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Existence proof for an exchange economy in the standard Arrow-Debreu Economy

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1 Introduction

This note have the aim of describe a ***pure exchange economy***. The focus is on characterizing scenarios where it's possible to find an ***equilibrium price***. The existence proof will done in two step: firstly we specify the necessary behavior consumer assumptions that guarantee the existence of an ***excess aggregate demand*** function with some particular properties; secondly we show how these properties implies the existence of an equilibrium price. The note is organized as follows: in the next section we give some basic definitions and recall some well known and useful results; in the last section we give the formal treatment of the existence of an equilibrium price in a pure exchange economy.

2 Preliminaries

2.1 Definitions

We consider an *economy* with a finite number of commodities C and a finite number of consumers or householders H. We denote with x_h^c the quantity of commodity c used by the consumer h, and when we omit index c we consider a vector of commodity. Each consumer h is described through a pair (u_h, e_h), where $u_h : \mathbb{R}_+^C \to \mathbb{R}, x_h \mapsto u_h(x_h)$ is the *utility function* representing his *preference relation* and $e_h \in \mathbb{R}_+^C$ is his initial endowment of commodities.
We assume:

1. our model of economy is a *model under certainity* so that the behaviour of the consumers is deterministic,

2. economy is a *pure exchange economy* with no production possibility,

3. consumers are *price takers* , i.e. they cannot influence prices.

Our assumptions mean that consumers with their initial endowments of the commodities go to the market where they see fixed prices, denoted by $p \in \mathbb{R}^C_+$, and decide to exchange their commodities so that maximizing their own utility.
We have an economy with an *equilibrium price* and an *allocation of commodities* if markets clear (i.e. every consumer gets what he wants) while everybody optimizes.

Definition 2.1 *The consumption set of each consumer is a subset of the commodity space $\mathbb{R}^C_+$, denoted by $X \subset \mathbb{R}^C_+$, whose elements are the consumption bundles that each consumer can conceivably consume given the physical constraints imposed by his environment.*

Before introducing the problem that every consumer faces in our economy and so that of maximizing his utility, we have to describe in some way the set within which each consumers can choose his consumption set given the prices p he sees on the market and the endowment e_h he has.
We, therefore,

Definition 2.2 *The* ***budget correspondence*** *β as*

$$\beta : \mathbb{R}^C_+ \times \mathbb{R}^C_+ \to\to \mathbb{R}^C_+ \tag{2.1}$$

that, for each pair (p, e_h), define the so called ***budget set:***

$$\beta(p, e_h) = \{x_h \in \mathbb{R}^C_+ \ : \ px_h \leq pe_h\} \tag{2.2}$$

Budget sets define, for each consumer, the consumption sets that such a consumer can afford given market prices and personal endowment. The aim of each consumer is the maximization of the personal utility u_h and so he solves the following:

Definition 2.3 *The* ***Utility Maximization Problem*** *(in short UMP) is :*

$$Max_{x_h \in \mathbb{R}^C} \quad u_h(x_h) \tag{2.3}$$

subject to

$$px_h \leq pe_h \tag{2.4}$$

Relations (2.3) and (2.4) represent the formal description of UMP and mean that each consumer chooses a consumption vector within his budget set in order to maximize his own utility. Now we characterize the the way consumers choose through the following definition :

Definition 2.4 *The **demand correspondence** $x_h(p, e_h)$ is:*

$$x_h : \mathbb{R}^C_+ \times \mathbb{R}^C_+ \to\to \mathbb{R}^C_+ \tag{2.5}$$

such that:

$$x_h(p, e_h) = argmax(UMP) \tag{2.6}$$

Observe that demand correspondence has teh following nice property:

Definition 2.5 *Demand correspondence $x_h(p, e_h)$ is **homogeneous of degree zero** if $x_h(\alpha p, \alpha e) = x_h(p, e)$ for any p, e and $\alpha > 0$.*

Homogeneity of degree zero says that if both prices and endowment change in the same proportion, then the individual's budget set does not change, as can be easily seen from its definition.

