Some background on formal structures

Mike Worboys
NCGIA, University of Maine, USA

1 Sets, functions, relations, posets, and lattices

We assume that basic material on sets, functions and relations is understood. Also, equivalence relations and their quotients.

Definition 1: A *tolerance* is a relation on a set that is reflexive and symmetric (but not necessarily transitive).

Definition 2: A *partial order* is a relation on a set that is reflexive, antisymmetric and transitive. A *poset* $\langle X, \leq \rangle$ is a set X on which is defined a partial order $\leq$.

Let $\langle X, \leq \rangle$ be a poset and $S \subseteq X$. An *upper bound* of S is an element $u \in X$ such that $s \leq u$ for all $s \in S$. A *supremum* is a least such upper bound. It can be shown that the supremum of a set is unique, if it exists. There are symmetrical definitions for *lower bound* and *infimum*

Definition 3: A *lattice* is a poset which has the property that every pair of its elements, say a and b has a supremum, called the *join* of a and b, and written $a \sqcup b$), and an infimum, called the *meetn* of a and b, and written $a \sqcap b$).

Other useful structures are weaker kinds of lattices, called *semilattices* and stronger kinds of lattices with complementation, called *Boolean algebras* and *Heyting algebras* if the complementation is weak.

2 Topology

Definition 4: A *topological space* is a pair $\langle X, O \rangle$, where X is a set and O is a collection of subsets of X, such that:

1. O contains X and $\emptyset$.
2. O is closed under union.
3. O is closed under finite intersection.

Definition 5:

1. A set in O is called *open*. The complement in X of an open set is called *closed*.
2. The *interior* of a set $A \subseteq X$ is the union of all open sets contained in A.
3. The *closure* of a set $A \subseteq X$ is the intersection of all closed sets containing A.
4. The *boundary* of a set $A \subseteq X$ is the set of points x with the property that any open set containing x overlaps both A and the complement of A.

Definition 6: A set is *regular* if it is equal to the closure of its interior.

Example 1: Indiscrete topology $\langle X, \{\emptyset, X\}\rangle$.

Example 2: Discrete topology $\langle X, \wp(X)\rangle$.

Example 3: Topologies deriving from metric spaces, including the usual topology.

Definition 7: A topological space X is *connected* if the only sets that are both open and closed are $\emptyset$ and X.

3 Topology preserving functions

Definition 8: Given topological spaces $\langle X, O\rangle$ and $\langle X', O'\rangle$. The function $f : X \rightarrow X'$ is *continuous* iff for all open sets $o' \in O'$, $f^{-1}(o') \in O$ (inverse images of open sets is open).

Definition 9: Given topological spaces $\langle X, O\rangle$ and $\langle X', O'\rangle$. The function $f : X \rightarrow X'$ is a *homeomorphism* iff for it is a continuous bijection with a continuous inverse.

4 Modal logic for topology

Assume the formulas P of the language $\mathcal{L}$ are made out of atomic propositions, connectives (including $\top$ and $\bot$), unary modal operators $\Box$ and $\Diamond$.

Semantics

The semantics of the logic is based on regions. Each formula represents a region, where the proposition expressed by the formula is true, and compositions of formulas represent combinations of regions, as follows.

1. $\top$ represents the universe X.
2. $\bot$ represents $\emptyset$.
3. $\neg\phi$ represents the complement of the region represented by ϕ.
4. $\phi \cup \psi$ represents the union of the regions represented by ϕ and ψ.
5. $\Box\phi$ represents the interior of the region represented by ϕ.
6. $\Diamond\phi$ represents the closure of the region represented by ϕ.

These ideas can be formalized by a valuation function $\nu : P \rightarrow \wp(X)$.

5 Axioms and rules of deduction

Proof system - Rules of deduction:

$$\frac{\phi \quad \phi \rightarrow \psi}{\psi} \text{ (modus ponens)} \tag{1}$$

$$\frac{\phi}{\Box\phi} \text{ (necessitation)} \tag{2}$$

Axioms

$$\Diamond\phi \leftrightarrow \neg\Box\neg\phi \tag{3}$$
$$\Box\top \tag{4}$$
$$\Box\phi \rightarrow \phi \tag{5}$$
$$\Box\phi \rightarrow \Box\Box\phi \tag{6}$$
$$\Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi) \tag{7}$$
$$\Box\phi \vee \Box\psi \rightarrow \Box(\Box\phi \vee \Box\psi) \tag{8}$$

For those who know about modal logics, the axioms above are stronger than the system S4.