Notes of the course
"An Introduction to Dynamical Systems"

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Doctoral Course *An Introduction to Dynamical Systems*

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Abstract

The present paper contains a summary of the lectures held by Prof. Federico Oliveira-Pinto from October 6 to October 20 2005. They have been drafted on the basis of some loose notes i wrote down during the lessons so they are surely full of errors, inaccuracies and gaps of which i am the sole responsible. Obviously corrections and suggestions are welcome.

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1 First lesson

1.1 Introduction

Main topics of the course (or course outline):

1. discrete models,
2. continuous models,
3. difference and differential equations and their solutions,
4. characterization of the solutions (stable and unstable),
5. points of equilibrium and their types.

Motivations: the course aims at presenting a basic introduction to difference and differential equations as tools for the modeling of, respectively, discrete and continuous dynamic systems (respectively DDS and CDS) and to give some applications. During the lessons we are going to examine:

1. the general solutions in closed form;
2. equilibria for first order DDS;
3. linear DDS with exponential growth;
4. economical examples:
   (a) bank account with compound interest,
   (b) mortage repayment,
5. a comparison between DDS and CDS;
6. linear bounded DDS with applications to marketing;
7. linear bounded CDS;
8. analytical solutions for bounded CDS;
9. quadratic DDS;
10. logistic CDS;
11. analytical solutions of the logistic equation;
12. closed form solutions for quadratic DDS;

13. stability of oscillating solutions;
14. from quadratic to cubic solutions of DDS;
15. models of populations.

We underline the fact that DDS are modeled with **difference equations** whereas CDS are modeled with **differential equations**.

1.2 First example

In case of weather forecast we are interested in the prevision of some quantities such as

1. temperature,
2. wind
3. humidity

but small variations can give rise to unstable combinations.

1.3 DDS

Now, at the end of the first introductory lesson, we state the general form of a first order DDS (we call it first order since next value depends only on one of the preceding values):

$$x_{k+1} = f(x_k) \tag{1}$$

with $k = 0, 1, 2, \ldots$ and with an initial known value x_0. If we have

$$x_{k+1} = f(x_k, x_{k-1}, \ldots, x_{k-m+1}) \tag{2}$$

with $k = m-1, m, m+1, \ldots$ and $m \in \mathbb{N}$ we speak of an m order DDS in which we need m initial values. In case of a second order DDS we have

$$x_{k+1} = f(x_k, x_{k-1}) \tag{3}$$

with two initial values x_0 and x_1.
Let's now go back to equation (1) and rewrite it as follows:

$$x_{k+1} = x_k + f(x_k) - x_k = x_k + g(x_k) \tag{4}$$

with

$$g(x_k) = f(x_k) - x_k \tag{5}$$

so that we have

$$x_{k+1} - x_k = g(x_k) \tag{6}$$

in which $g(x_k)$ plays the role of a correction since we have

$$\Delta x_{k+1} = x_{k+1} - x_k = g(x_k) \tag{7}$$

It is easy to switch from a discrete model, such as that represented by equation (7), to a continuous time model that, in case of equation (7), is represented as:

$$\frac{dy}{dt} = g(y) \tag{8}$$

The transformation is obtained through easy steps that will be examined in greater detail in the course of the following lessons.
Going again back to equation (1) if we want a closed form solution we look for a solution in which x_{k+1} depends only from the initial value x_0 and on k (the iteration counter) so that we can write:

$$\Delta x_{k+1} = F(k, x_0) \tag{9}$$

so that we have no recursion at all. There are cases in which such a general closed form solution cannot be found.

1.3.1 Equilibrium values for first order DDS

If we have

$$x_{k+1} = f(x_k) \tag{10}$$

with $k = 0, 1, 2, \ldots$ we can find an equilibrium point x^* such that

$$x^* = f(x^*) \tag{11}$$

and so x^* is a fixed point of f. To find such an equilibrium value we have to solve

$$x - f(x) = 0 \tag{12}$$

The simplest case is the one in which f is a polynomial with constant coefficients so that the roots of (12) can be either reals or complex conjugates. In any case we find solutions that can be either stable (e. g. decreasing exponential) or unstable (e. g. increasing exponential) or oscillating.
If, given $f(x)$, we use Taylor expansion formula around x^* we obtain (stopping at first order):

$$f(x) = f(x^*) + (D_x f_{x=x^*})(x - x^*) \tag{13}$$

We have two cases:

1. $\mid D_x f \mid < 1$ so that if we consider $x = x_k$ (in this case we get $x_{k+1} = f(x_k)$ and, from the definition of equilibrium, $x^* = f(x^*)$)

$$\mid x_{k+1} - x^* \mid < \mid x_k - x^* \mid \tag{14}$$

so that x^* is a stable point since at each iteration the distance from the equilibrium point decreases.

2. $\mid D_x f \mid > 1$ so that (see above)

$$\mid x_{k+1} - x^* \mid > \mid x_k - x^* \mid \tag{15}$$

In this case x^* is an unstable point since at each iteration the distance increases.

If $\mid D_x f \mid = 1$ we have $\mid x_{k+1} - x^* \mid = \mid x_k - x^* \mid$ so the distance remains unchanged and the equilibrium is said indifferent.

1.3.2 Example

Suppose we have

$$x_{k+1} = x_k^2 \tag{16}$$

with an initial value x_0. In this case we have $x = x^2$ (in general it is $x = f(x)$) so that (in order to find fixed points) we have to solve

$$x - x^2 = 0 \tag{17}$$

Equation (17) has two solutions that are:

1. $x_1^* = 0$,
2. $x_2^* = 1$

and they represent two equilibrium points. Now we have to understand what kind of equilibrium points they are since we they can belong to three possible types:

1. **stable**, if following a small perturbation the system, after a short evolution, recovers the equilibrium state;
2. **unstable**, if following a small perturbation the system goes away from the equilibrium state;
3. **indifferent**, if the equilibrium does not belong to any of the aforesaid categories.

To check to which type an equilibrium point belongs to we have an easy tool that uses the absolute value of the derivative of f evaluated in an equilibrium point. If such an absolute value is less that one the point is of stable equilibrium, if it is greater than one that point is of unstable equilibrium and if it is equal to one that point is of indifferent equilibrium.
In our case we have $D_x f = 2x$ so that:

1. $D_{x_1^*} = 0$ so that $x_1^* = 0$ is a stable equilibrium point,
2. $D_{x_2^*} = 2$ so that $x_2^* = 1$ is an unstable equilibrium point.

Another way to understand the nature of an equilibrium point is to find the closed form solution of equation(16). It is easy to see that, by successive substitutions (and by starting with the $k-$th term), we have

$$x_k = (x_{k-1})^2 = (x_{k-2})^{2^2} = \cdots = (x_0)^{2^k} \tag{18}$$

so that

1. if $x_0 < 1$ we have
$$lim_{k \to \infty} x_k = 0 \tag{19}$$
so that the equilibrium is stable;
2. if $x_0 > 1$ we have
$$lim_{k \to \infty} x_k = \infty \tag{20}$$
so that the equilibrium is unstable.

It is possible to use a graphic analysis by studying the following system of two difference equations:

$$\begin{cases} y_{k+1} = x_k^2 \\ x_{k+1} = y_{k+1} \end{cases} \tag{21}$$

with a known initial value x_0. By drawing the graphs of the two curves and computing successive values starting from different x_0 it is easy to verify the presence of two equilibrium points, one of which is stable whereas the other one is unstable.

2 Second lesson

2.1 Introduction

We start going over some of the topics of the first lesson.
During the first lesson we spoke about stability of equilibrium values i. e. of the fixed points of the function that describes the behavior of a DDS and we examined the following case:

$$x_{k+1} = x_k^2 \tag{22}$$

with a known initial value x_0. For equation (22) it is easy to plot the graphs for distinct values of x_0.
Now we face the somewhat inverse problem: we know a graph and we want to know if it can be associated with a equation of the kind of equation (22) for a given value of x_0. The problem is not easy to solve because the data we have may be misleading since (for instance) they may represent only a partial plot of a function that seems to behave like an exponential function but, really, tends to saturate from one value of k on. We are going to deal again with this problem in the future.
As a general form we have

$$x_{k+1} = f(x_k) \tag{23}$$

and such equation can be written as a system of two equations, each representing a curve in the $x \div y$ plane:

$$\begin{cases} y_{k+1} = f(x_k) \\ x_{k+1} = y_{k+1} \end{cases} \tag{24}$$

with an initial known value x_0. The couple of equations (24) represents a sort of an algorithm for the definition of a trajectory that characterizes the evolution of a DDS starting from an initial value x_0. A point x^* is an equilibrium point iff we have

$$x^* = f(x^*) \tag{25}$$

What we are looking for is the intersection point of the two curves of equations (24) (a straight line and a generic curve): so to get equilibrium values for which we have

$$x^* = f(x^*) \tag{26}$$

After that, in order to study the nature of equilibrium, we can either look for a closed form solution of equation (25) or study the absolute value of the derivative of $f(x)$ with respect to x. If we have

$$f(x) = x^2 \tag{27}$$

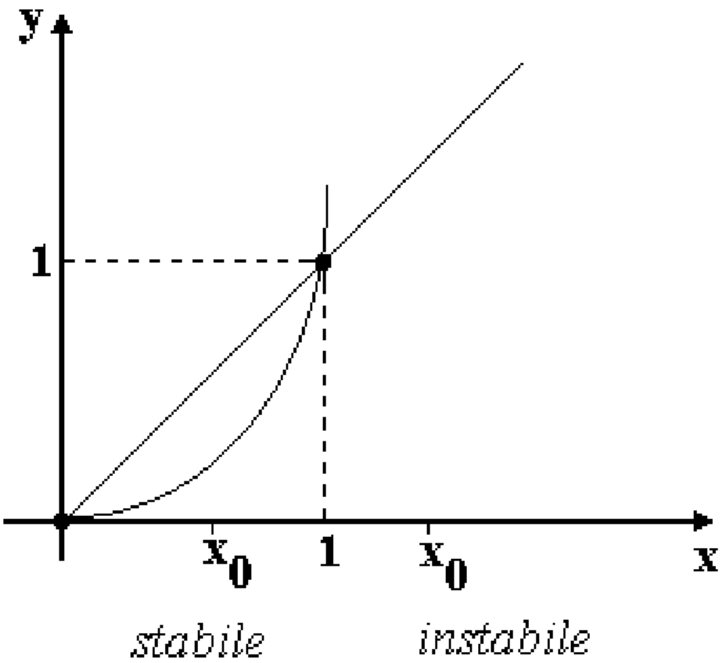

Figure 1: *Curves and stability (see text for details)*

(cf. figure 1) it is easy to see that equation (27) has two fixed points (those showed as dots on figure 1): $x_1^* = 0$ and $x_2^* = 1$.
The first of those points is a stable one (how can be easily seen by examining the absolute value of the derivative in that point, whose value is lower that one) whereas the second one is unstable (how can be easily seen by examining the absolute value of the derivative in that point, whose value is greater that one). In case of $x_1^* = 0$ if we perturb the system we have that it goes back to the equilibrium state (maybe with some dumped oscillations around it) whereas in case of $x_2^* = 1$ we have an even small perturbation drives away the system from such equilibrium and pushes it either toward $x_1^* = 0$ or toward ∞.

2.2 Another example

Let's step to a more complex example such as the following:

$$x_{k+1} = \frac{16}{5}x_k - \frac{4}{5}x_k^2 \tag{28}$$

with a known initial value x_0. In equation (28) we have

$$f(x) = \frac{16}{5}x - \frac{4}{5}x^2 \tag{29}$$

We can work out the problem in two steps:

1. we define fixed points (or equilibrium points) by solving

$$x - f(x) = 0 \tag{30}$$

so to get (with easy calculations) $x_1^* = 0$ and $x_2^* = 2.75$;

2. we evaluate the absolute values of the derivative $D_x f$ in such points so to get:

   (a) $D_x f_{|x_1^*} = \frac{16}{5} > 1$

   (b) $D_x f_{|x_2^*} = \frac{6}{5} > 1$

   so that both equilibrium points are unstable.

In case of equation (28) it is a little bit hard to define a closed form solution. We know the fixed points and also know their nature. It easy to see (cf.

1	1	0,1	-0,1	3	4,1
2	2,4	0,31	-0,33	2,4	-0,33
3	3,07	0,92	-1,14	3,07	-1,14
4	2,28	2,27	-4,67	2,28	-4,67
5	3,14	3,14	-32,35	3,14	-32,35
6	2,17	2,16	-940,61	2,17	-940,61
7	3,18	3,18	-710807,32	3,18	-710807,32
8	2,09	2,08	-404199916724,91	2,09	-404199916724,93
9	3,19	3,19	-1,31E+023	3,19	-1,31E+023
10	2,06	2,06	-1,37E+046	2,06	-1,37E+046
11	3,2	3,2	-1,49E+092	3,2	-1,49E+092
12	2,05	2,05	-1,79E+184	2,05	-1,79E+184
13	3,2	3,2	Err:503	3,2	Err:503

Figure 2: *Different behaviors (see text for details)*

figure 2), by doing some simple calculations either by hand or using any spread sheet, how do such DDS behaves with variable initial value x_0. We have the following cases:

1. $x_0 = 0$ all successive values are 0 so there is no evolution at all and the same holds also for $x_0 = 2.75$;

2. for all different initial values provided that they fall between 0 and 4 the system shows a transient period followed by a periodic oscillation of period 2 between the values 2.05 and 3.2;

3. for any other initial value $x_0 < 0$ or $x_0 > 4$ we have that the system diverges to $-\infty$.