2.2 Useful Results

We list here, with some comments, three theorems and a definition that prove very useful in the field of Walrasian equilibrium, even in the simplified version we are describing in the present notes:

Definition 2.6 *Consumers are supposed to be **rationals** so that their preference relations $\succeq$ are both complete and transitive so to allow an ordering of the consumption sets. Continuous preference relations can be represented with continuous functions called* utility functions. *Formally we have that an utility function for consumer h, $u_h : \mathbb{R}^C_+ \longrightarrow \mathbb{R}$, represents preference relation $\succeq_h$ if, for all x_1 and $x_2 \in \mathbb{R}^C_+$, we have*

$$x_1 \succeq_h x_2 \Longleftrightarrow u_h(x_1) \geq u_h(x_2) \tag{2.7}$$

so that u_h is a numeric representation of $\succeq_h$.

Theorem 2.1 (Weierstrass theorem) *Let be*

$$u_h \ \mathcal{A} :\longrightarrow \mathbb{R} \tag{2.8}$$

a continuos function, if A is a compact set then u attains a maximum and a minimum value.

We use such a theorem, in the characterization theorem, to prove that z is a well defined function and, in the existence theorem, to prove that a correspondence we define there is not empty valued.

Theorem 2.2 (Maximum Theorem) *Consider a budget correspondence β, an utility function u_h, a demand correspondence x_h and the indirect utlity function $v : R^C_{++} \times R^C_{+} \to R$, $v : (p,e) \mapsto max(UMP)$.*
Assume that β is (non-empty valued), compact valued and continuous, u_h continuous. Then

1. *x_h is (non-empty valued), compact valued, upper hemicontinuous (UHC) and closed;*

2. *v is continuous.*

Observe that Weierstrass theorem assure us that UMP at least a solution from since $\beta(p, e_h)$ is a compact set (it is closed and limited) and function u is supposed continuous.

Theorem 2.3 (Kakutani's fixed-point theorem) *Suppose that $A \subset \mathbb{R}^{\mathbb{N}}$ is a nonempty, compact, convex set, and that $\phi : A \rightarrow\rightarrow A$ is a closed correspondence with the property that set $\phi(x) \subset A$ is nonempty and convex for every $x \in A$. Then $\phi(\cdot)$ has a fixed point; that is there is an $x \in A$ such that $x \in \phi(x)$.*

We use Kakutani's fixed-point theorem in the existence theorem to show that the correspondence we have defined there defines an equilibrium price vector p^*. To show that we define a correspondence $\mu : S \rightarrow\rightarrow S$ and than prove that S is *convex*, *compact*, μ is *not empy valued*, *convex valued*, *closed graph* so that we can use the theorem and be sure that $\exists s^* \in S$ such that $s^* \in \mu(s^*)$.

Definition 2.7 (Walras' law) *We have it in various versions and we list them here one after the other. We denote with $x_h(p, w_h)$ the demand correspondence of consumer h, $w_h > 0$ the consumer's wealth, e_h the consumer's endowment, $p \gg 0$ the vector of the prices and $z(p)$ the* excess aggregate demand correspondence.

$$pz(p) = 0 \, \forall p \tag{2.9}$$

$$px_h = w_h \, \forall x_h \in x(p, w_h) \tag{2.10}$$

$$px_h(p, w_h) = w \,\forall p\, and w_h \tag{2.11}$$

$$px_h = pe_h \,\forall x_h \in x(p, e_h) \tag{2.12}$$

We can, indeed, describe a consumer either in terms of a monetary wealth w_h or of an endowment of goods e_h. We use the definition as a thesis in the characterization theorem as an hypothesis in the existence theorem.

