Computing Hilbert bases for network design problems

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Abstract

We present a linear time in the size of the output algorithm to compute the right hand side vectors which correspond to atomic fibers of a network design problem where the flows and the capacities are constrained to be integral.We present a linear time in the size of the output algorithm in order to compute the right handside vectors which correspond to atomic fibers of a network design problem where the flows and the capacities are constrained to be integral.

Keywords: *Polyhedral cone, Hilbert basis, Fiber.*

1 Introduction

Network design problems have been extensively studied from a polyhedral point of view. For instance we can cite the seminal papers by M. Grötschel, C. Momna and M. Stoer [7], [8], [9]. More recently A. Atamtürk, [1], D. Bienstock, O. Günlück, G. Muratore and S. Chopra investigated the capacited network design problem, see [2], [3] and [4]. In 2006 E. Eisenschmidt & al. introduced in [6] a new approach based on integer Minkowski programs . Let us recall some definition about Minkowski programs.

Definition 1 *Let $\mathcal{A} : \mathbb{Z}^n \rightarrow 2^{\mathbb{R}^{d,+}}$ be a superadditive set-valued mapping, i.e. for any z_1 and $z_2 \in \mathbb{Z}^n$ we have $\mathcal{A}(z_1) + \mathcal{A}(z_2) \subseteq \mathcal{A}(z_1 + z_2)$. An integer Minkowski program is an optimization program with the following form:*

$$\begin{array}{ll} Min & w^t z \\ s.t. & \mathcal{A}(z) \neq \emptyset \\ & Bz = v \\ & z \in \mathbb{Z}^n \end{array}$$

where w is a cost vector, $v \in \mathbb{Z}^m$ and B is an $m \times n$ matrix.

A finite set $\mathcal{Z} = \{z_j \in \mathbb{Z}^n, j \in J\}$ such that for any $z \in \mathbb{Z}^n$ the two following conditions hold:

$$z = \sum_{j \in J} \lambda_j z_j \text{ and } \mathcal{A}(z) = \mathcal{A}(\sum_{j \in J} \lambda_j z_j) = \sum_{j \in J} \lambda_j \mathcal{A}(z_j),\ \lambda_j \in \mathbb{N},\ \forall j \in J. \tag{1}$$

is called a finite generating set. The following theorem has been proved in [6]

Theorem 2 *If there exists a finite generating set for the family $\mathcal{A}(z),\ z \in \mathbb{Z}^n$ then an integer Minkowski program can reformulated as an integer linear program in the following way.*

$$\begin{array}{ll} Min & w^t(\sum_{j\in J}\lambda_j z_j) \\ s.t. & \sum_{j\in J}\lambda_j B z_j = v \\ & \lambda_j \in \mathbb{N},\ \forall j \in J \end{array}$$

At this stage this approach comes across the problem of computing the finite generating set $\mathcal{Z}$, which is very time consumming if it is done using standard Buchberger algorithm. In [10] we investigated an alternative approach based on extended truncated fibers and this approach gave rise to a linear time algorithm in the size of the set $\mathcal{Z}$ for the flow problem. We recall the main result in next section and we give explicitely the extension of our approach to the multicommodity flow problem. In section 3 we apply this material to the capacited network design problem where the flows and the capacities are subject to integrality constraints.

Now let us define the flow and multicommodity flow problem in the form of interest according to the approach that we propose in this article. Let $G = (V, E)$ be a digraph which will support the routing of the demands and N_G be its vertex-arc incidence matrix, Id_n denotes the $n \times n$ identity matrix. We denote b_k the vector with all its components equal to 0 except the one related to the source of demand $k \in \{1, \ldots, K\}$ which equals $+1$ and the one related to the sink of this demand which equals -1, and r_k will denote the intensity of the demand k. At last $C \in \mathbb{N}^{|E|}$ is the capacity vector. In order to complete the description of the problem, let us define the two following sets of variables, $f_k \in \mathbb{N}^{|E|}$ is the flow vector associated to the demand k and $s \in \mathbb{N}^{|E|}$ is the vector of slack variables. According to these definitions we can formulate both flow problem and multicommodity flow problem. The set of the feasible flows circulating from s to $t \in V$ with intensity r is provided with:

$$P^I_{rb_{st},C} = \{(f,s)^t \in \mathbb{N}^{2|E|},\ \mathcal{C}(f,s)^t = (rb_{st}, C)^t\},\ \text{where } \mathcal{C} = \begin{pmatrix} N_G & 0 \\ Id_{|E|} & Id_{|E|} \end{pmatrix}.$$

Thus the set of st-flows is the fiber of the matrix $\mathcal{C}$ in $(rb_{st}, C)^t$.