In the general case of equation (23) we start with x_0 then we evaluate, one after the other:

1. $x_1 = f(x_0)$
2. $x_2 = f(x_1) = f^2(x_0)$
3. $x_3 = f(x_2) = f^3(x_0)$
4. $x_4 = f(x_3) = f^4(x_0)$

and so on, where with f^i we mean the function f composed with itself i times, if $i = 0$ we have, by definition, the identity function.
In this way we may guess if the system's behavior diverges, converges or oscillates.

2.3 Equilibrium values

If we have a DDS described by the following equation:

$$x_{k+1} = f(x_k) \tag{31}$$

we can restate such defining equation so that it can be described in terms of two copies of the same function:

$$\begin{cases} y_{k+1} = f(x_k) \\ x_{k+2} = f(y_{k+1}) \end{cases} \tag{32}$$

In this case we can use the graphic method sketched in figure 3 to evaluate the successive points starting from an initial point x_0 (called 0 in that figure). Starting from 0 we evaluate point 1 as the intersection of the straight line $x = x_0$ with $y = f(x)$ then we use 1 to evaluate 2 as the intersection of the straight line $y = y_1$ with $x = f(y)$ and so on. We so obtain a trajectory that is usually called an **orbit** that can show either a stable or an unstable behavior. In the first case it gets stuck on a point or tends toward a point

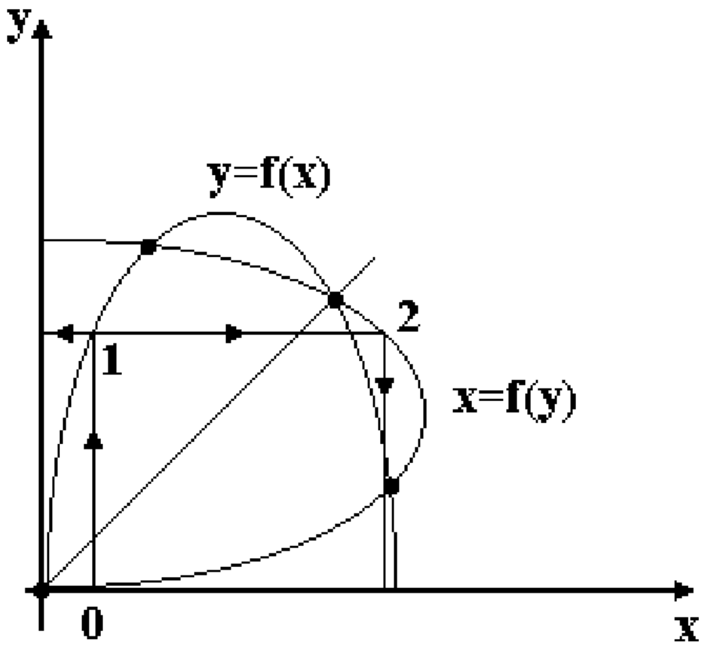

Figure 3: *Another graphic method (see text for details)*

(maybe without ever reaching it) whereas in the second case it diverges going more and more away from the starting point.
In figure 3 we suppose we have the curves whose equations are:

1. $y = -(x - a)^2 + b$
2. $x = -(y - a)^2 + b$

(with $a, b \in \mathbb{R}$) for which we have the four intersection points shown as small dots on the graph. The points that lie also on the straight line are stable equilibrium points whereas the others deserve further investigation.
It is very easy to step from the system of equations (32) to the following:

$$\begin{cases} y_{k+1} = f(f(x_k)) \\ x_{k+2} = y_{k+1} \end{cases} \tag{33}$$

2.4 Linear DDS with exponential growth

In this subsection we examine some simple models of linear DDS with exponential growth. Of such models we will present also applications in economics.

2.4.1 Simple case, single constant

Let us suppose we have

$$x_{k+1} = \lambda x_k \tag{34}$$

with an initial known value x_0 and with a constant $\lambda > 0$.
If we want to evaluate the fixed points (with $f(x) = \lambda x$) we must solve:

$$x = \lambda x \tag{35}$$

The only solution of equation (35) is $x = 0$. If we evaluate the absolute value of the derivative of equation $f(x) = \lambda x$ we get λ and so:

1. if $\lambda < 1$ we get that the only fixed point represents a stable equilibrium point,
2. if $\lambda > 1$ we get that the only fixed point represents a unstable equilibrium point.

Stability and instability must be understood in this sense: if we impose an even small perturbation to the system (so, in this case, we start with an initial value $x_0 \neq 0$) in the former case the evolution of the system is toward the fixed point (in this case 0) whereas in the latter case we have that the evolution of the system tends to go more and more away from the initial value and from the fixed point.
Another way of solving the problem is to find the closed form solution of equation (34). In the present case it is easy to see that such a closed form is:

$$x_k = \lambda^k x_0 \tag{36}$$

(for any $k > 0$ and $k \in \mathbb{N}$) so that

1. if $\lambda < 1$ we have that $x_k \to 0$ as $k \to +\infty$,
2. if $\lambda > 1$ we have that $\mid x_k \mid \to +\infty$ as $k \to +\infty$.

If $\lambda = 1$ the system has no evolution from the initial value that is, indeed, the only fixed point of the system since we have $x_{k+1} = x_k = x_0$.

2.4.2 More complex case, two constants

Now suppose we have

$$x_{k+1} = \lambda x_k + \mu \tag{37}$$

with $\lambda, \mu > 0$ and a known initial value x_0. Again we have a first order system (since the next value depends only on one previous value) of first degree but not linear (owing to the presence of the constant μ).
Performing by hand some simple repeated evaluations we have:

1. $x_1 = \lambda x_0 + \mu$
2. $x_2 = \lambda x_1 + \mu = \lambda^2 x_0 + \lambda\mu + \mu$

and so on, up to

$$x_k = \lambda^k x_0 + \mu(1 + \lambda + \lambda^2 + \cdots + \lambda^{k-1}) \tag{38}$$

By noting the presence of a geometric series we can simplify equation (38) as follows:

$$x_k = \lambda^k x_0 + \mu(\frac{1-\lambda^k}{1-\lambda}) \tag{39}$$

that is sought-for general solution. From equation (39) we have the constraint:

$$\lambda \neq 1 \tag{40}$$

If $\lambda = 1$ we have, indeed, that the general solution has the following form:

$$x_k = x_0 + k\mu \tag{41}$$

so that the system is unstable since the orbit goes more and more away from the initial state and do not approach any final state. If we see equation (41) as $x_k = g(k)$ we have that g represents an increasing and unbounded function of k so that an unstable behavior easily follows.
If we go back to equation (39) under the constraint of equation (40) we can simplify it as follows:

$$x_k = \lambda^k(x_0 - c) + c \tag{42}$$

with

$$c = \frac{\mu}{1-\lambda} \tag{43}$$

a constant depending on λ and μ. equation (42), apart from the presence of constant c, describes an exponential behavior that is

1. stable if $0 < \lambda < 1$ so that $\lambda^k \to 0$ and $x_k \to c$ as $k \to \infty$ with $x^* = c$;
2. unstable if $\lambda > 1$ so that $\lambda^k \to \infty$ and $x_k \to \infty$ as $k \to \infty$.

In the stable case we have that the fixed point is

$$x^* = c = \frac{\mu}{1-\lambda} \tag{44}$$

how it can be easily shown by solving (according to equation (37)):

$$x = \lambda x + \mu \tag{45}$$

2.4.3 More and more complex: three constants

Now we suppose to have three strictly positive constants λ, μ, r with the constraint $\lambda \neq r$ so that the system we have can be described with the following first order, first degree non linear equation

$$x_{k+1} = \lambda x_k + \mu r^k \tag{46}$$

with a known initial value x_0. As usually we have two paths to follow:

1. find fixed points and evaluate, for those points, the absolute value of the derivative;

2. find the closed form solution and study its behavior as a function of k.

Let's start with the first path.
This path is not really promising owing to the presence of the term r^k. All we can obtain in this way is that, owing to the constraints on the constants, if $r > 1$ the system shows an unstable behavior.
Let's step to the second path and write down a tentative general solution of the form:

$$x_k = b\lambda^k + cr^k \tag{47}$$

with b, c constants to be determined. From equation (47) we determine:

$$x_{k+1} = b\lambda^{k+1} + cr^{k+1} \tag{48}$$

By substituting equations (47) and (48) in equation (46) we get:

$$b\lambda^{k+1} + cr^{k+1} = b\lambda^{k+1} + c\lambda r^k + \mu r^k \tag{49}$$

By a simplification we get

$$cr = c\lambda r + \mu \tag{50}$$

so that

$$c = \frac{\mu}{r - \lambda} \tag{51}$$

(from which we see the necessity of the constraint $\lambda \neq r$). To evaluate the other constant (b) we use equation (47) and make use of the initial known value x_0 so to write:

$$x_0 = b + c \tag{52}$$

and finally:

$$b = x_0 - c = x_0 - \frac{\mu}{r - \lambda} \tag{53}$$

From equation (47), that represents the general solution in closed form (and on the ground of equations (51) and (53)), we can conclude that:

1. the system shows an unstable behavior if either $r > 1$ or $\lambda > 1$;
2. the system shows a stable behavior if $r < 1$ and $\lambda < 1$.

We note that the tentative general solution of equation (47) is modeled on the form of the equation (46): by an examination of such equation we see that through recursion we have an accumulation on λ so the necessity of λ^k. The need of the other term comes from the presence of an analogous term in equation (46).
Let's now examine briefly the case $\lambda = r$. In this case we have (from equation (46)):

$$x_{k+1} = \lambda x_k + \mu \lambda^k \tag{54}$$

with a known initial value x_0. Now we can guess a general solution of the form:

$$x_k = b\lambda^k + c \tag{55}$$

By substituting and simplifying we get

$$c = \frac{\mu \lambda^k}{1 - \lambda} \tag{56}$$

with $\lambda \neq 1$. Such a solution does not represent a real solution since c depends on k. The problem lies in the wrong choice of the tentative solution of equation (54) and will be examined in detail in the next section. Anyway we can say that:

1. if $0 < \lambda < 1$ then $c \to 0$ as $k \to +\infty$,
2. if $\lambda > 1$ then $c \to -\infty$ as $k \to +\infty$.

If, moreover, we have $\lambda = 1$ from equation (54) we get:

$$x_{k+1} = x_k + \mu \tag{57}$$

from which we get the following general solution in closed form:

$$x_k = x_0 + k\mu \tag{58}$$

and so (since by hypothesis $\mu > 0$) a diverging behavior.

2.4.4 Last example: a DDS with two constants (or what happens if $\lambda = r$)

Suppose we have λ, μ both strictly positive and the following equation:

$$x_{k+1} = \lambda x_k + \mu \lambda^k \tag{59}$$

In this case as a tentative solution we can try:

$$x_k = (b + ck)\lambda^k \tag{60}$$

with b, c to be determined. Substituting equation (60) in equation (59) and simplifying we get:

$$c = \frac{\mu}{\lambda} \tag{61}$$

To find b we use x_0 and put $k = 0$ in equation (60) so that:

$$b = x_0 \tag{62}$$

As a general solution we get again an exponential growth of the form:

$$x_k = (x_0 + k\frac{\mu}{\lambda})\lambda^k \tag{63}$$

In this case we have:

1. if $\lambda > 1$ the system shows an unstable behavior,
2. if $\lambda < 1$ the system shows a stable behavior.

If $\lambda = 1$ we get:

$$x_k = x_0 + k\mu \tag{64}$$

and, again, a diverging behavior.