3 Characterization and Existence

We have two theorems, one of characterization and one that allows the definition of an equilibrium price vector p. The first theorem, actually, characterizes the *excess aggregate demand correspondence* $z(p)$ in terms of:

1. the *demand correspondence* $x_h(p, e_h)$ [1]
2. the *endowment* e_h

We call $z(p)$ *excess aggregate demand correspondence* because it represents the difference between the overall consumers willings(at a certain price level) and what is really available in the markets(the sum of consumer endowments). More formally we have:

$$z :\mapsto\mapsto \sum_{h=1}^{H} (x_h(p) - e_h) \tag{3.13}$$

The basic hypotheses involve the utility function u_h and the demand correspondence $x_h(p)$ of each consumer: the first must be *continuous* and *strictly increasing* wheras the latter must take up values in $\mathbb{R}^C_+$ and be *homogeneous of degree zero* so to "absorb" scalings of the parameter p by a factor $\alpha > 0$. Through such a theorem we prove that $z(p)$ is *well defined* (so it is not a pure formalism but has a real meaning), *continuous*, *homegeneous of degree zero*, *bounded from below* and *satisfies boundary conditions* (so that if one of the prices go to 0 the $z()$ is unbounded).
The second theorem is a *theorem of existence* and, by using as hypotheses the conclusions of the first theorem about *excess aggregate demand correspondence*, aims at saying that an equilibrium price vector $p^* \gg 0$ exists such that $z(p^*)$ where with the term *equilibrium price* we mean a price vector such that *demand* equals *supply*. The proof strongly rely on *Kakutani's fixed point theorem* we introduced in the previous subsection.
We note that the theorem proves that an equilibrium price vector exists but in no way gives tools for its determination or says something about its uniqueness.

[1] In what follows argument e_h will be disregarded in the notation for $x_h()$.

3.1 The Excess Aggregate Demand

For sake of simplicity from now on we omit the argument e_h in the demand correspondence and we write $x_h(p)$ for $x_h(p, e_h)$.

Theorem 3.1 *In a pure exchange economy if for any household $h \in \{1, \ldots, H\}$:*

1. $u_h : \mathbb{R}^C_+ \to \mathbb{R}$,
2. u_h *is continuous,*
3. u_h *is strictly increasing,*
4. $x_h(p) \in \mathbb{R}^C_+$ *and homogeneous of degree zero*

then, the ***excess aggregate demand map***

$$z : \mathbb{R}^C_+ \to \mathbb{R}^C$$

$$z : p \mapsto \sum_{h=1}^{H} x_h(p) - \sum_{h=1}^{H} e_h$$

i. is a well defined continuous function,

ii. satisfies Walras law,

iii. is homogeneous of degree zero

iv. is bounded from below,

v. satisfies the boundary condition:$\{p_n\}_n \subset \mathbb{R}^C_+$ *and* $p_n \to \bar{p} \in \partial\mathbb{R}^C_+$ $\Longrightarrow z(p_n)$ *is unbounded.*

Proof.

ia.) z is a well defined function. The existence follows from the Extreme Value Theorem apply to the household's maximization problem. The uniqueness follows from the strictly monotonicity of u_h.

ib.) u_h is continuous.

ii.)

$$pz(p) = \sum_{j=1}^{C} p_j \big(\sum_{h=1}^{H} x_h(p) - \sum_{h=1}^{H} e_h \big) = \sum_{h=1}^{H} p \cdot x_h(p) - p \cdot e_h = 0$$

where the last equality follows since x_h binds the budget constraint.

iii.) It follows from the homogeneity of $x_h(p)$.

iv.) For any $c \in \{1, \dots, C\}$ and any p, $m^c \leq z^c(p)$ with $m \in \mathbb{R}^C$,

$$z(p) = \sum_h x_h(p) - \sum_h e_h \geq -\sum e_h = m \tag{3.14}$$

where we use the facts that $x_h(p) \in \mathbb{R}^C_+$ and $e_h \in \mathbb{R}^C_+$ for any h.

