Ranking electoral systems through hierarchical properties ranking*

Lorenzo Cioni

Department of Computer Science, University of Pisa

e-mail: lcioni@di.unipi.it

Abstract

Electoral systems are complex entities composed of a set of phases that form a process to which performance parameters can be associated. On of the key points of every electoral system is represented by the electoral formula that can be characterized by a wide spectrum of properties that, according to Arrow's Impossibility Theorem and other theoretical results, cannot be all satisfied at the same time. Starting from these basic results the aim of the paper is to examine such properties within a hierarchical framework, based on Analytic Hierarchy Process (AHP) proposed by T. L. Saaty ([Saa80]), performing pairwise comparisons at various levels of a hierarchy so to get a global ranking of such properties. Since any real electoral system is known to satisfy some of such properties but not others it should be possible, in this way, to get a ranking of the electoral systems according also to the political goals both of the voters and the candidates. In this way it should be possible to estimate the relative importance of each property with respect to the final ranking of every electoral formula.

1 Introduction

The present paper contains both the description of a ranking method and some applications of that ranking method on the properties we wish our voting systems satisfy.

*I wish to thank Dr. Franco Vito Fragnelli and Prof. Giorgio Gallo for their help and advices.

Our aim is to investigate if through a hierarchic ranking of properties we can devise a ranking of electoral methods (or even an electoral method).
The paper is structured as follows. After a very short description of the ranking method we propose some notes on electoral systems and list, with some comments, the properties we wish the various electoral systems satisfy. Afterwards we are going to perform a simple ranking exercise and introduce the actors whose points of view and interests we are going to consider, voters and candidates. The next step is the application of the method to more complex cases so to get a certain number of ranked orderings. The last step is the association between some orderings and some more or less classical voting methods. The paper, in a classic way, closes with some remarks and plans for future works.

2 The mathematical tool

AHP is both a method and a tool developed by T. L. Saaty ([Saa80]) and used by himself and others in many fields (see for instance [SK85] and [BR04])[1]. It represents a useful investigation tool in all cases we have to rank n alternatives depending on their order of importance or preference (with respect to some actors and to a general or main goal) and on the basis of qualitative valuations expressed using numerical values on a ratio scale.
After an **analysis phase** that allows the identification of a set of elements, we define a hierarchy among these elements under the form of a rooted hierarchy[2]. At the root (level $l = 0$) we have the main goal, in many cases of political nature, at level 1 we may have the actors , at level 2 the policies, at level 3 the criteria and, last but not least, at the level of the leaves we have the alternatives[3]. In Figure[4] 1 we show a somewhat simplified hierarchy with

[1]The topic is really complex and wide. It is obvious that in this Section we cannot scratch but the very surface. Anyway [Saa80], though hard to find, is a good starting point.

[2]We have a hierarchy with a root at level $l = 0$ and a set of elements at the deepest level from the root that we call leaves and that contain the alternatives we want to rank with respect to the root. The hierarchy does not contain any cycle because the arcs are covered either from the root to the leaves (analysis phase) or from the leaves to the root (synthesis phase).

[3]Any of these levels may be missing but, on the other hand, more levels can be added if this is required by the problem at hand. The minimal number of levels is three: the main goal at level $l = 0$ and two more levels.

[4]A hierarchy is defined **complete** if all the elements at two contiguous levels are connected by exactly one arc (so that if they have respectively n_1 and n_2 elements between the two levels we have $n_1 \times n_2$ arcs) otherwise it is termed as **incomplete**. In this paper we are going to consider only hierarchies of the first type.