3 Third lesson

3.1 Introduction

Before stepping to some easy economic applications of what we have seen so far we review some concepts we have examined in the past lessons so we go back to the following system of equations:

$$\begin{cases} y_{k+1} = f(x_k) \\ x_{k+1} = y_{k+1} \end{cases} \tag{65}$$

System (65) must be solved having a starting point x_0 (cf. figure 4 in which

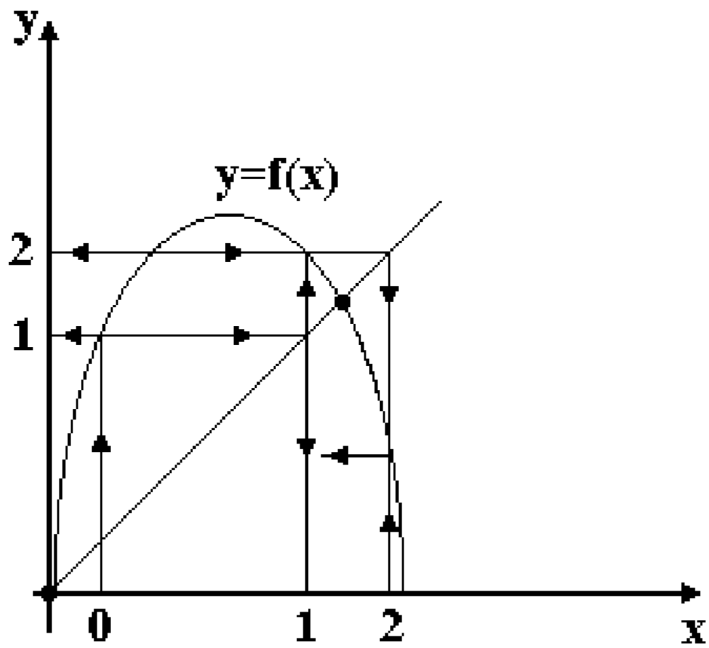

Figure 4: *Again on graphic methods (see text for details)*

points are enumerated only with the increasing value of k).
More complex models are those involving difference equations of the first order but of second degree such as the following:

$$x_{k+1} = \varepsilon[1 - x_k]x_k \tag{66}$$

in which ε is a parameter. Some instances are the followings:

1. if $\varepsilon = 3.5$ we have $x_{k+1} = 3.5[1 - x_k]x_k$;

2. if $\varepsilon = 3.84$ we have $x_{k+1} = 3.84[1 - x_k]x_k$.

Figure 5 shows two tests, one for each of the aforesaid equations, the first is for $\varepsilon = 3.5$ whereas the second is for $\varepsilon = 3.84$. We easily see that in the first case we have an orbit with a period of three whereas in the second case we have an orbit with period four.
To define fixed points for equation (66) we have to solve:

$$x = \varepsilon(1 - x)x \tag{67}$$

The solutions of equation (67) are easily evaluated in

1. $x_0 = 0$,
2. $x_1 = \frac{\varepsilon - 1}{\varepsilon}$

so that

1. if $\varepsilon = 3.5$ we have $x_1 = 0,7$
2. if $\varepsilon = 3.84$ we have $x_1 = 0,74$.

1	0,5	0,5
2	0,88	0,96
3	0,38	0,15
4	0,83	0,48
5	0,5	0,96
6	0,87	0,15
7	0,38	0,49
8	0,83	0,96
9	0,5	0,15
10	0,87	0,48
11	0,38	0,96
12	0,83	0,15
13	0,5	0,49
14	0,87	0,96
15	0,38	0,15
16	0,83	0,49
17	0,5	0,96
18	0,87	0,15
19	0,38	0,49
20	0,83	0,96
21	0,5	0,15
22	0,87	0,49
23	0,38	0,96
24	0,83	0,15
25	0,5	0,49

Figure 5: $\varepsilon = 3.5$ *(first column),* $\varepsilon = 3.84$ *(second column, see text for details)*

3.2 Examples from economics: bank account and mortgage

We consider a **bank account** with an initial capital x_0 and we want a model of the compound interest assuming that no money is withdrawn from that account. We can use one of our previously seen models and precisely the simplest model of exponential growth:

$$x_{k+1} = \lambda x_k \tag{68}$$

with a known initial value x_0 that we suppose, for the present case, strictly positive. If we pose

$$\lambda = 1 + \alpha \tag{69}$$

with α denoting the interest defined as

$$\alpha = \frac{I}{T} \tag{70}$$

where I is the interest rate and T is the time period used for the calculation, both being known quantities. We already know that equation (68) general closed form solution has the following form:

$$x_k = \lambda^k x_0 \tag{71}$$

so that by a simple substitution we get:

$$x_k = (1 + \frac{I}{T})^k x_0 \tag{72}$$

This unrealistic model describes an exponential unbounded growth to $+\infty$ if $x_0 > 0$ but also an exponential unbounded growth to $-\infty$ if $x_0 < 0$.
Let us examine the case of a **mortgage repayment**. In this case the model is represented by the following equation:

$$x_{k+1} = \lambda x_k + \mu \tag{73}$$

with

1. $\lambda = (1 + \frac{I}{T})$,
2. $\mu = -R$

where I, T and x_0 are known entities (with the same meaning they had in the previous case) and R is the entity of the repayment and is a fixed quantity. In this case the general solution in closed form is the following:

$$x_k = \lambda^k (x_0 - c) + c \tag{74}$$

with

$$c = R\frac{T}{I} \tag{75}$$

so that we can write:

$$x_k = (1 + \frac{I}{T})^k (x_0 - R\frac{T}{I}) + R\frac{T}{I} \tag{76}$$

For equation (76) to have an economic validity we must impose that:

$$x_0 - \frac{RT}{I} \tag{77}$$

has a negative sign (since the amount to be paid must decrease with time) so that the following inequality must be satisfied:

$$R > x_0\frac{I}{T} \tag{78}$$

I this way we have:

1. for $k = 0$ we have $x_0 = x_0$,
2. for $k > 0$ the amount to be paid decreases,
3. the payment stops for a value of k such that $x_k = 0$.

3.3 More complex models

When we use difference equations we refer to discrete models in which time rolls in discrete units fro k to $k + 1$ and so on. Now we put all this in relation with the continuous counterpart so to define the so called Continuous Dynamic Systems (or CDS). The natural model for CDS is made by **differential equations**. Let's now see how we can switch from **difference equations** to **differential equations** with a very simple example.
The starting point is a difference equation:

$$x_{k+1} = \lambda x_k = x_k + \lambda x_k - x_k \tag{79}$$

Equation (79) can be rewritten as:

$$\Delta x_{k+1} = x_{k+1} - x_k = (\lambda - 1)x_k \tag{80}$$

In both the first and the last member of the chain of equalities (80) we can imagine the presence a $\Delta k = (k+1) - k = 1$ as a divisor so that switching to the continuous counterpart (and using y to denote the continuous dependent variable and t the independent variable time) we can rewrite equation (80) as follows:

$$\frac{dy}{dt} = (\lambda - 1)y \tag{81}$$

with a known initial value

$$y_0 = y(t_0) \tag{82}$$

in which t_0 represents an arbitrarily fixed initial time.
Equation (81) is a type of differential equation also known as with separable variables and represents a model for CDS. The solution of such a family of equations springs from the following steps:

1. we start with equation (81) and rewrite it as follows:

$$\frac{dy}{y} = (\lambda - 1)dt \tag{83}$$

2. integrating equation (83) we get:

$$ln\, y = (\lambda - 1)t + k^* \tag{84}$$

3. using exponentials we obtain:

$$y = Ke^{(\lambda-1)t} \tag{85}$$

   with $K = e^{k^*}$

4. putting $t = 0$ and using the known value y_0 we get $y_0 = K = e^{k^*}$.

In general (so for a generic initial time t_0) we have:

$$y = y_0 e^{(\lambda-1)(t-t_0)} \tag{86}$$

in which $\frac{1}{\lambda-1}$ plays the role of a time constant. Equation (86) describes the so called Malthusian law of growth and describes:

1. a stable behavior if $\lambda - 1 < 0$ and so $0 < \lambda < 1$,

2. an unstable behavior if $\lambda - 1 > 0$ and so $\lambda > 1$.

In the first case we have:

$$lim_{t\to+\infty} y = 0 \tag{87}$$

for any finite initial value y_0.

3.4 Bounded linear DDS

For this family of systems we make use of models of the following form:

$$x_{k+1} = x_k + \varepsilon(x_\infty - x_k) \tag{88}$$

where:

1. x_0 is a known initial value,
2. x_∞ is the saturation value theoretically reached at $t = \infty$,
3. ε is a known parameter.

We have two cases:

1. the initial value is lower than the saturation value or $x_0 < x_\infty$, in this case the curve that describes the behavior increases till it reaches x_∞;
2. the initial value is greater than the saturation value or $x_0 > x_\infty$, in this case the curve that describes the behavior decreases till it reaches x_∞.

We can rewrite equation (88) as follows:

$$x_{k+1} = (1 - \varepsilon)x_k + \varepsilon x_\infty \tag{89}$$

or

$$x_{k+1} = \lambda x_k + \mu \tag{90}$$

with $\lambda = (1 - \varepsilon)$ and $\mu = \varepsilon x_\infty$. A we have already seen, the general solution in closed form of equation (90) is:

$$x_k = \lambda^k x_0 + \mu(\frac{1 - \lambda^k}{1 - \lambda}) = \lambda^k(x_0 - \frac{\mu}{1 - \lambda}) + \frac{\mu}{1 - \lambda} \tag{91}$$

we require $\lambda > 0$ so that we must impose $\varepsilon < 1$.
If we substitute $\lambda = (1 - \varepsilon)$ and $\mu = \varepsilon x_\infty$ in the last member of equation (91) we obtain:

$$x_k = (1 - \varepsilon)^k(x_0 - x_\infty) + x_\infty \tag{92}$$

Since we want a stable evolution we must impose:

$$\mid 1 - \varepsilon \mid < 1 \tag{93}$$

or (since $\varepsilon > 0$)

$$0 < \varepsilon < 2 \tag{94}$$

Disequations (94) define a model with saturation (with or without dumped oscillations around the saturation value) whereas if $\varepsilon > 2$ we have no stable solution and so the corresponding model is no more with saturation.
The continuous counterpart of equation (88) has the following form:

$$\frac{dy(t)}{dt} = \varepsilon(x_\infty - y(t)) \tag{95}$$

(we remind that we use y to denote a continuous variable, function of time , as in this case). On this ground we can say that:

1. if $x_\infty > y(t)$ the derivative is positive and so the variable $y(t)$ is increasing with time;

2. if $x_\infty < y(t)$ the derivative is negative and so the variable $y(t)$ is decreasing with time.

Such conclusions cab be contradicted by what happens in case of discrete functions (cf. below for some more comments). Figure 6 shows some calculations made with a spread sheet and applying equation (88) in the following cases:

1. **column A**: $\varepsilon = 0,5$, $x_0 = 0,25$;

2. **column B**: $\varepsilon = 1,9$, $x_0 = 0,25$;

3. **column C**: $\varepsilon = 2,1$, $x_0 = 0,25$.

In all cases we have $x_\infty = 1$. It is worth noting what follows:

1. in the first case ($\varepsilon = 0,5$) we can observe a quick growth to the saturation value without any oscillations,

2. in the second case ($\varepsilon = 1,9$) we can observe an initial overshoot followed by a dumped oscillation around the saturation value that represents the asymptotic value,

3. in the third case ($\varepsilon = 2,1$) we can observe an unbounded oscillation around the saturation value.

Such behaviors represent an empirical corroboration of the theoretical observations made in the foregoing paragraphs. Some more data are contained in figure 7. We only note that:

1. in all the columns D, E, F we have put $x_0 = 1,2$ so the initial value is above the saturation value (which is $x_\infty = 1$),

2. **column D**: $\varepsilon = 0,5$,
3. **column E**: $\varepsilon = 1,9$,
4. **column F**: $\varepsilon = 2,1$,
5. the shown behaviors, apart from some initial differences (only due to the values of x_0) coincide (from a qualitative point of view) with those shown in figure 6.

	A	B	C
1	0,25	0,25	0,25
2	0,63	1,68	1,83
3	0,81	0,39	0,09
4	0,91	1,55	2
5	0,95	0,51	-0,1
6	0,98	1,44	2,21
7	0,99	0,6	-0,33
8	0,99	1,36	2,46
9	1	0,68	-0,61
10	1	1,29	2,77
11	1	0,74	-0,95
12	1	1,24	3,14
13	1	0,79	-1,35
14	1	1,19	3,59
15	1	0,83	-1,85
16	1	1,15	4,13
17	1	0,86	-2,45
18	1	1,13	4,79
19	1	0,89	-3,17
20	1	1,1	5,59

Figure 6: ***column A**: $\varepsilon = 0,5$; **column B**: $\varepsilon = 1,9$; **column C**: $\varepsilon = 2,1$; all columns $x_0 = 0,25$; (see text for details)*

D	E	F
1,2	1,2	1,2
1,1	0,82	0,78
1,05	1,16	1,24
1,03	0,85	0,73
1,01	1,13	1,29
1,01	0,88	0,68
1	1,11	1,35
1	0,9	0,61
1	1,09	1,43
1	0,92	0,53
1	1,07	1,52
1	0,94	0,43
1	1,06	1,63
1	0,95	0,31
1	1,05	1,76
1	0,96	0,16
1	1,04	1,92
1	0,97	-0,01
1	1,03	2,11
1	0,97	-0,22

Figure 7: ***column D***: $\varepsilon = 0,5$, ***column E***: $\varepsilon = 1,9$, ***column F***: $\varepsilon = 2,1$*(see text for details)*

4 Fourth lesson

4.1 Introduction

In the third lesson we have examined bounded linear DDS either with an oscillating behavior or not. We have also looked for equilibrium points and we have, at least, examined their continuous counterpart.
As to the bounded linear DDS we note that we call them linear in a somewhat imprecise manner since they do not behave linearly neither their descriptive equation is (strictly speaking) linear. One model for bounded linear DDS is, indeed, the following:

$$x_{k+1} = x_k + \varepsilon(x_\infty - x_k) \tag{96}$$

with a known initial value $x_0 > 0$, a known (or guessed) saturation value x_∞ and a known parameter ε. Equation (96) can be rewritten as

$$x_{k+1} = (1 - \varepsilon)x_k + \varepsilon x_\infty \tag{97}$$

so hat its general solution in closed form is:

$$x_k = (1 - \varepsilon)^k (x_0 - x_\infty) + x_\infty \tag{98}$$

with x_∞ representing the saturation value or the value that (in cases of stability) is attained (theoretically) for $k = +\infty$.
From equation (98) it is easy to see that:

1. if $0 < \varepsilon < 2$ then the system shows a stable behavior;
2. if $\varepsilon > 2$ then the system shows an unstable behavior (so it oscillates around x_∞).