v.) Suppose otherwise i.e. $\{x_h(p^n)\}_n$ is bounded. then there exists a converging subsequence:

$$x_h(p^r) \to \bar{x}_h \in \mathbb{R}^C_+ \tag{3.15}$$

Being $x_h(p)$ continuous,

$$\lim_{r \to +\infty} x_h(p^r) = x_h(\bar{p}) = \bar{x}_h \tag{3.16}$$

Since $\bar{p} \in \partial\mathbb{R}^C_+$, $\exists c$ such that $\bar{p}^c = 0$ but then for any $\varepsilon > 0$ w.l.o.g. we take $c = 1$:

$$\bar{p}(\bar{x}_h + (\varepsilon, 0, \dots, 0)) = \bar{p}\bar{x}_h \leq \bar{p}e_h \tag{3.17}$$

From the strictly monotonicity of $u_h(\cdot)$,

$$u_h(\bar{x}_h + (\varepsilon, 0, \dots, 0) > u_h(\bar{x}_h) \tag{3.18}$$

and (4), (5) contradict the definition of $x_h(\bar{p}) \equiv \bar{x}_h$. ■

3.2 Existence

Theorem 3.2 *If*

$$z : \mathbb{R}^C_+ \to \mathbb{R}^C$$
$$z : p \mapsto \sum_h (x_h(p) - e_h)$$

1. *is a continuous function,*

2. *satisfies Walras law: $pz(p) = 0$,*

3. *is homogeneous of degree zero,*

4. *is bounded from below,*

5. *satisfies the boundary condition:*$\{p_n\}_n \subset \mathbb{R}^C_+$ *and* $p_n \to \bar{p} \in \partial\mathbb{R}^C_+ \Longrightarrow$ $z(p_n)$ *is unbounded,*

then $\exists p^* \in \mathbb{R}^C_{++}$ *such that* $z(p^*) = 0$

Proof.

From the homogeneity we can normalize the prices and define:

$$\Delta \equiv \{p \in \mathbb{R}^C_+ : \sum_{c=1} p^c = 1\} \tag{3.19}$$

and consider this as the domain of z. Observe that Δ is a convex and compact set and each $p \in \partial\delta$ has at least one zero component. So we will prove the thesis showing that $\exists p^* \in \partial\Delta$ such that $z(p^*) = 0$.

The main steps of proof will be:

1. define a correspondence $\mu : \Delta \to\to \Delta$;
2. show that $\exists p^* \in Int\Delta$ such that $p^* \in \mu(p^*)$;
3. show that $z(p^*) = 0$.

STEP 1) Definition of the correspondence μ.

$$\mu(p) = \begin{cases} \arg\max_{q \in \Delta} qz(p) & if \quad p \in Int\Delta \\ \{q \in \Delta : pq = 0\} & if \quad p \in \partial\Delta \end{cases} \tag{3.20}$$

STEP 2) We have to apply Kakutani's fixed point theorem:

If

1. $\mu : S \to\to S$,
2. $S \neq \emptyset$,
3. S is convex,
4. S is compact,
5. $\mu \neq$ empty valued,
6. μ is convex valued,
7. μ is closed graph,

then $\exists s^* \in S$ such that $s^* \in \mu(s^*)$.

Let's verify (1)-(7)

1. Let be $S = \Delta$,

2. Take for example $p = (1, 0, \ldots, 0)$,

3. It follows from definition (3.19),

4. It follows from definition (3.19),

5. We have to consider the definition (3.20) and distinguish

   i. If $p \in Int\Delta$
   then the maximization problem has at least a solution from the Weirstrass theorem. Moreover if $z(p) \neq 0$ then $\mu(p) \subseteq \partial\Delta$ otherwise if $z(p) = 0$ then $\mu(p) = \Delta$

   ii. if $p \in \partial\Delta$
   take any $q \in \Delta$ such that $q \perp p$, moreover $\mu(p) \subseteq \partial\Delta$,

   CLAIM A fixed point $p^* \notin \partial\Delta$.
   PROOF. Consider $p \in \partial\Delta$. Then:

   i. $q \in \mu(p) \Rightarrow q \perp p$,

   ii. $p \neq 0$

   So $p^* \notin \mu(p^*)$ because a non-zero vector cannot be orthogonal to itself.

