongs to C then $2 + x(\delta(u_i)) \leq x(u_i, A) + 4$ (otherwise the capacity

can decrease by putting u_i in B). Similarly, if $u_i \in B$ then $x(u_i, A) + 4 \leq 2 + x(\delta(u_i))$. Consequently, the capacity of the cut is equal to $\sum\limits_{u_i \in A} 2x(\delta(u_i)) + \sum\limits_{u_i \in V \setminus A} \min(2 + x(\delta(u_i)), 4 + x(u_i, A))$. After some algebra the capacity becomes $\sum\limits_{u_i \in A}(x(\delta(u_i)) - 2) + 2|A| + \sum\limits_{u_i \in A} x(\delta(u_i)) + \sum\limits_{u_i \in V \setminus A} \min(0, 2 - x(\delta(u_i) \setminus \delta(A))) + 2|V \setminus A| + \sum\limits_{u_i \in V \setminus A} x(\delta(u_i))$. In other words, the capacity of the cut is equal to $f(A)$ up to an additive constant $(4n)$.

5   Conclusion and further research

A cutting plane algorithm was proposed in this paper to solve the network design problem with unicyclic connected components and a girth constraint. Computational experiments will be presented during the conference.

This work can be extended in different ways. One may, for example, integrate other technical constraints (number of connected components, size of connected components, some nodes that must be in the same component, degree constraints, etc.). Another research direction is related to the complexity of the design problem for small values of p and special graph classes. More details are given in [3].

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