4.2 Bounded linear CDS

In this section we step to the continuous counterpart of the present calss of DDS and so we talk about CDS. Rewriting equation (96) as follows:

$$\Delta x_{k+1} = x_{k+1} - x_k = \varepsilon(x_\infty - x_k) \tag{99}$$

and using y to denote the continuous dependent variable (that varies as a continuous function of time) we can write:

$$D_t y = \varepsilon(y_\infty - y) \tag{100}$$

(we remind that with $D_t y$ we denote the derivative of y with respect to t). The general solution of equation (100) (if we have a known initial value $y_0 = y(t_0)$) is:

$$y = y_\infty - (y_\infty - y_0)e^{-\varepsilon(t-t_0)} \tag{101}$$

In equation (101) quantity $1/\varepsilon$ plays the role of a time constant. We can easily see that the solution described by equation (101):

1. can never show an oscillating behavior as it can happen with the corresponding DDS;

2. is stable iff $\varepsilon > 0$ and the greater it is the faster the solution reaches its saturation value x_∞;

3. if $t = t_0$ we have $y = y_0$ whereas as $t \to \infty$ we get $y \to y_\infty$.

If we evaluate the derivative (with respect to time) of equation (101) we obtain:

$$\frac{dy}{dt} = t(y_\infty - y_0)e^{-\varepsilon(t-t_0)} > 0 \tag{102}$$

so the graph is that of an increasing function but with a decreasing slope (since the function described by equation (102) gets lower and lower values as t increases).
A more complex model is represented by that described with the **logistic** function.
In this case we have:

1. a slow initial (or t near to t_0) growth (small values of the derivative);

2. a faster growth as t increases;

3. an inflection point;

4. after the inflection point, a slower and slower growth toward a saturation value.

All this under the hypothesis that $y_\infty > y_0$. Logistic curve allows the inclusion of reality effect in our model since, for instance if we model a population, the slow initial growth accounts for the small number of individuals that are in the reproductive age whereas the approach to a saturation value takes into account the effect of some physical limiting factor (food, land and so on) on the growth of the population (that, otherwise, would be unbounded). To describe a logistic behavior, in either DDS or CDS, we have to switch to more complex expressions, as we will see in the next section.

4.3 Quadratic DDS

In this case we have a first order second degree difference equation of the form:

$$x_{k+1} = x_k + \varepsilon(1 - x_k)x_k \tag{103}$$

where we have $x_\infty = 1$ (such a value is usually called **carrying capacity**). We note that:

1. fixed points of equation (103) are $x_0^* = 0$ and $x_1^* = 1$ for any value of ε;
2. $x_0 = 0$ is an unstable point at 0 (see later on in the present section);
3. if we have $x_0 = 1$ in one step the system reaches the same condition as before.

Again we can rewrite equation (103) as follows:

$$\Delta x_{k+1} = x_{k+1} - x_k = \varepsilon(1 - x_k)x_k \tag{104}$$

In equation (103) we have $\varepsilon > 0$ and known and the aforesaid fixed points. To study the nature of those fixed points we can evaluate the derivative of:

$$f(x) = x + \varepsilon(1 - x)x \tag{105}$$

and study its absolute value in such points. In the present case we have:

$$\frac{df}{dx} = 1 + \varepsilon - 2\ x \tag{106}$$

and then

1. in $x_0^* = 0$ $\frac{df}{dx} = 1 + \varepsilon > 1$ since $\varepsilon > 0$ so that $x_0^* = 0$ is not a stable equilibrium point;
2. in $x_1^* = 1$ $\frac{df}{dx} = 1 - \varepsilon$ so that we must impose that $\mid 1 - \varepsilon \mid < 1$ (since $\varepsilon > 0$) therefore $x_1^* = 1$ is a stable equilibrium point. In this case we have that, solving $\mid 1 - \varepsilon \mid < 1$, the constraints $0 < \varepsilon < 2$ allow the the fixed point $x_1^* = 1$ to be a stable equilibrium point;
3. until ε remains far from the limiting value 2 we have no oscillation and the more it approaches to 2 we get more and more wide oscillations around the equilibrium point 1 and also the transitory phase tends to last longer;

4. when $\varepsilon > 2$ the system shows a more and more unstable behavior with wider and wider oscillations around the point 1 that cannot be classified as an equilibrium point anymore.

Going back to equation

$$x_{k+1} = x_k + \varepsilon(1 - x_k)x_k \tag{107}$$

we note that it represents a parabola of the family:

$$y = x + \varepsilon(1 - x)x = -\varepsilon x^2 + x(1 - \varepsilon) \tag{108}$$

so that we can evaluate its maximum value that occurs at (easily evaluate the derivative and equate it to 0):

$$x = \frac{1 - \varepsilon}{2\varepsilon} \tag{109}$$

and is

$$y = \frac{(1 - \varepsilon)^2}{2\varepsilon} \tag{110}$$

Using a method already seen in the past we can rewrite $x = f(x)$ as a system of equations:

$$\begin{cases} y = f(x) \\ x = y \end{cases} \tag{111}$$

so that we can work out by using two intersecting curves and an initial point: in this way we can guess the value of the saturation value to be used as either x_∞ or y_∞ depending on which type of system (DDS or CDS) we are studying.

4.4 Logistic CDS

At this point (with the usual techniques) we find the continuous counterpart of equation (103):

$$D_t y = \varepsilon(1 - y)y \tag{112}$$

with

1. $y_0 = y(t_0)$ known;
2. $y_\infty = 1$ for the sake of simplicity.

If we rewrite equation (112) as follows:

$$\frac{1}{y(1 - y)} dy = \varepsilon dt \tag{113}$$

and integrate it with usual integration methods we get (for $t_0 = 0$):

$$y = \frac{y_0}{y_0 + (1 - y_0)e^{-\varepsilon t}} \tag{114}$$

whereas if $t_0 \neq 0$ we get:

$$y = \frac{y_0}{y_0 + (1 - y_0)e^{-\varepsilon(t - t_0)}} \tag{115}$$

If we consider again equation (112) and evaluate its derivative we get:

$$D_t^2 y = \varepsilon(1 - 2\varepsilon y) D_t y \tag{116}$$

so that the maximum value of $D_t y$ is obtained when $D_t^2 y = 0$ and so for:

$$y = \frac{1}{2\varepsilon} \tag{117}$$

5 Fifth lesson

5.1 Logistic CDS

We go back again to the equation describing a CDS with a saturating behavior:

$$D_t y = \varepsilon(1-y)y \tag{118}$$

with

1. $y_0 = y(t_0)$ known;
2. $\varepsilon > 0$ known;
3. $y_\infty = 1$ for the sake of simplicity.

The general solution of equation (118) is

$$y = \frac{y_0}{y_0 + (1-y_0)e^{-\varepsilon(t-t_0)}} \tag{119}$$

We can use equation (119) to model the growth of a population form a starting time and an initial value for a certain period of time.

5.2 CDS and DDS

We now restate the equation for a CDS as follows:

$$\frac{dy}{dt} = \varepsilon(1-\frac{y}{K})y \tag{120}$$

with:

1. $\varepsilon > 0$ known;
2. K estimated in some way and representing the saturation value;
3. if $y < K$ we have $\frac{dy}{dt} > 0$ so that for values lower than he saturation value the behavior is of increasing type;
4. if $y > K$ we have $\frac{dy}{dt} < 0$ so that for values higher than he saturation value the behavior is of decreasing type.

For the DDS we have:

$$x_{k+1} = x_k + \varepsilon(1-\frac{x_k}{K})x_k \tag{121}$$

that can be rewritten as:

$$\Delta x_{k+1} = x_{k+1} - x_k = \varepsilon(1-\frac{x_k}{K})x_k \tag{122}$$

5.3 Quadratic DDS

We now step to a class of DDS for which no general solution exists, the so called **quadratic DDS** whose descriptive equation has the following form:

$$x_{k+1} = Ax_k^2 + 2Bx_k + C \tag{123}$$

with:

1. A, B, C known constants;
2. x_0 known initial value.

The technique that is usually used in these cases is that of guessing a general form of a tentative solution, write it down using some parameters to be determined and then:

1. substitute such a tentative solution in equations such as (123) ,
2. perform all the possible calculations and
3. evaluate the unknown parameters by equating the coefficients of analogous terms.

We note that such a procedure can effectively work since as composing elements we use functions that form a basis of a vector space. In our case the tentative solution of equation (123) has the form:

$$x_k = \alpha e^{i\chi_k} + \beta e^{-i\chi_k} + \gamma \tag{124}$$

with:

1. $\chi_k = 2^k\theta_0$;
2. θ_0 known;
3. $e^{i\chi_k} = cos\chi_k + isen\chi_k$;
4. α, β, γ constants to be calculated.

If we suppose $\alpha = \beta$ we get (by using Euler's rules):

$$\alpha e^{i\chi_k} + \beta e^{-i\chi_k} = cos\chi_k + isen\chi_k + cos\chi_k - isen\chi_k = 2cos\chi_k \tag{125}$$

and so a periodic oscillating solution. If we observe that if for a given k (and so x_k) we have:

$$\chi_k = 2^k\theta_0 \tag{126}$$

then for $k+1$ (and so x_{k+1}) we have:

$$\chi_{k+1} = 2^{k+1}\theta_0 = 2 \times 2^k\theta_0 = 2\theta_0 \tag{127}$$

If now we substitute equation (124) in the equation (123) we get:[1]

$$\alpha e^{i2\chi_k} + \beta e^{-i2\chi_k} + \gamma = A[\alpha e^{i\chi_k} + \beta e^{-i\chi_k} + \gamma]^2 + 2B[\alpha e^{i\chi_k} + \beta e^{-i\chi_k} + \gamma] + C \tag{129}$$

If we consider that the functions $e^{2i\chi_k}$, $e^{-2i\chi_k}$, $e^{i\chi_k}$ and $e^{-i\chi_k}$ belong to a base of a vector space (and so are linearly independent one from the others) if we perform the calculations and equate the coefficients of the corresponding terms we get:

1. $\alpha = A\alpha^2$ so that $\alpha = 1/A$;
2. $\beta = A\beta^2$ so that $\beta = 1/A$;
3. $A\gamma + B = 0$ since we have no terms involving $e^{i\chi}$.

From such equations we see that:

1. we must impose $A \neq 0$ (and tis is obvious since we want to use a quadratic model);
2. we have $\alpha = \beta$;
3. $\gamma = -\frac{B}{A}$.

We have also one more equation that represents a constraint on C:

$$\gamma = 2A\alpha\beta + A\gamma^2 + 2B\gamma + C \tag{130}$$

If we substitute in equation (130) the values we have found for α, β and γ we get the following relation among A, B and C:

$$C = \frac{B^2 - B - 2}{A} \tag{131}$$

Such an equation represents an added constraints on the values of the given constants A, B, C.

[1] We use the following equivalence:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \tag{128}$$

with $a = \alpha e^{-i\chi_k}$, $b = \alpha e^{i\chi_k}$ and $c = \gamma$

At this point (on the ground of the foregoing equations and relations) we rewrite equation (124) for $k+1$

$$x_{k+1} = \alpha e^{i2\chi_k} + \beta e^{-i2\chi_k} + \gamma \tag{132}$$

In this way we have that the general solution of equation (123) with C given by equation (131) is:

$$x_k = \frac{e^{i2\chi_k} + e^{-i2\chi_k}}{A} - \frac{B}{A} \tag{133}$$

or

$$x_k = \frac{2cos\chi}{A} - \frac{B}{A} = \frac{2}{A}cos(2^k\theta_0) - \frac{B}{A} \tag{134}$$

Now we use the initial value x_0 to evaluate θ_0. If we put $k = 0$ in equation (134) we get:

$$x_0 = \frac{2}{A}cos\theta_0 - \frac{B}{A} \tag{135}$$

and, at last (with easy mathematics):

$$\theta_0 = arccos(\frac{Ax_0 + B}{2}) \tag{136}$$

In this way we have obtained the initial value of the angle. At this point, putting all things together, we obtain:

$$x_k = \frac{2}{A}cos(2^k\, arccos(\frac{Ax_0 + B}{2})) - \frac{B}{A} \tag{137}$$

and this gives rise to the following constraint: $\mid Ax_0 + B \mid \leq 2$ owing to the interval of variation of cosine function.
In equation (137) we have that x_0, A, B represent known values that must satisfy the aforesaid constraint and the condition on C of equation (131).