6. i. If $p \in Int\Delta$
   then If q^1 and q^2 are maximizers, then $[\lambda q^1 + (1-\lambda)q^2]z(p) = \lambda q^1 z(p) + (1-\lambda)q^2 z(p) = q^1 z(p) = q^2 z(p)$;

   ii. if $p \in \partial\Delta$
   then If $pq^1 = pq^2 = 0$ then $p[\lambda q^1 + (1-\lambda)q^2] = \lambda pq^1 + (1-\lambda)pq^2 = 0$;

7. WTS:$\langle p^n \to \overline{p}, q^n \to \overline{q}, q^n \in \mu(p^n)\rangle \Rightarrow \langle \overline{q} \in \mu(\overline{p})\rangle$
   We examinate separately the two cases:

   a) $\overline{p} \in Int(\Delta)$

   b) $\overline{p} \in \partial\Delta$

   a) $\overline{p} \in Int\Delta \Rightarrow \overline{p} \gg 0$
   So we have for n sufficiently large that $p^n \gg 0$ and $q^n \in \text{argmax}_{q\in\Delta} qz(p)$, then:

$$q^n z(p^n) \geq q' z(p^n) \quad \forall q' \in \Delta$$

When $n \to \infty$, continuity of z implies that

$$qz(p) \geq q'z(p) \quad \forall q' \in \Delta$$

$\Rightarrow q \in \mu(p)$.

b) $\overline{p} \in \partial\Delta$
If, for n sufficiently large $p^n \in \partial\Delta$, it must be $q_c^n = 0$ for some c. Then $\overline{q}_c = 0$ and $\overline{q} \in \mu(\overline{p})$.
If $p^n \in Int\Delta$, i.e. $p^n \gg 0 \ \ \forall n$, take c with $p_c > 0$, there is an $\varepsilon > 0$ such that $p_c^n > \varepsilon$ for n sufficiently large.
We want to show that for n sufficiently large

$$z_c(p^n) < \max\{z_1(p^n), \ldots, z_C(p^n)\}$$

For the boundary condition the right side of the above inequality goes to infinity. But the left side is bounded from above:

$$z_c(p^n) \leq \frac{1}{\varepsilon} p_c^n z_c(p^n) = -\frac{1}{\varepsilon} \sum_{c' \neq c} p_c^n z_c(p^n) < \frac{m}{\varepsilon} \sum_{c' \neq c} p_c^n < -\frac{m}{\varepsilon}$$

where the equality is done by Walras' law and m is the lower bound of z.
Then, for large n, any $q^n \in \mu(p^n)$ will have nonzero weight only on commodities whose prices approach zero. Hence $\overline{pq} = 0$ and $\overline{q} \in \mu(\overline{p})$.

Step 3)The fixed point p^* is an equilibrium price.

Remark

$$\left\langle \begin{array}{l} z(p) = ku \text{ with } u = (1, \ldots, 1) \in \mathbb{R}^C \\ k \in \mathbb{R}^1 \end{array} \right\rangle \Longrightarrow \langle k = 0 \rangle \tag{3.21}$$

Proof of remark from Walras law:

$$0 = pz(p) = kpu = k \sum_{c=1}^{C} p^c = k \tag{3.22}$$

Now we know from the claim that $p^* \in Int\Delta$ i.e. $p^* \gg 0$ and $p^* \in \arg\max_q qz(p^*)$. Then $z(p^*) = uz^*$ where $u = (1, \ldots, 1) \in \mathbb{R}^C$ (by definition of the max problem $\max_q qz(p^*)$). From Walras' law:

$$0 = p^* z(p^*) = \sum_c p^{c*} z^* = z^* 1 = z^*$$

so that $z(p^*) = u0 = 0$. ■