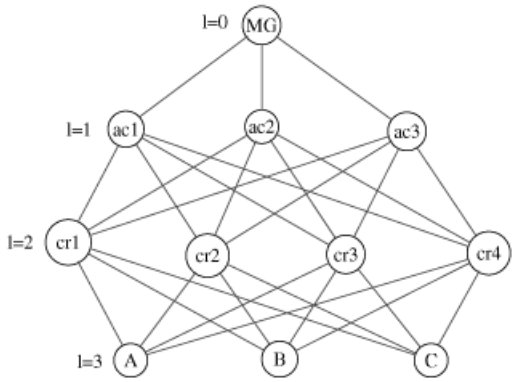

Figure 1: *Example of a complete hierarchy*

a main goal (MG), three actors ($ac1$, $ac2$ and $ac3$) four criteria (the cri) and three alternatives (A, B and C). Our example rooted hierarchy has therefore a depth[5] $d = 3$.
Given any level $i \in [0, d-1]$, if we want to evaluate the importance of the elements at level $i+1$ with respect to those at level i we can build m matrices $n \times n$ where n is the number of elements at level $i+1$ and m is the number of elements at level i. In case of Figure 1 we have one matrix 3×3 to weight the importance of the actors with respect to the main goal, three matrices 4×4 to weight the importance of the criteria with respect to the actors and four matrices 3×3 to weight the alternatives with respect to the criteria. All this represents what Saaty calls the **analysis phase**. Such phase is carried out by the actors that have a common goal and that, either individually or in co-operation, evaluate the matrices of the pairwise comparisons.
Between each pair of consecutive levels i and $i+1$ each matrix is evaluated performing pairwise comparisons between the elements of level $i+1$. If we call A one of those matrices we have that its elements a_{ij} (with $i, j = 1, \ldots, n$) assume positive values from an a priori defined scale and satisfy the following conditions:

1. $a_{ii} = 1$
2. $a_{ji} = \frac{1}{a_{ij}}$

If matrix A satisfies such properties it is called positive reciprocal. We note that a_{ij} can assume a value (that represents the relative importance of element i with respect to element j) from the following scale of values ([Saa80]):

[5] With the term **depth** we define the number of arcs from an element of the hierarchy to the root along the shortest path.

1 to denote equal importance, 3 to denote a weak importance of one over the other, 5 to denote essential or strong importance of one over the other, 7 to denote very strong or demonstrated importance of one over the other, 9 to denote absolute importance of one over the other, 2, 4, 6 and 8 to denote intermediate values whereas a_{ji} assumes the reciprocal value (or vice versa). At this point (see Figure 1) we have to switch to the **synthesis phase**[6] whose aim is the definition of a normalized vector of priorities of the three alternatives with respect to the main goal. The calculation of such vector turns into a series of eigenvalue/eigenvector problems. To see how this can hold we need some preliminary steps.
Once the matrices have been defined we have to define for each of them a normalized vector[7] w of weights $w_i \in [0, 1]$. Such weights are obviously not known in advance otherwise we could write:

$$a_{ij} = \frac{w_i}{w_j} \tag{1}$$

with $i, j = 1, \ldots, n$. For the moment let us suppose we live in an ideal world so that the weights are known. From (1) we can get:

$$a_{ij}\frac{w_j}{w_i} = 1 \tag{2}$$

or (through simple algebra):

$$\sum_{j=1}^{n} a_{ij} w_j = n w_i \tag{3}$$

with $i = 1, \ldots, n$. In compact form we can write equation (3) as:

$$Aw = nw \tag{4}$$

with $w = (w_1, \ldots, w_n)$. It is easy to see that (4) is an eigenvalue/eigenvector problem where n is the eigenvalue and w is the associated eigenvector. Owing to the particular form of the matrix A, if we denote with λ_i $(i = 1, \ldots, n)$ its eigenvalues, we have:

$$\sum_{i=1}^{n} \lambda_i = n \tag{5}$$

We know from equation (4) that n in an eigenvalue of A so that (from equation (5)) all the other eigenvalues are equal to 0. The case where we

[6] The synthesis phase is a purely computational phase whose aim is the evaluation of a vector of priorities with the highest accuracy.

[7] As a normalization condition we have $\sum_{i=1}^{n} w_i = 1$.

know the elements of A through the elements of w is the so called **consistent case**. In this case matrix A is said consistent[8]. If we now suppose to know the elements of the matrix A (through a set of pairwise comparisons) but not the weights w we can solve the problem[9] (4) and obtain the maximum eigenvalue λ_{max} and the associated eigenvector w. We note that if A is consistent, from the preceding remarks we have that $\lambda_{max} = n$ is the only non null eigenvalue to which the required eigenvector of the weights is associated.
If, on the other hand, A is not fully consistent we have that $\lambda_{max} \approx n$ and the other eigenvalues are such that $\lambda_i \approx 0$. In this case the eigenvector[10] w' represents a proxy of the "real" eigenvector w and such approximation is the better the more λ_{max} tends to n. The methods has been indeed endowed by Saaty ([Saa80]) with a criterion that allows the evaluation of the consistency both of the matrix A and of eigenvector w we obtain from it. Such criterion, if it is violated, does not prevent the use of such results but simply gives a strong hint that pairwise comparisons must be carefully revised so to attain to a better set of pairwise rankings.
The criterion is basically grounded on the definition of a **consistency index** $C.I.$ and a **consistency ratio** $C.R.$ The former is defined as:

$$\frac{\lambda_{max} - n}{n - 1} \tag{8}$$

Such index is compared with the average **random index** $R.I.$ $R.I.$ represents the consistency index of a randomly generated reciprocal matrix on the scale $1 \div 9$. It allows us to obtain the $C.R.$ index as a ratio:

$$C.R. = \frac{C.I.}{R.I.} \tag{9}$$

Values of $C.R.$ lower than 0.10 defines the matrix A we are working with as acceptable, slightly higher values must be considered with care, really higher

[8] We note that in the general case the matrix A is consistent if and only if its elements satisfy conditions 1. 2. and the following transitive relation:

$$a_{ij} = a_{ik} a_{kj} \tag{6}$$

with $i, j, k = 1, \ldots, n$.

[9] We note that owing to the structure of a consistent matrix A all the eigenvalues are non negative. In the general case a problem such as:

$$Aw = \lambda w \tag{7}$$

can be solved by imposing $det(A - \lambda I) = 0$ so to define the eigenvalues and the associated eigenvectors.

[10] Again such a vector must satisfy the normalization condition $\sum_{i=1}^{n} w'_i = 1$.

values should turn into the rejection of the matrix A. On [Saa80] the following table of averages $R.I.$ values is provided:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	.58	.9	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.48	1.56	1.57	1.59

where the values on the first row are the dimension of A whereas those on the second row are the values of the corresponding average[11] $R.I.$
At this point we have the matrices A_i and the associated eigenvalues λ_i and eigenvectors w_i and of each matrix we can say if it is enough consistent or not. We have to combine all these bricks together so to obtain a ranking of the alternatives with respect to the main goal.
If we call A_1 the matrix of the pairwise comparisons between the n_1 elements at level 1 with respect to the main goal (level 0) we have a vector of the weights of n_1 components that we may call L_1. If at level 2 we have n_2 elements through pairwise comparisons we get n_1 matrices $n_2 \times n_2$ and therefore n_1 eigenvectors of n_2 elements each. In this way we can construct a matrix $n_2 \times n_1$ and call it L_2. At this point if we want to evaluate the weights of the elements at level 2 with respect to the main goal we can simply evaluate the product:

$$L_2 L_1 \tag{10}$$

so to get a normalized vector of n_2 elements. In a similar way we can define the matrices of the pairwise comparisons of the elements at level 3 with respect to those at level 2, be it A_2, and define the matrix L_3 of the vectors of the weights. In order to get the weights of the elements at level 3 with respect to the main goal we can evaluate:

$$L_3 L_2 L_1 \tag{11}$$

so to get a normalized vector of n_3 elements. Further practical details will be given in the next Sections. For the moment we only note that through equations such as (10) and (11) we flatten the hierarchy by evaluating the priorities of the elements of any level with respect to the main goal.
The last step is a set of computationally "light" procedures for the evaluation of the normalized eigenvectors from the matrices A_i without solving the associated characteristic equations. In [Saa80], pages 19 and 20, five methods of increasing precision and complexity are provided. All such methods are based on the particular form of matrix A.

[11] It is easy to understand why in cases $n = 1$ and $n = 2$ the problem of consistency cannot arise. More precisely the problem of consistency arises only when the dimension of the matrices is greater than 2.

1. The less precise÷less complex. We sum the elements of each row and divide such a value with the sum of all the elements of the matrix. The ratio for the i-th row gives the i-th element of the eigenvector w that is normalized by construction.

2. Higher precision÷higher complexity. We sum the elements of each column and then we evaluate the reciprocal of each sum. To normalize we divide each reciprocal with the sum of the reciprocals.

3. Good precision÷higher complexity. We evaluate the sum of the elements of each column and divide each element of a column for that sum (we normalize each column) so to obtain a new matrix. At this point we sum the elements on each row of the new matrix and divide the sum for the dimension of the matrix. In this way we evaluate an average over the normalized columns.

4. Good precision÷higher complexity. We multiply the elements of each row among themselves, evaluate the $n-$th root (if n is the dimension of the matrix) of that value and, lastly, normalize each of such values.

5. Exact solution÷highest complexity. We raise the matrix A to an arbitrarily large power and then divide the sum of the elements of each row of the resulting matrix by the sum of the elements of such matrix.