Now we give a similar formulation for the multicommodity flow. Let $\mathcal{D}$ be the constraint matrix as below:

$$\mathcal{D} = \begin{pmatrix} N_G & 0 & \ldots & 0 & 0 \\ 0 & N_G & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & N_G & 0 \\ Id_{|E|} & \ldots & Id_{|E|} & Id_{|E|} & Id_{|E|} \end{pmatrix}.$$

Let $f_1^K = (f_1, \ldots, f_K)$ and $b = (r_1 b_1, \ldots, r_K b_K)$, then the set of feasible multicommodity flows is provided with the fiber:

$$P^I_{b,C} = \{(f_1^K, s)^t \in \mathbb{N}^{(K+1)|E|},\ \mathcal{D}(f_1^K, s)^t = (b, C)^t\}.$$

Notice that for the network design problem the C part of the right hand side vector represents the capacity variables.

The remainder of the article is organized as follows. In section 2 we define the material useful to compute finite generating set for integral flow problems and give a linear time in the size of the output in order to compute such a set of generators. In section 3 we formulate network design problem as a Minkowski program using the finite generating set previously defined.

2 Computing finite generating set for multiflow problem

In this section we introduce extended and truncated fibers, in particular we are interested with non-decomposable truncated fibers as finite generating set.

We call an extended fiber of a $m \times n$ matrix A in $b \in \mathbb{Z}^m$ the discrete set $Q^I_b = \{x \in \mathbb{Z}^n,\ Ax = b\}$. An extended fiber Q^I_b is decomposable if there exist b_1 and $b_2 \in cone(A)$ such that $Q^I_b = Q^I_{b_1} + Q^I_{b_2}$ and $b = b_1 + b_2$.

Proposition 3 *Let b_1 and $b_2 \in cone(A)$. Then $Q^I_{b_1} + Q^I_{b_2} = Q^I_{b_1+b_2}$.*

The proof of the previous proposition comes down directly form the definition of extended fibers and from the linearity of the product matrix vector. Applying extended fibers to flow problems leads to consider flows with loops or negative values on the arcs, which is without any interest in the scope of network design. Thus we must truncate the extended fiber with an appropriate set. We call truncated fiber of A in b with respect to the set M the set $Tr_M(Q^I_b) = \{x \in Q^I_b; \nexists m \in M,\ x = y + m,\ y \in Q^I_b\}$. Let $cone(A)$ denote the cone spanned by the columns of A. A truncated fiber $Tr_M(Q^I_b)$ is said to be decomposable if there exist b_1 and $b_2 \in cone(A)$ such that $Tr_M(Q^I_b) \subseteq Tr_M(Q^I_{b_1}) + Tr_M(Q^I_{b_2})$, otherwise it is said an atomic truncated fiber. Notice that $0 \notin M$ otherwise $Tr_M(Q^I_b) = \emptyset$. Now let us establish some relation between atomic extended fibers and atomic truncated fibers on one hand, and on the other hand let us show that both sets are finite sets.

Proposition 4 *If $Tr_M(Q^I_b)$ is atomic then Q^I_b is also atomic.*

Proof : Consider Q^I_b a decomposable extended fiber. Then there exist b_1 and b_2 such that $Q^I_b = Q^I_{b_1} + Q^I_{b_2}$. Let $x \in Tr_M(Q^I_b) \subseteq Q^I_b = Q^I_{b_1} + Q^I_{b_2}$, thus $x = x_1 + x_2,\ x_i \in Q^I_{b_i},\ i \in \{1, 2\}$. Suppose there exist $m \in M$ and $x'_1 \in Q^I_{b_1}$ such that $x_1 = x'_1 + m$, then $x = x'_1 + m + x_2$ contradicting $x \in Tr_M(Q^I_b)$, therefore $x_1 \in Tr_M(Q^I_{b_1})$. The same holds for x_2. Hence we can conclude $Tr_M(Q^I_b) \subseteq Tr_M(Q^I_{b_1}) + Tr_M(Q^I_{b_2})$. ∎