5.4 The Chebyshev polynomials

We now introduce the definition of Chebyshev polynomials (CPs) as follows. A CP is a function, defined between -1 and $+1$, of the form:

$$T_i(x) = cos(i \times arccos(x)) \tag{138}$$

that defines a polynomial of degree i since the parameter i defines the number of the zeroes. If we put $i = 2^k$ we have that such a number is determined by k in such a way that, for instance:

1. if $k = 5$ we have a polynomial of degree 5 and so with 5 zeroes;

2. if $k = 21$ we have a polynomial of degree 21 and so with 21 zeroes and so on.

By using such a definition we can rewrite equation (137) as

$$x_k = \frac{2}{A} T_{2^k}(\frac{Ax_0 + B}{2}) - \frac{B}{A} \tag{139}$$

If now we pose $A = 2$ and $B = 0$ (from equation (131) we have $C = -1$) equation (123) becomes:

$$x_{k+1} = 2x_k^2 - 1 \tag{140}$$

so that we can study its evolution as a function of x_0. Fixed points of equation (140) can be easily evaluated and are:

1. $x_1 = -\frac{1}{2}$,
2. $x_2 = 1$

so we can study its behavior starting, for instance, with $x_0 = 0.25$. We get a so called chaotic behavior since it is non periodic and shows very quick variations.
Last but not least we have to take care of the constraint represented by equation (131) that we rewrite here for the sake of convenience:

$$C = \frac{B^2 - B - 2}{A} \tag{141}$$

In such equation all the elements represent known values so that if, for instance, we have $B = 2$ then we must necessarily have $C = 0$.
In this case equation (123) becomes:

$$x_{k+1} = Ax_k^2 + 4x_k \tag{142}$$

and can be rewritten as:

$$x_{k+1} = (4 + A)x_k - A(1 - x_k)x_k \tag{143}$$

that shows a logistic behavior.
Next lesson we will investigate what happens if $\mid Ax_0 + B \mid > 2$ and we will find a general solution of the problem modeled with equation (123) under the aforesaid condition.

6 Sixth lesson

Now we step to the examination of the following expression:

$$x_k = \frac{1}{2}[1 - T_{2^k}(1 - 2x_0)] \tag{144}$$

with an initial value $0 < x_0 < 1$. We can write:

$$x_k = \alpha e^{\chi_k} + \beta e^{-\chi_k} + \gamma \tag{145}$$

with $\chi_k = 2^k\theta_0$. The general expression is:

$$x_{k+1} = Ax_k^2 + 2Bx_k + C \tag{146}$$

and with an initial value x_0 such that $\mid Ax_0 + B \mid > 2$. In equation (145) we have that x_0 is known whereas α, β, γ must be determined with some calculations on equation (146).
If we substitute equation (145) in equation (146), perform calculations and simplifications and equate the coefficients of the corresponding terms we, finally, get:

1. $\gamma = \frac{-B}{A}$,
2. $\alpha = \frac{1}{A}$,
3. $\beta = \frac{1}{A}$.

We can easily see that $\alpha = \beta$ so that equation (145) becomes:

$$x_k = \frac{1}{A}(e^{\chi_k} + e^{-\chi_k}) - \frac{B}{A} = \frac{1}{A}2cosh(\chi_k) - \frac{B}{A} \tag{147}$$

so that at the end we get:

$$x_k = \frac{1}{A}2cosh(2^k\theta_0) - \frac{B}{A} \tag{148}$$

If we evaluate θ_0 at $k = 0$ and use the initial value x_0 (as we have already done in the past lessons) we get:

$$x_k = \frac{1}{A}2cosh(2^k arccosh(\frac{A}{2}x_0 + \frac{B}{2})) - \frac{B}{A} \tag{149}$$

Again we must impose:

1. $A \neq 0$,
2. $C = (B^2 - B - 2)A^{-1}$

In this case we do not perform any analysis of stability.

6.1 Quadratic DDS, particular form

We use the following form of quadratic DDS:

$$x_{k+1} = 4(1 - x_k)x_k \tag{150}$$

with a known initial value x_0.
Now we can study stability. Given

$$f(x) = 4(1 - x)x \tag{151}$$

we can study the behavior of the derivative or use it to write:

$$\Delta y = \mid \frac{df}{dx} \mid_{x=x^*} \Delta x \tag{152}$$

where x^* is a fixed point for function f. At this point we rewrite here equation (139):

$$x_k = \frac{2}{A} T_{2^k}(\frac{Ax_0 + B}{2}) - \frac{B}{A} \tag{153}$$

to see how the initial condition propagate to the solutions. If we evaluate the derivative of the expression of x_k and we remember that the derivative of $arccos(x)$ is

$$\frac{1}{\sqrt{1 - x^2}} \tag{154}$$

we can see haw the solutions show a highly oscillating behavior. The derivative of equation (153) is:

$$\frac{2^k sen 2^k arccos(\frac{Ax_0+B}{2})}{1 - (\frac{Ax_0+B}{2})^2} \tag{155}$$

We have:

1. if $\mid Ax_0 + B \mid \leq 2$ we have stable solutions;

2. if $\mid Ax_0 + B \mid = 2$ the denominator is equal to 0.

In case of the following quadratic DDS:

$$x_{k+1} = 4(1 - x_k)x_k \tag{156}$$

we can (since $f(x) = 4(1 - x)x$):

1. evaluate its fixed points;

2. study their nature (stable or unstable).

As to the first point we have that equation $x = 4(1-x)x$ has the following solutions:

1. $x_1^* = 0$
2. $x_2^* = 0.275$

As to the second point we evaluate

$$f'(x) = 4 - 8x \tag{157}$$

In this case in both fixed points the absolute value of the derivative is greater than one so that both fixed points are unstable equilibrium points.

6.2 Cubic DDS

We start with the following general form:

$$x_{k+1} = Ax_k^3 + Bx_k^2 + Cx_k + D \tag{158}$$

with A, B, C, D constants (of which A, B can vary freely while C, D are somewhat limited) and an initial know value x_0.
Using a "classic" technique we guess a solution of equation (158) as having the form:

$$x_k = \alpha e^{i\chi_k} + \beta e^{-i\chi_k} + \gamma \tag{159}$$

with

$$\chi_k = 3^k \theta_0 \tag{160}$$

In equation (160) we have that θ_0 is known whereas in equation (159) α, β, γ must be determined in the usual way. The steps we follow are:

1. substitution of the expressions for x_{k+1} and x_k in equation (158);
2. execution of the calculations and all the possible simplifications;
3. comparison of the corresponding coefficients.

Acting this way (calculations left to the reader) we get:

1. $\alpha = \pm \frac{1}{A}^{\frac{1}{2}}$,
2. $\beta = \pm \frac{1}{A}^{\frac{1}{2}}$,

3. $\gamma = -\frac{1}{3}\frac{A}{B}$,

4. $C = \frac{B^2}{3A} - 3$,

5. $D = \frac{B}{9A}(\frac{B^2}{9A} - 4)$.

We note that:

$$\chi_{k+1} = 3^{k+1}\theta_0 = 33^k\theta_0 = 3\chi_k \tag{161}$$

Substituting the expressions for C and D we get:

$$x_{k+1} = Ax_k^3 + Bx_k^2 + (\frac{1}{2}\frac{B^2}{A} - 3)x_k + \frac{1}{9}\frac{B}{A}(\frac{1}{9}\frac{B^2}{A} - 4) \tag{162}$$

whose general solution is:

$$x_k = \pm\frac{1}{\sqrt{A}}[e^{i\chi_k} + e^{-i\chi_k}] - \frac{A}{3B} \tag{163}$$

or

$$x_k = \pm\frac{2}{\sqrt{A}}[cos(3^k\theta_0] - \frac{A}{3B} \tag{164}$$

As the sign we choose the + sign since the values x_k are positive even if the effective choice depends on the chosen initial point. At this point we can obtain θ_0 as a function of the known initial value x_0. In this way we get (retaining the two signs to get a general solution):

$$x_0 = \pm\frac{2}{\sqrt{A}}cos(\theta_0) - \frac{A}{3B} \tag{165}$$

or

$$\theta_0 = arccos(\pm\frac{\sqrt{A}}{2}(x_0 + \frac{A}{3B})) \tag{166}$$

By substituting equation (166) in equation (164) we get x_k as a function of the initial value x_0.

7 Seventh lesson

7.1 Introduction

For one moment we go back to logistic equation for CDS. In this case the equation that describes the model has the following form:

$$\frac{dy}{dt} = \varepsilon(1 - \frac{y}{K})y \tag{167}$$

with two parameters:

1. K the carrying capacity,
2. ε

and a known initial value $y_0 = y(t_0)$ evaluated in an initial time t_0. Such a model can be easily solved and gives a relation among the aforesaid quantities to which corresponds a graph with an s-shape that starts at a low level, increases with a change in concavity from convex to concave and reaches a saturation value. Such a model can be used both to describe historical series of data so to ground them on a solid theoretical background and to predict the behavior of a model and so of a modeled system.

7.2 Continuous models: theory and practice

In this case we are sure that solutions exist and the same holds for equilibrium points that prove to be stable. We will, for the moment, examine some classical model that can be used for describing harvesting and fishing strategies starting with logistic CDS.
We rewrite here both the descriptive equation:

$$\frac{dy}{dt} = \varepsilon(1 - \frac{y}{y_\infty})y \tag{168}$$

(with two positive parameters ε and y_∞ and a given initial value y_0, usually non negative) and the general solution:

$$y = \frac{y_0}{\frac{y_0}{y_\infty} + (1 - \frac{y_0}{y_\infty})e^{-\varepsilon(t-t_0)}} \tag{169}$$

where y_∞ is the so called **carrying capacity** and represents the theoretical size that a modeled population will reach at $t = \infty$. For equation (168), indeed, we have:

1. when $t = t_0$ we have $y = y_0$,
2. when (theoretically) $t = \infty$ or (in practice) after an enough long period of time we have $y = y_\infty$.

We note that equation (167) represents a parabola and attains its maximum value for

$$y = \frac{\varepsilon y_\infty}{2} \tag{170}$$

and such a value is

$$\frac{\varepsilon^2 y_\infty}{2} > 0 \tag{171}$$

We can use this model to control the growth of a population through two strategies:

1. constant fishing or harvesting (so any time we fish or harvest we are bounded by a quota or a maximum quantity),
2. proportional fishing or harvesting: is an ex-post strategy so that each time we can fish or harvest a fraction of what we fished or harvested on the previous time.