Precision of every method is measured by comparing the results with those obtained by solving the corresponding characteristic equations. We note that if we evaluate the principal eigenvector w we can evaluate the associated eigenvalue by solving directly equation (7). In this way, if the matrix A is consistent, we get n identical values otherwise we get n slightly different values that we can average to get the "true" value of λ_{max} to be used to evaluate the degree of consistency of the matrix.

3 A few short notes on electoral systems

At this point we present some short notes and comments on electoral systems[12]. An electoral system represents a process ([dCMP+99]) that can be decomposed in four phases:

1. the definition of the electoral rules,

[12]Further and better details can be found on [dCMP+99], [Saa01] and [BMMP+00] among the many.

2. the vote expression,

3. the vote-to-seat translation,

4. the government formation.

As such it is a very complex process, nevertheless well suited for a unified formal description with the language of elementary set theory ([dCMP$^+$99]), and whose performance can be "measured" with a set of criteria and indicators ([dCMP$^+$99]).
Though complex, an electoral system, starting from each voter's ranking of a set of alternatives (the candidates) from the best to the worse without ties[13], aims at aggregating such rankings in a global social ranking. Unfortunately this is a very hard task and literature is full of impossibility results (see [Saa01] for many paradoxes and some possible solutions). The main results we want to cite in passing here are:

1. Arrow's [im]possibility Theorem[14] that ([BMMP$^+$00]), roughly speaking, states that, with more than three candidates, there is no aggregation methods that can satisfy simultaneously the properties of Universal Domain, Transitivity, Unanimity or Pareto condition (or principle), Binary Independence and Non-dictatorship (see Section 4 for the definitions);

2. Sen's Theorem ([Saa01]) that is based on a condition of Minimal Liberalism (ML)[15] and states that with three or more alternatives and two or more voters with a Social Welfare Function, if Universal Domain, ML and Pareto are satisfied we are damned to have profiles (or sets of preferences) that have cyclic outcomes (and so fall in the Condorcet paradox of voting);

3. Gibbard-Satterthwaite Theorem ([BMMP$^+$00]) that concerns strategic voting (or the convenience of not expressing one's true preferences) and that states that with more than two candidates there exists no aggregation method that satisfies simultaneously the properties of Universal Domain, Non-manipulability and Non-dictatorship.

[13] As it will be evident from our examples, in this paper we are going to relax such a hypothesis and use also an indifference relation among the alternatives.

[14] Such a theorem comes in three versions ([Tay05]): one for the Social Choice Functions, one for the Social Welfare Functions and one for the Voting Rules. Anyway, in all the versions it prevents even a minimal set of properties from being satisfied at the same time.

[15] A Social Welfare Procedure or Function, or a procedure for the ranking of a set of alternatives, is said to satisfy ML if ([Saa01]) each of at least two voters is decisive over a pair of alternatives so that his/her ranking of such pair determines that pair's societal ranking.

We note that:

1. the properties we have listed with Arrow's [im]possibility theorem are really minimal for any real democratic process and that things are even worse ([BMMP$^+$00]) if we wished to define a method that satisfied additional properties such as Neutrality, Separability, Monotonicity, Non-manipulability and so on;

2. similar considerations hold also for Gibbard-Satterthwaite Theorem and Sen's Theorem. Both are hard to accept (this is true also for Arrow's Theorem, see [Saa01] for a deep discussion and some tentative solutions) and stir up our hope of designing a perfect voting system. Anyway ranking alternatives is needed in many fields so that many "imperfect" voting systems have been devised and used since a long time. Here we only note that ML maybe should be put in context with a new examination of the Condorcet voting paradox and that the danger of manipulability can be reduced by imposing constraints that make harder the presentation of stray (dummy) candidates.

4 The desired properties or the "wish lists"

In this Section we start with the "wish lists" of the electoral systems. Unfortunately such lists, as it should be clear after Section 3, are nothing more than impossible dreams. Anyway, our main goal is to recall some basic definitions so to frame them in the context of the present paper. We start with a first "wish list" or a first group of basic properties that are involved in Arrow's Theorem. We derive our definitions essentially from [BMMP$^+$00] and [dCMP$^+$99].

1. **Universal Domain** means that the chosen aggregation method must be universally applicable so that from any rankings of the voters it must yield an overall ranking of the candidates.