Proposition 5 *Let $H(A)$ be an Hilbert basis of $cone(A)$. There exists a one-to-one correspondance between the atomic extended fibers of A and the vectors of $H(A)$. If $Q^I_b = \sum_{h_i \in H(A)} \beta_i Q^I_{h_i}, \beta_i \in \mathbb{N}$ is a decomposable extended fiber, then $Tr_M(Q^I_b) = Tr_M(\sum_{h_i \in H(A)} \beta_i Tr_M(Q^I_{h_i})), \beta_i \in \mathbb{N}$.*

Proof : Let Q^I_b be a decomposable extended fiber with $b \in H(A)$. Then there exist b_1 and b_2 such that $Q^I_b = Q^I_{b_1} + Q^I_{b_2}$ and $b = b_1 + b_2$, which leads to $b \notin H(A)$. Now suppose that Q^I_b is atomic and $b \notin H(A)$. Then b can be expressed as the sum of elements of $cone(A)$, $b = b_1 + b_2$, according to proposition 3 we have $Q^I_b = Q^I_{b_1} + Q^I_{b_2}$, then there is a contradiction with the hypothesis Q^I_b is atomic.
Consider $x \in Tr_M(Q^I_b)$, since Q^I_b is decomposable $x \in Tr_M(\sum_{h_i \in H(A)} \beta_i Q^I_{h_i})$. Clearly $Tr_M(Q^I_{h_i}) \subseteq Q^I_{h_i}$ and then $\sum_{h_i \in H(A)} \beta_i Tr_M(Q^I_{h_i}) \subseteq Q^I_b$. Suppose that $x \notin Tr_M(\sum_{h_i \in H(A)} \beta_i Tr_M(Q^I_{h_i}))$ then two alternatives can occur. On the one hand $\exists m \in M$ such that $x = y + m$ with $y \in \sum_{h_i \in H(A)} \beta_i Tr_M(Q^I_{h_i}) \subseteq \sum_{h_i \in H(A)} \beta_i Q^I_{h_i} = Q^I_b$, contradicting $x \in Tr_M(Q^I_b)$. On the other $x \notin \sum_{h_i \in H(A)} \beta_i Tr_M(Q^I_{h_i})$, since we know that $Tr_M(Q^I_b) \subseteq \sum_{h_i \in H(A)} \beta_i Tr_M(Q^I_{h_i})$ we can conclude that $x \notin Tr_M(Q^I_b)$, a contradiction. The reverse inclusion comes down from the definition of truncation. ∎

As a corollary of the previous proposition we can say that the set of atomic extended fibers and the set of atomic truncated fibers are finite since $H(A)$ is finite.
Now we give a characterization of atomic fibers devoted to flow and multicommodity flow problems, this is done with respect to the Hilbert basis of $cone(N_G)$. In order to model integral flow sets, let us define an appropriate truncating set M as below :

$$M = \{(x, y) \in \mathbb{R}^{2|E|};\ x \in Ker(N_G)\} \cup \{x \notin \mathbb{R}^{|E|,+},\ y \notin \mathbb{R}^{|E|,+}\} - \{0\}.$$

Proposition 6 *Let $Q^I_{rb_{st},C} = \{(f, s)^t \in \mathbb{Z}^{2|E|};\ \mathcal{C}(f, s)^t = (rb_{st}, C)^t\}$ then $Tr_M(Q^I_{rb_{st},C})$ is the set of non negative elementary integral flow vectors available between the vertices s and t on the graph G.*

Proof : Non negativity comes down from the definition of M. Since the incidence vectors of the circuits of G are contained in $Ker(N_G)$, flows in $Tr_M(Q^I_{rb_{st},C})$ are elementary. ▮

Applying the result contained in proposition 5 to the atomic fibers of the matrix $\mathcal{C}$ we obtain a characterization of the right hand side vectors corresponding to these atomic fibers. This is done in the following proposition.

Proposition 7 *Let $H(N_G) = \{n_1, \ldots, n_p\}$ be an Hilbert basis of cone(N_G). Let $\bar{P}^I_{n_j} = \{u^{n_j}_1, \ldots, u^{n_j}_{q_{n_j}}\} = \{u \in \mathbb{N},\ N_G u = n_j\}/Ker(N_G)$. Then $H(\mathcal{C}) = \{(n_j, u^{n_j}_k), j \in \{1, \ldots, p\},\ k \in \{1, \ldots, q_{n_j}\}\}$.*

Sketch of proof :
The proposition is a consequence of the two following facts. A vector $(v, w) \in H(\mathcal{C})$ has the following form $(N_G f, f)$ and if $s \geq 0$ then $(N_G f, f) \leq (N_G f, f + s)$.