7.2.1 Strategy number 1: constant fishing or harvesting

We try to keep the overall size of a population between two boundary values (a lower and an upper boundary) by removing at a given instant the net increase of the population defined as the difference between births and deaths (but this only if that balance is positive). So what we remove (we use this term for both fishing and harvesting) is dy/dt and, in practice, what we need is an evaluation of the time gap (or Δt) between two successive removals: usually such a value is modeled on biological parameters (such as the interval from one reproductive period and the successive). Since every time we remove a fixed quota of the population we can restate the equation that describes its variation (and so equation (168)) as:

$$\frac{dy}{dt} = \varepsilon(1 - \frac{y}{y_\infty})y - H \tag{172}$$

(with two positive parameters ε and y_∞ and a given initial value y_0, usually non negative). In this case we only know that H is strictly positive but we don't know its value. H represents the amount we remove each time from the total population. Since we don't want that the population from which we are removing risks extinction we want to maximize H under that constraint so we look for a point in which the growth of the population stops. In other words we want to solve

$$\frac{dy}{dt} = 0 \tag{173}$$

or

$$\varepsilon \frac{y^2}{y_\infty} - \varepsilon u + H = 0 \tag{174}$$

Equation (174) has the following solutions:

$$y_{1,2} = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - \frac{4\varepsilon}{y_\infty}H}}{\frac{2\varepsilon}{y_\infty}} \tag{175}$$

We now have that if we put $H = 0$ we obtain $y_1 = 0$ and $y_2 = y_\infty$. Since we want real solutions we must impose that the argument of the square root is non negative and therefore we impose:

$$\varepsilon^2 - \frac{4\varepsilon}{y_\infty}H \geq 0 \tag{176}$$

From equation (176) we get an upper limitation on the value of H:

$$H \leq \frac{\varepsilon}{4}y_\infty \tag{177}$$

In case of disequation (177), if we impose an equality we get

$$H = \frac{\varepsilon}{4}y_\infty \tag{178}$$

and, therefore, in equation (175) the square root vanishes and so we get:

$$y_1 = y_2 = \frac{1}{2}y_\infty \tag{179}$$

Figure 8 shows the limiting situation in which equations (178) and (179) hold. From that figure we can also see that the real parameter is ε so that since we remove a quantity equal to H each time and starting from a level equal to the one specified by equation (179) we must have:

$$\frac{\varepsilon}{4}y_\infty \leq \frac{y_\infty}{2} \tag{180}$$

so that the following relation must be verified:

$$\varepsilon \leq 2 \tag{181}$$

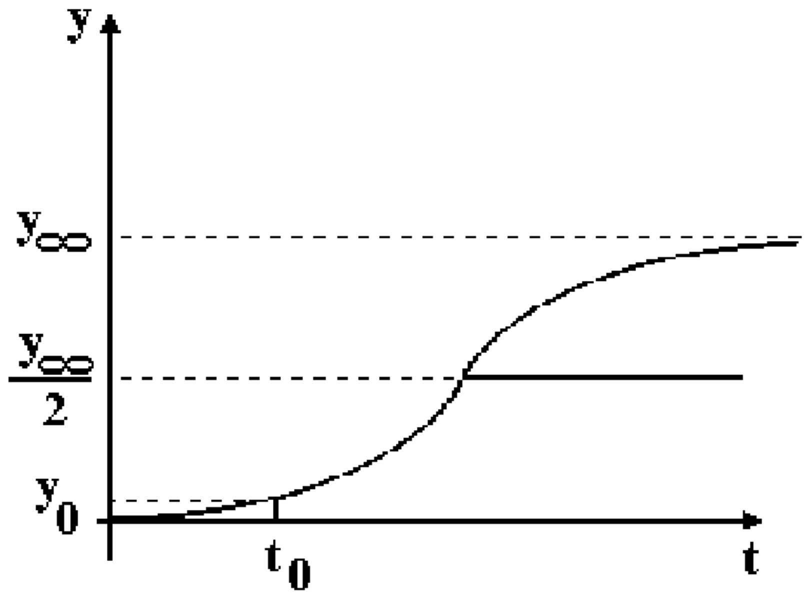

Figure 8: *Logistic curve with limiting level (see text for details)*

Figure 9 shows the behavior of the successive removal campaigns, each campaign collecting the maximum quota H with an interleave (Δt in figure 9) fixed on biological bases. With a maximum removal equal to H every each Δt population is kept at the level given by equation (179) with a removal that cannot be greater than the quantity specified by equation (178). The main risk in this case is that if we fix a wrong value for Δt the population can slope down till the extinction. We note that the behavior shown in figure 9 can be described also by the equations derived so far that give:

1. the equilibrium level (equation (179)),
2. the maximum possible removal (equation (178)),
3. instant of time at which start the removal, when the derivative attains its maximum value.

Each of the spikes can be seen as the part of a delayed copy of the basic curve with an upper cut of amplitude H above the line $y_\infty/2$. In a real world application every removal cannot occur in a zero time so each gap of amplitude H, in reality, is a segment with a slope and models a removal that occurs in a finite time $\delta(t)$.

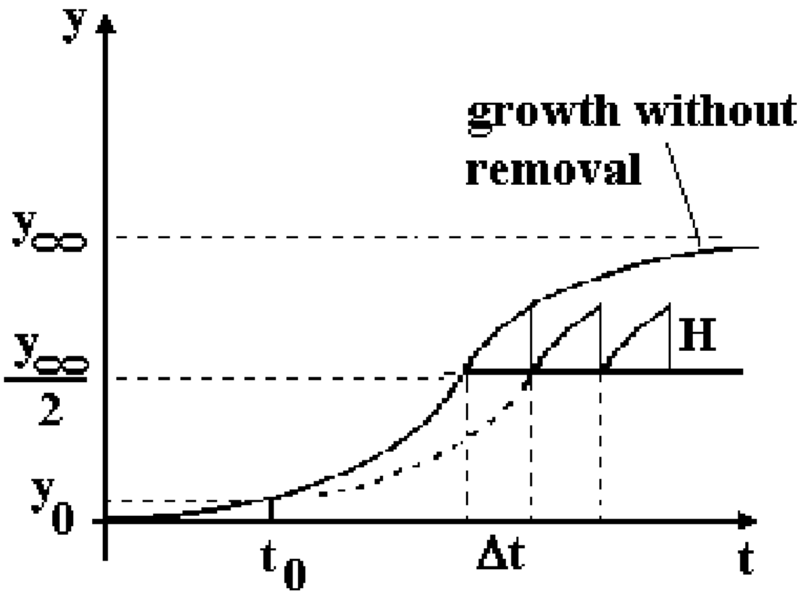

Figure 9: *Logistic curve with free growth and removal campaigns (see text for details)*

As a final note we say that the parameter H is usually called the **maximum sustainable yield of a dynamical system**.

7.2.2 Strategy number 2: proportional fishing or harvesting

In this case we suppose that the entity of the removal is proportional to the amount of the current population through a parameter λ so that we can rewrite the logistic equation in the following form:

$$\frac{dy}{dt} = \varepsilon(1 - \frac{y}{y_\infty})y - \lambda y \tag{182}$$

or

$$\frac{dy}{dt} = (\varepsilon(1 - \frac{\lambda}{\varepsilon}) - \frac{y}{y_\infty})y \tag{183}$$

in which we have $\varepsilon, \lambda, y_\infty$ that are strictly positive parameters and y_0 is a known initial value. We want to define the extremal points so we look for the zeroes of the derivative and so of equation (183). In this case we get the following value:

$$y_\chi = (1 - \frac{\lambda}{\varepsilon})y_\infty \tag{184}$$

in which the derivative vanishes. Such a value is smaller than the one we get in the usual logistic curve. Now we have to fix the value of he parameter λ. Since we want

$$0 < y_\chi < y_\infty \tag{185}$$

we must have

$$0 < (1 - \frac{\lambda}{\varepsilon}) < 1 \tag{186}$$

or

$$0 < \frac{\lambda}{\varepsilon} < 1 \tag{187}$$

and, at last:

$$0 < \lambda < \varepsilon \tag{188}$$

Since we want to maximize

$$\frac{dy}{dt} \tag{189}$$

we evaluate the second derivative and equate it to zero so that we obtain:

$$\frac{d^2y}{dt^2} = \varepsilon(1 - \frac{\lambda}{\varepsilon} - 2\frac{y}{y_\infty})\frac{dy}{dt} = 0 \tag{190}$$

and, at last:

$$y = \frac{1}{2}(1 - \frac{\lambda}{\varepsilon})y_\infty \tag{191}$$

where y_∞ represents the maximum possible removal. We can easily see (from equations (184) and (191)) that the point in which the derivative attains its maximum value coincides with half of the equilibrium value so that

$$y = \frac{1}{2}y_\chi \tag{192}$$

If we substitute equation (191) in equation (182) we get

$$\frac{dy}{dt} = \frac{\varepsilon}{4}(1 - \frac{\lambda}{\varepsilon})^2 y_\infty \tag{193}$$

in which

$$(1 - \frac{\lambda}{\varepsilon}) \tag{194}$$

acts as a reduction factor. In this case we have that the maximum growth decreases quadratically (with an increase of λ) so that the bigger is λ the smaller is the equilibrium value and as $\lambda \longrightarrow \varepsilon$ we have that both the equilibrium value in equation (192) and y_χ tend to 0.
Since we want

$$0 < y_\chi < y_\infty \tag{195}$$

we must have

$$0 < (1 - \frac{\lambda}{\varepsilon}) < 2 \tag{196}$$

or (taking into account the constraints on the parameters):

$$0 < \frac{\lambda}{\varepsilon} < 1 \tag{197}$$

and, at last:

$$0 < \lambda < \varepsilon \tag{198}$$

If we fix

$$\lambda = \frac{\varepsilon}{2} \tag{199}$$

we get:

$$y_\chi = \frac{1}{2}y_\infty \tag{200}$$

and

$$\frac{dy}{dt} = \frac{\varepsilon}{16}y_\infty \tag{201}$$

7.2.3 Brief comparison between the strategies 1 and 2

We only note that:

1. strategy number 1 is more effective (with regard to the quantities we can remove from a population) in removing items from a population but, since it is based on an estimated maximum growth, it can bring the population to extinction;

2. strategy number 2 is less effective but brings the system to an equilibrium value that is lower of that is reached in strategy 1 and that is preferable from an environmental point of view.

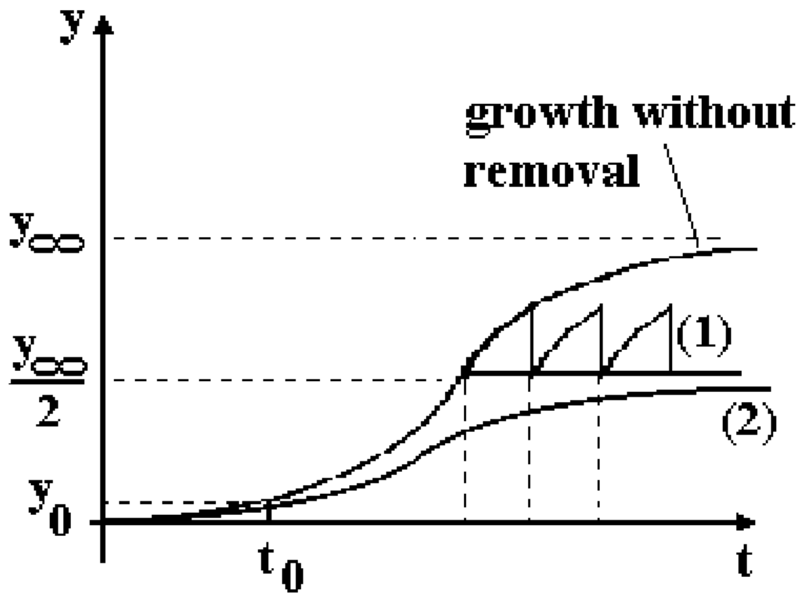

Figure 10: *Comparison of strategies (see text for details)*

Figure 10 contains a comparison among:

1. a growth without removal;

2. a growth (1) with a succession of removals according to strategy 1;

3. a growth (2) with removals according to strategy 2.

Figure 11, last but non least, shows a comparison among the logistic growth without any removal (in this case there is a limit to the population growth due to external factors such as limited availability of food and space and so on . . .) and a growth with removals governed by strategy number 2 (under the hypothesis that $\lambda = \varepsilon/2$): it is clear that the system reaches an equilibrium point. To know the amount of what can be removed we proceed as shown in figure 11. Starting from an instant of time t^* we find the corresponding value on the lower curve then we find the corresponding point on the upper curve: the value of the derivative in that point is the amount of what can be removed.

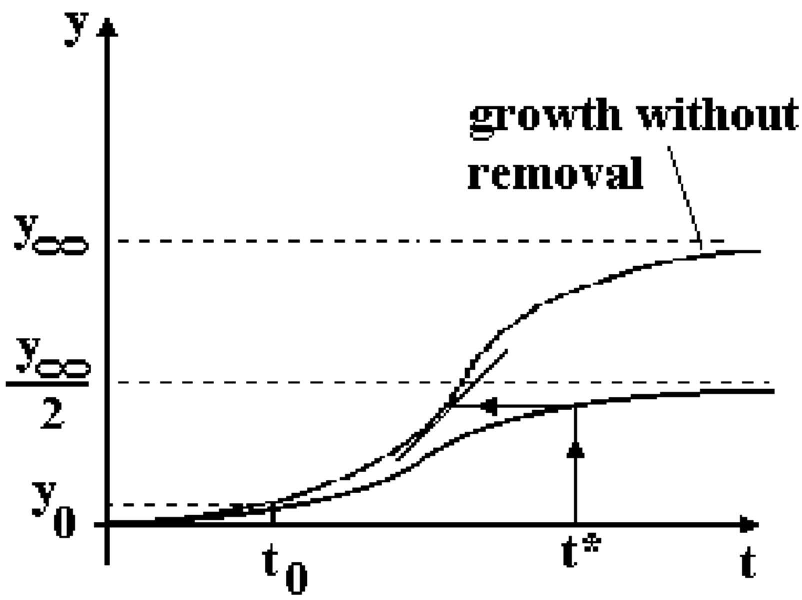

Figure 11: *Strategy number* 2 *(see text for details)*

8 Eighth lesson

8.1 Introduction

In this lesson we are going to examine some model of increasing complexity and relating to one population or two populations competing for the same environment.The more complex competitive model we are going to examine today is the model of two populations, one of preys and another of predators, that compete in a somewhat violent way, the former representing the reservoir of food for the latter. In what follows we use $y(t)$ or $y_1(t)$ and $y_2(t)$ to denote the numerical evolution of either one population or two populations as a continuous function of time.