The extension to multicommodity flow problem is direct, thus we have the following corollary.

Corollary 8 *Let $H(\mathcal{D})$ be an Hilbert basis of cone$(\mathcal{D})$, then $v \in H(\mathcal{D})$ if and only if it is of the following form $(n_j, 0, \ldots, 0, u^{n_j}_k), (0, n_j, \ldots, 0, u^{n_j}_k), \ldots, (0, \ldots, 0, n_j, u^{n_j}_k),\ \forall j \in \{1, \ldots, p\},\ \forall k \in \{1, \ldots, q_{n_j}\}$.*

In order to achieve a complete description of both Hilbert bases $H(\mathcal{C})$ and $H(\mathcal{D})$ we give a description of $H(N_G)$ just depending on columns of N_G.

Proposition 9 *Let $\{N_G(1), \ldots, N_G(p)\}$ be a set of columns of the matrix N_G such that the subgraph $G[\{N_G(1), \ldots, N_G(p)\}]$, induced by the arcs related to these columns has the same transitive closure than G. Then $h \in H(N_G)$ is equivalent to $h \in \{N_G(1), \ldots, N_G(p)\}$.*

Sketch of proof :
This result arises from the fact that if an arc of G can be replaced by a path in G, then the related column in N_G is the sum of the columns related to the arcs of the path.

One can check that a set $\{N_G(1), \ldots, N_G(p)\}$ can be computed in a greedy way. Then in order to compute $H(\mathcal{D})$ we the following algorithm :

$H(\mathcal{D}) := \emptyset$,

Compute $\{N_G(1), \ldots, N_G(p)\}$,

For $j \in \{1, \ldots, p\}$

Compute $\bar{P}^I_{N_G(j)} := \{u \in \mathbb{N},\ N_G u = N_G(j)\} = \{u^{N_G(j)}_1, \ldots, u^{N_G(j)}_{q_{N_G(j)}}\}$,
For $k \in \{1, \ldots, q_{N_G(j)}\}$
$H(\mathcal{D}) := H(\mathcal{D}) \cup \{(N_G(j), 0, \ldots, 0, u^{N_G(j)}_k), (0, N_G(j), 0, \ldots, 0, u^{N_G(j)}_k), \ldots, (0, \ldots, 0, N_G(j), u^{N_G(j)}_k)\}$

It is easy to verify that this algorithm compute $H(\mathcal{D})$ in a linear time of the size of this set.

3 Atomic fibers and network design problem

After establishing a relation between optimal solutions over truncated fibers and optimal solution over atomic truncated fibers we will show how to use this material in order to formulate some network design problems. We saw previously that $H(N_G)$ is a subset of the columns of N_G and thus is rather small. So $H(\mathcal{C})$ is easily tractable and even $H(\mathcal{D})$ is tractable.

Proposition 10 *If $Q^I_b = \sum_{h_i \in H(A)} Q^I_{h_i}$ be a decomposable extented fiber A. Then x^* is an optimal solution of the program $Min\{d^t x,\ x \in Tr_M(Q^I_b)\}$ if and only if $x^* = (\sum_{h_i \in H(A)} x^*_{h_i})/M,\ x^*_{h_i} \in Tr_M(Q^I_{h_i})$ implies that $x^*_{h_i}$ is an optimal solution of the program $Min\{d^t x,\ x \in Tr_M(Q^I_{h_i})\},\ \forall h_i \in H(A)$.*