8.2 First model: one population

This situation is easily described by the following model:

$$\frac{dy}{dt} = \varepsilon(1 - \gamma y)y \tag{202}$$

with ε, γ constant strictly positive parameters and a known initial value y_0. In this case we have:

$$\gamma = \frac{1}{y_\infty} \tag{203}$$

Equation (202) has the already known solution:

$$y = \frac{y_0}{[\gamma y_0 + (1 - \gamma y_0)e^{-\varepsilon(t-t_0)}]} \tag{204}$$

In this case we have that:

$$lim_{t\longrightarrow\infty} y = y_\infty = \frac{1}{\gamma} \tag{205}$$

so that the equilibrium value is stable and is independent from both ε and y_0.

8.3 Second model: CDS of two "friendly " populations

In this case we have two populations that share an environment and compete for a set of resources without trying to destroy each other. We use $y_1(t)$ and $y_2(t)$ to describe the size of each population with time. The descriptive model is represented by the following equations:

$$\begin{cases} \frac{dy_1}{dt} = \varepsilon_1(1 - \gamma_1(y_1 + y_2))y_1 \\ \frac{dy_2}{dt} = \varepsilon_2(1 - \gamma_2(y_1 + y_2))y_2 \end{cases} \tag{206}$$

with the strictly positive constants $\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2$ and known initial values y_1^0 and y_2^0. In this case there is a full symmetry between the two populations, this is, indeed, the simplest case. We note that the growth rate of one population is affected by the presence of both populations in equal proportion. In general we can find a general solution of system (206). A solution can be found by integrating after a separation of the variables. We start from the first equation of system (206).
We can rewrite it as follows:

$$\frac{dy_1}{dt} = \varepsilon_1 y_1 - \varepsilon_1 \gamma_1 (y_1 + y_2) y_1 \tag{207}$$

and then

$$\frac{1}{\varepsilon_1 y_1 \gamma_1} \frac{dy_1}{dt} = \frac{1}{\gamma_1} - (y_1 + y_2) \tag{208}$$

With similar transformations on the second equation of system (206) we get:

$$\frac{1}{\varepsilon_2 y_2 \gamma_2} \frac{dy_2}{dt} = \frac{1}{\gamma_2} - (y_1 + y_2) \tag{209}$$

Subtracting side by side equations (208) and (209) we get:

$$\frac{1}{\varepsilon_1 y_1 \gamma_1} \frac{dy_1}{dt} - \frac{1}{\varepsilon_2 y_2 \gamma_2} \frac{dy_2}{dt} = \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \tag{210}$$

or

$$\frac{1}{\varepsilon_1 y_1 \gamma_1} dy_1 - \frac{1}{\varepsilon_2 y_2 \gamma_2} dy_2 = (\frac{1}{\gamma_1} - \frac{1}{\gamma_2}) dt \tag{211}$$

Integrating we get

$$\frac{1}{\varepsilon_1 \gamma_1} ln y_1 - \frac{1}{\varepsilon_2 \gamma_2} ln y_2 = (\frac{1}{\gamma_1} - \frac{1}{\gamma_2}) t + K \tag{212}$$

Using some simple properties of logarithmic functions we obtain:

$$ln \frac{y_1^{\frac{1}{\varepsilon_1 \gamma_1}}}{y_2^{\frac{1}{\varepsilon_2 \gamma_2}}} = (\frac{1}{\gamma_1} - \frac{1}{\gamma_2}) t + K \tag{213}$$

At the end, using an exponentiation, we arrive at the general solution in the form:

$$\frac{\left(\frac{y_1}{y_1^0}\right)^{\frac{1}{\varepsilon_1 \gamma_1}}}{\left(\frac{y_2}{y_2^0}\right)^{\frac{1}{\varepsilon_2 \gamma_2}}} = e^{(\frac{1}{\gamma_1} - \frac{1}{\gamma_2})(t - t_0)} \tag{214}$$

We note explicitly that:

1. equation (214) does not describe a prey-predator model;
2. equation (214) cannot be used to describe the evolution of the populations over time.

At this point we are going to examine some special cases of equation (214).

1. if the two populations show an equal aggressivity so that $\gamma_1 = \gamma_2$ we have a stable solution of the form

$$(\frac{y_1}{y_1^0})^{\frac{1}{\varepsilon_1}} = (\frac{y_2}{y_2^0})^{\frac{1}{\varepsilon_2}} \tag{215}$$

   Now we have two sub-cases:

   (a) if $\varepsilon_1 = \varepsilon_2$ we obtain

$$\frac{y_1}{y_1^0} = \frac{y_2}{y_2^0} \tag{216}$$

   or

$$\begin{cases} y_1 = k y_2 \\ y_2 = \frac{1}{k} y_1 \end{cases} \tag{217}$$

   with $k = y_1^0/y_2^0$. We see easily that starting from the initial values the sizes of the two populations remain constant at those values, there is no evolution at all.

   (b) if $\varepsilon_1 \neq \varepsilon_2$ we have

$$(\frac{y_1}{y_1^0})^{\frac{1}{\varepsilon_1}} = (\frac{y_2}{y_2^0})^{\frac{1}{\varepsilon_2}} \tag{218}$$

   and so there is an evolution of the two populations.

2. if the two populations show an different aggressivity so that $\gamma_1 \neq \gamma_2$ we, again, have two sub-cases:

   (a) $\frac{1}{\gamma_1} > \frac{1}{\gamma_2}$: in this case the exponential on the right side of equation (214) is positive so that the right size tends to $+\infty$ at t tend to $+\infty$. This implies that since y_1 assumes finite values we must have $y_2 \longrightarrow 0$ so the less aggressive population declines to extinction.)

   (b) $\frac{1}{\gamma_1} < \frac{1}{\gamma_2}$: we can reason as before by exchanging the roles of y_1 and y_2.

From all this we learn that a population survives the other if it obtains all the resources independently from the relative magnitude of the growth coefficients ε_1 and ε_2 and of the initial sizes of the populations (y_1^0 and y_2^0). We note that the condition $\gamma_1 = \gamma_2$ represents more an exception than a rule.

8.4 The most complex model: prey-predator system

In this case we have two populations with one population that act as a killer of individuals of the other so to gain environmental control and supremacy. We obtain the following system of differential equations:

$$\begin{cases} \frac{dy_1}{dt} = \varepsilon_1(1-\gamma_1 y_1 - \lambda_1 y_2)y_1 \\ \frac{dy_2}{dt} = -\varepsilon_2(1-\gamma_2 y_1 - \lambda_2 y_2)y_2 \end{cases} \tag{219}$$

in which ε_1 and ε_2 represent the two growth factor and the first equation refers to the prey population whereas the second one refers to the predator population. In this case we have no more a condition of symmetry between the two populations and the predator population tends to extinction if the preys are not enough.
In this case we have

1. the following strictly positive coefficients: $\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2, \lambda_1, \lambda_2$;
2. two known initial values: y_1^0 and y_2^0;

and the following special cases:

1. the population size y_1 increases according to the logistic function if there is no predator so that $y_2 = 0$;
2. the population size y_2 decreases if there are not enough preys.

As to the parameters we note that γ_1, γ_2 depend on the environment whereas λ_1, λ_2 depend one on the size of the other population so that we can make the simplifying assumptions that the growth of one populations depends only on the growth of the other and is independent from the environment. In this way we can:

1. disregard γ_1 with respect to λ_1;
2. disregard γ_2 with respect to λ_2;

so that equations of system (219) can be rewritten as:

$$\begin{cases} \frac{dy_1}{dt} = \varepsilon_1(1-\lambda_1 y_2)y_1 \\ \frac{dy_2}{dt} = -\varepsilon_2(1-\lambda_2 y_2)y_2 \end{cases} \tag{220}$$

with the aforesaid sign constraints on the parameters. The first equation relates to the prey population so that if there is no predator (i. e. $y_2 = 0$) it describes an exponential growth (instead of the expected logistic behavior).

The second equation relates to the predator population so that in absence of preys (i. e. $y_2 = 0$) it describes an exponential decrease. The underlying hypothesis is that the environment plays a minor role in the evolution of the populations that depends only on the presence of the predators (for the preys) and of the preys (for the predators).
The basic idea to get a solution of equations of system (220) is to proceed with successive manipulations and simplifications till we arrive at a form like the one shown in equation (214). As a firs step we rewrite the equations of system (220) as

$$\begin{cases} \frac{\lambda_2}{\varepsilon_1}\frac{dy_1}{dt} = \lambda_2 y_1 - \lambda_1\lambda_2 y_1 y_2 \\ -\frac{\lambda_1}{\varepsilon_2}\frac{dy_2}{dt} = \lambda_1 y_2 - \lambda_1\lambda_2 y_1 y_2 \end{cases} \tag{221}$$

If we subtract side by side the equations of system (221) we get:

$$\frac{\lambda_2}{\varepsilon_1}\frac{dy_1}{dt} + \frac{\lambda_1}{\varepsilon_2}\frac{dy_2}{dt} = \lambda_2 y_1 - \lambda_1 y_2 \tag{222}$$

Now we go back to equations (220) and rewrite them as:

$$\begin{cases} \frac{1}{\varepsilon_1 y_1}\frac{dy_1}{dt} = (1 - \lambda_1 y_2) \\ -\frac{1}{\varepsilon_2 y_2}\frac{dy_2}{dt} = (1 - \lambda_2 y_1) \end{cases} \tag{223}$$

so that subtracting again side by side we get

$$\frac{1}{\varepsilon_1 y_1}\frac{dy_1}{dt} + -\frac{1}{\varepsilon_2 y_2}\frac{dy_2}{dt} = \lambda_2 y_1 - \lambda_1 y_2 \tag{224}$$

so that if we compare equations (223) and (222) we obtain:

$$\frac{\lambda_2}{\varepsilon_1}\frac{dy_1}{dt} + \frac{\lambda_1}{\varepsilon_2}\frac{dy_2}{dt} = \frac{1}{\varepsilon_1 y_1}\frac{dy_1}{dt} + \frac{1}{\varepsilon_2 y_2}\frac{dy_2}{dt} \tag{225}$$

Now we can rewrite such equation as:

$$\frac{\lambda_2}{\varepsilon_1}dy_1 + \frac{\lambda_1}{\varepsilon_2}dy_2 = \frac{1}{\varepsilon_1}\frac{dy_1}{y_1} + \frac{1}{\varepsilon_2}\frac{dy_2}{y_2} + K \tag{226}$$

with K a constant of integration. At this point we can multiply both sides for dt, integrate and obtain:

$$ln y_1^{\frac{1}{\varepsilon_1}} + ln y_2^{\frac{1}{\varepsilon_2}} = \frac{\lambda_2}{\varepsilon_1}y_1 + \frac{\lambda_1}{\varepsilon_2}y_2 + K \tag{227}$$

ad, at last, using some simple properties of logarithmic functions:

$$(\frac{y_1}{y_1^0})^{\frac{1}{\varepsilon_1}}(\frac{y_2}{y_2^0})^{\frac{1}{\varepsilon_2}} = e^{\frac{\lambda_2}{\varepsilon_1}(y_1 - y_1^0)}e^{\frac{\lambda_1}{\varepsilon_2}(y_2 - y_2^0)} \tag{228}$$

We note that in equation (228) does not appear parameter t so that we cannot use that equation to model the behavior over time of the sizes of the two populations. Equations such as (228) are called phase plane solutions of prey-predator CDS. Such equations describe a periodically and indefinitely oscillating behavior.

9 Ninth lesson

9.1 Phase plane solutions

We go back to equation (228) and rewrite it here for convenience:

$$(\frac{y_1}{y_1^0})^{\frac{1}{\varepsilon_1}}(\frac{y_2}{y_2^0})^{\frac{1}{\varepsilon_2}} = e^{\frac{\lambda_2}{\varepsilon_1}(y_1-y_1^0)}e^{\frac{\lambda_1}{\varepsilon_2}(y_2-y_2^0)} \tag{229}$$

We can rewrite it as follows:

$$(\frac{y_1}{y_1^0}e^{-\lambda_2(y_1-y_1^0)})^{\frac{1}{\varepsilon_1}}(\frac{y_2}{y_2^0}e^{-\lambda_1(y_2-y_2^0)})^{\frac{1}{\varepsilon_2}} = 1 \tag{230}$$

In this way if we put:

$$U = (\frac{y_1}{y_1^0}e^{-\lambda_2(y_1-y_1^0)})^{\frac{1}{\varepsilon_1}} \tag{231}$$

$$V = (\frac{y_2}{y_2^0}e^{-\lambda_1(y_2-y_2^0)})^{\frac{1}{\varepsilon_2}} \tag{232}$$

we get the equation of an hyperbola:

$$UV = 1 \tag{233}$$

Such an hyperbola allows us to define a phase plane solution of prey-predator CDS and to see that such a solution oscillates periodically and indefinitely. In order to study one independently from the other equations (231) and (233) we can perform some tricky calculations. As to equation (231) we can evaluate:

$$\frac{dU^{\varepsilon_1}}{dt} = \frac{d}{dt}(\frac{y_1}{y_1^0}e^{-\lambda_2(y_1-y_1^0)}) \tag{234}$$

By equating it to 0 we have:

$$\frac{1}{y_1^0}(1-\lambda_2 y_1)e^{-\lambda_2(y_1-y_1^0)} = 0 \tag{235}$$

so that when

$$y_1 = \frac{1}{\lambda_2} \tag{236}$$

we have a zero of U^{ε_1}. Furthermore we have that $y_1 = 0 \Rightarrow U = 0$ and $U^{\varepsilon_1} = 0$: we have thus that U starts from 0 in $y_1 = 0$, reaches a maximum in $y_1 = \frac{1}{\lambda_2}$ and then decreases to 0 as $y_1 \longrightarrow \infty$. To see why this happens it is sufficent to examine the structure of equations (231) and (232). If we act in a similar way for V (so we evaluate V^{ε_2}, we derive respect to y_2 and equate the derivative to 0) we find a maximum for:

$$y_2 = \frac{1}{\lambda_1} \tag{237}$$

and a similar behavior, with the proper changes. Figure 12 shows (on the SE part) the phase plane. Phase plane is a set of orbit that are described by variables related each other by some function (as equation (233) in this case). It is derived by the graphs of the functions of the tho involved variables (in our case U and V) as functions of the corresponding variable and by the graph of their link (and so of U as a function of V). Such graphs are in the positions shown in figure 12 and their point-by-point composition gives rise to the phase plane graph for such pair of variables each one varying according to the depicted behaviors. To see how it can be used let's suppose to know one value of the first variable y_1 so that we are in point 0. We evaluate U for such a value than, using the link shown in the NW part of figure 12 we find the corresponding value of V (point 1). At this point we draw a straight line down to intersect the graph of V as a function of y_2 and obtain the two points 2 and 3. To such points correspond two values:

1. the first one (2) is in the decreasing part of the y_2, V curve and to this point correspond point p of the orbit in the phase plane;

2. the second one (3) is in the increasing part of the y_2, V curve and to this point correspond point q of the orbit in the phase plane.

Both the two points are admissible points so to distinguish between them it is necessary to have some additional information. We note that the two variables y_1 and y_2 vary between their extreme values in an oscillating forever lasting way. We note, also, that since the two variables are linked by the phase plane orbit their periods of oscillations T_1 and T_2 are bound to be equal so that we have:

$$T_1 = T_2 \tag{238}$$

The system is closed so that, since it is cyclic, it is of no importance which of the two points (so values of y_2) correspond to a given value of y_1 (on condition that it is between the extremal values $miny_1$ and $maxy_1$) or which of the two points (so values of y_1) correspond to a given value of y_2 (on condition that it is between the extremal values $miny_2$ and $maxy_2$) and moreover the graph does not depend on the initial values provided that those values are within the allowed intervals, and so within, respectively, $miny_1$ and $maxy_1$ and $miny_2$ and $maxy_2$.

9.2 Other prey-predator models

9.2.1 Samuelson's model

Such a model is based on the following set of equations:

$$\begin{cases} \frac{dy_1}{dt} = \varepsilon_1 y_1(1 - \gamma_1 y_1 - \lambda_1 y_2) \\ \frac{dy_2}{dt} = -\varepsilon_2 y_2(1 - \lambda_2 y_1) \end{cases} \tag{239}$$

The first equation describes the preys and the second one the predators. All the coefficients are strictly positive and we have also the initial values:

1. $y_1^0 = y_1(t_0)$,
2. $y_2^0 = y_2(t_0)$.

We note that:

1. if predators are absent so that $y_2 = 0$ the equation of the preys is a well known logistic;
2. if we impose
$$\frac{dy_2}{dt} = 0 \tag{240}$$
we get
$$y_1^* = \frac{1}{\lambda_2} \tag{241}$$
3. under the assumption of equation (241),if we impose
$$\frac{dy_1}{dt} = 0 \tag{242}$$
we get
$$y_2^* = \frac{1 - \frac{\gamma_1}{\lambda_2}}{\lambda_1} \tag{243}$$

If we evaluate the Jacobian we find that its eigenvalues are complex conjugates with negative real parts so that we have dumped oscillations and, in the phase plane, we get spiral like behaviors.

9.2.2 Kolmogorov's model

It is based on the following equations:

$$\begin{cases} \frac{dy_1}{dt} = f_1(y_1, y_2)y_1 \\ \frac{dy_2}{dt} = f_2(y_1, y_2)y_2 \end{cases} \tag{244}$$

with the initial values:

1. $y_1^0 = y_1(t_0)$,
2. $y_2^0 = y_2(t_0)$.

We note that f_1 and f_2 are functions to be characterized.

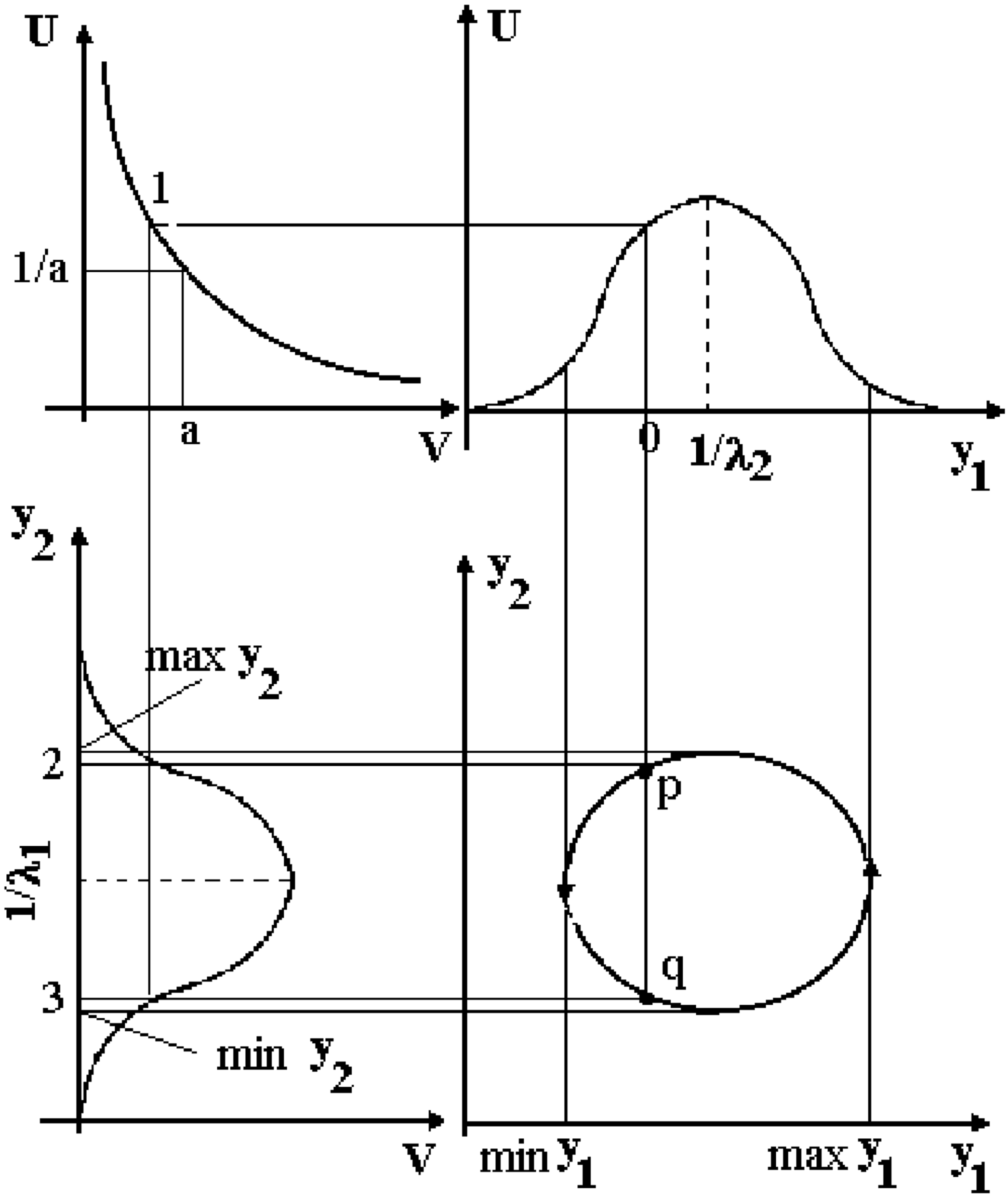

Figure 12: *Phase plane (see text for details)*

10 Tenth lesson

10.1 Some loose notes on Kolmogorov's model

We now go back to Kolmogorov model that can be used in epidemic and diffusion theory. Such a model is described by the following equations:

$$\begin{cases} \frac{dy_1}{dt} = f_1(y_1, y_2)y_1 \\ \frac{dy_2}{dt} = f_2(y_1, y_2)y_2 \end{cases} \tag{245}$$

with the initial values:

1. $y_1^0 = y_1(t_0)$,
2. $y_2^0 = y_2(t_0)$.

Functions f_1 and f_2 must be differentiable. For any point $P = (y_1, y_2)$ of the phase plane we can define a direction vector s that links the origin O to the moving point P. We remind that:

1. $y_1(t)$ is the function describing the preys,
2. $y_2(t)$ is the function describing the predators.

Kolmogorov's model is based on the following assumptions:

1. $$\frac{\partial f_1}{\partial y_2} < 0 \tag{246}$$
   so that f_1 decreases with increasing y_2;
2. $$\frac{\partial f_1}{\partial s} < 0 \tag{247}$$
   so that prey decreases either because of a logistic effect or because of an increase of prey-predators encounters;
3. $$f_1(0,0) > 0 \tag{248}$$
   so for small sizes of prey and predators sizes the prey tend to increase;
4. $$f_1(0,A) > 0 \tag{249}$$
   so exists a constant A such that for a sufficient big size of predators, prey cannot increase any more;

5.
$$f_1(B, 0) = 0 \tag{250}$$

6.
$$\frac{\partial f_2}{\partial y_2} < 0 \tag{251}$$

the predator growth rate decreases with the size of predator population;

7.
$$\frac{\partial f_2}{\partial s} > 0 \tag{252}$$

for a fixed *prey/predator* ratio an increase in both sizes is an advantage for the predators;

8.
$$f_2(C, 0) = 0 \tag{253}$$

so exists C such that the predators size can increase iff the size of the prey population is above C.

Under such assumptions we can have dumped growth rates or undamped oscillations on the predator-prey solutions. As to the aforesaid constants we have $C < B$ where B represents a limitation due to environmental resources and C depends on the size of the prey population. Under the assumptions from 1. to 8. we have that a fixed point can be a point of either stable or unstable equilibrium.

10.2 Piecewise-linear pharmacology CDS

We want a continuous model of drug usage with the aim of getting a given maximum concentration in a patient's blood stream. With T we define a period of drug supply that allows the definition of a succession of instances of drug supply $T, 2T, \ldots$.
The model is given by the following differential equation:

$$\frac{dy}{dt} = -\varepsilon y \tag{254}$$

with a given initial condition $y(t = t_0) = y_0$. It is easy to solve equation (254) so to get:

$$y = y_0 e^{-\varepsilon(t-t_0)} \tag{255}$$

in which ε is a constant not dependent on t. We note that:

$$lim_{t \to +\infty} y = 0 \tag{256}$$

Since in equation (255) a total decay requires an infinite time if, after a time interval equal to T, we give to the patient another dose of drug we start from a new initial condition:

$$y_1 = y_0(1 + e^{-\varepsilon T}) > y_0 \tag{257}$$

and after another T we have:

$$y_1 = y_0(1 + e^{-\varepsilon T} + e^{-2\varepsilon T}) \tag{258}$$

until

$$y_k = y_0(\sum_{i=0}^{k} e^{-i\varepsilon T}) \tag{259}$$

In equation (259) we have a geometrical series of reason $r = e^{-\varepsilon T}$ whose partial sum is:

$$S = \frac{1 - r^{k+1}}{1 - r} \tag{260}$$

and so (substituting $r = e^{-\varepsilon T}$ in equation (260) we get:

$$y_k = y_0(\frac{1 - e^{-\varepsilon T(k+1)}}{1 - e^{-\varepsilon T}}) \tag{261}$$

so that:

$$lim_{k \longrightarrow \infty} y_k = \frac{y_0}{1 - e^{-\varepsilon T}} = y_\infty \tag{262}$$

We have a known saturation value so that we can give to the patient an amount of drug $y_0 = y_\infty$ at once: we know there can be no damage, otherwise we can act on either y_0 or T so to modify y_∞. We note that if $\varepsilon \longrightarrow 0$ or $T \longrightarrow 0$ we have $y_\infty \longrightarrow \infty$.