Proof : Let x^* be an optimal solution of the program $Min\{d^t x,\ x \in Tr_M(Q^I_b)\}$. Since $Tr_M(Q^I_b) \subseteq \sum_{h_i \in H(A)} Tr_M(Q^I_{h_i})$ we can say that there exist $x^*_{h_i} \in Tr_M(Q^I_{h_i}),\ h_i \in H(A)$ such that $x^* = \sum_{h_i \in H(A)} x^*_{h_i}$. Now suppose that there exists i_0 such that $x^*_{h_{i_0}}$ is not an optimal solution of the program $Min\{d^t x,\ x \in Tr_M(Q^I_{h_{i_0}})\}$. We suppose in addition that $Tr_M(Q^I_{h_{i_0}})$ is not reduced to $x^*_{h_{i_0}}$ otherwise the case is trivial. Then there exists $y_0 \in Tr_M(Q^I_{h_{i_0}})$ such that $d^t y_0 < d^t x^*_{h_{i_0}}$ which implies that $d^t x^* > d^t(\sum_{h_i \in H(A)-\{h_{i_0}\}} x^*_{h_i} + y_0)$. If $\sum_{h_i \in H(A)-\{h_{i_0}\}} x^*_{h_i} + y_0 \in Tr_M(Q^I_b)$ then there is a contradiction with the hypothesis x^* is an optimal solution of the program $Min\{d^t x,\ x \in Tr_M(Q^I_b)\}$. Now suppose that $\sum_{h_i \in H(A)-\{h_{i_0}\}} x^*_{h_i} + y_0 \notin Tr_M(Q^I_b)$, then we can say that there exists $m \in M$ such that $x^* + m = \sum_{h_i \in H(A)-\{h_{i_0}\}} x^*_{h_i} + y_0$. Moreover $Tr_M(Q^I_{h_{i_0}})$ being not reduced to $x^*_{h_{i_0}}$ we can say that there exists $g \in Ker(A)$ such that $y_0 = x^*_{h_{i_0}} + g$. Hence we have $x^* + m = \sum_{h_i \in H(A)} x^*_{h_i} + g$ implying $m = g$. Thereby $y_0 = x^*_{h_{i_0}} + m \notin Tr_M(Q^I_{h_{i_0}})$, which shows that there exists no y_0 satisfying the hypothesis. ∎

Here we present an application to network design problems. Let d be a vector of demands. Then according to previous results d is feasible over the topology represented by the graph G if and only if there exists a capacity vector C such that :

$$(d, C)^t = \sum_{h_i \in H(\mathcal{D})} \lambda_i h_i,\ \lambda_i \in \mathbb{N}\ \forall h_i \in H(\mathcal{D}).$$

If γ is the cost vector for installing unit capacities, then the minimum cost dimensioning problem can be formulated as below :

$$\begin{array}{ll} Min & \gamma^t C \\ & \sum_{h_i \in H(\mathcal{D})} \lambda_i h_i = (d, C)^t \\ & \lambda_i \in \mathbb{N}\ \forall h_i \in H(\mathcal{D}). \end{array}$$

Since forall $h_i \in H(\mathcal{D})$ we have $h_i = (d_i, C_i)^t$ the above program can be written in the following form :

$$\begin{array}{ll} Min & \gamma^t C \\ & \sum_{h_i \in H(\mathcal{D})} \lambda_i (d_i, C_i)^t = (d, C)^t \\ & \lambda_i \in \mathbb{N}\ \forall h_i \in H(\mathcal{D}). \end{array}$$

Notice that we reach an integral formulation analogous to the one obtained in the continuous case when applying the japanese theorem, [11] and [13]. This formulation is known as the metric formulation. One can remark that for the integral case the constraints do not concern paths only but subnetworks. Separating over the Hilbert basis may be difficult. Consider now the minimum cost routing problem, $\alpha \in \mathbb{R}^{|E|}$ being the cost vector. Since according to proposition 10 an optimal solution over the truncated fiber is the sum of optimal solution over atomic truncated fibers, thus if $f^*_i := Argmin\{\alpha^t f, f \in Tr_M(Q^I_{h_i})\},\ h_i \in H(\mathcal{D})$ then the minimum cost routing problem can be formulated as follows :

$$\begin{array}{ll} Min & \alpha^t f \\ & f = \sum_{h_i \in H(\mathcal{D})} \lambda_i f^*_i \\ & \sum_{h_i \in H(\mathcal{D})} \lambda_i (d_i, C_i)^t = (d, C)^t \\ & \lambda_i \in \mathbb{N}\ \forall h_i \in H(\mathcal{D}). \end{array}$$

4 Conclusion

In this articles we propose an approach which leads to a linear time in the size of the output algorithm in order to compute the right hand sides related to the atomic fibers for flow problems. This provides a significant improvement in comparison with the previous standard Buchberger type algorithms as used in [5]. This comes from the small size of the Hilbert basis of network matrices, which can be determined in a greedy way. After we reformulate the network design problems and these formulations look like the metric formulation for the continuous case. Using functionals as defined in [5] we can investigate survivability problems. In the future we will try to find some properties of the elements of $H(\mathcal{D})$ in order to determine the conditions of the survivability of the network.
In a forthcoming work we will show some similar results for computing Graver basis for flow and multi-commodity flow problems, the approach being closely related to N-fold programming as it is in [12].

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