2. **Transitivity** requires that the aggregation of the rankings must be a ranking, with possible ties, that satisfies transitivity.

3. **Unanimity** or **Pareto condition**[16] implies that, if each voter ranks a candidate higher than another, this ranking must be reflected in the overall ranking.

[16] We note that though [Tay05] differentiates the two properties we consider them as synonyms.

4. **Binary Independence** (or Independence from Irrelevant Alternatives) requires that the relative position of two candidates in the overall ranking depends only on their relative position in each voter's ranking so that all the other alternatives are seen as irrelevant with regard to those candidates.

5. **Non-dictatorship** means that there is no voter that can impose his/her ranking as the overall social ranking.

In passing we note that.

1. Condorcet method (the one of the pairwise comparisons between candidates) satisfies properties 1., 3., 4. and 5. so that, by Arrow's Theorem, it must fail property 2. and indeed a Condorcet winner does not necessarily exists owing to the existence of cycles among candidates;

2. Borda method (the one of the global ranking of the candidates) satisfies properties 1., 2., 3. and 5. so that, by Arrow's Theorem, it must fail property 4. and indeed Borda method suffers from this drawback that can be exploited to manipulate the overall ranking[17].

At this point we could enlarge the basic list (and make things even worse) by adding the following properties[18].

1. **Anonymity** ([Tay05]) requires that voters are treated the same way so that the overall ranking is independent from any permutation of the voters.

2. **Neutrality** ([Tay05]) means that alternatives are treated the same way so that the overall ranking is independent from any permutation of the alternatives.

3. **Separability** ([BMMP$^+$00]) requires that if we perform an election with two separate set of voters and obtain a winner candidate on each set such candidate remains a winner if we repeat the election with the same method on the union of the two set of voters.

4. **Monotonicity** ([Tay05]) requires that a winner remains a winner when a voter interchanges the winning alternative with the one that voter ranks immediately above it.

[17] We note however that [Saa01], at page 148, states that "when ways to circumvent the difficulties of Arrow's Theorem are examined ... only the Borda Count survives all of the different requirements".

[18] We note that some of these properties may take a different meaning when we will examine proportional and majoritarian methods.

5. **Non-manipulability** is a very complex issue ([Tay05] but essentially it means that the overall ranking of a set of candidates does not depend either on the agenda or on the presence or stray candidates or on the expression of non true preferences.

At this point, in order to deepen our examination of electoral systems, we can give the "wish lists" of properties for both majoritarian methods (where only one seat is assigned in every district) and proportional methods (where S seats are assigned in every district). We takes the definitions from [dCMP$^+$99] and list them very concisely mainly as a help for a better understanding of subsequent Sections.
Majoritarian methods are characterized by the following properties.

1. **Condorcet winner**: it is the winner of all pairwise comparisons, if it exists it should be the winner of the electoral competition.

2. **Condorcet loser**: a method should not choose the candidates that loses every pairwise comparison with all the other candidates.

3. **Monotonicity**: a method is monotone if the number of seat assigned to a party does not decrease if the number of its supporters grow.

4. **Pareto principle**: if all the voters prefer a candidate to another the latter cannot be chosen.

5. **Weak Axiom of Revealed Preference**: it requires that, **(a)**, if a candidate is a winner on a set X it must remain a winner also on any subset $X' \subseteq X$ to which he/she belongs and that, **(b)**, if there are ties among candidates in $X' \subseteq X$ those candidates at par must be all either included or excluded from X. This axiom is used to get voting methods immune from manipulations on the set of candidates through the addition of stray candidates.

6. **Path independence**: a method satisfies path independence if the outcome is independent from the ordering of the phases that are used for the selection of the candidates.

We note, in passing, that **First-past-the-post**[19] method satisfies anonymity (as already defined), 4. and 5. whereas **Double ballot** and **Single transferable vote** methods satisfy anonymity, 2. and 4.
Proportional methods are characterized by the following properties.

1. **House monotonicity** means that if the number of seats passes from S to $S+1$ no party gets fewer seats.

2. **Quota satisfaction** requires that the number of seats each party receives is as close as possible to its exact quota and so to a percentage of the total seats that is almost equal to the percentage of the votes it receives.

3. **Population monotonicity** ([dCMP+99]) "if a party (or state) with a growing weight cannot lose a seat in favour of a party (or state) with a declining weight".

4. **Consistency** requires that any partial assignment is itself proportional.

5. **Stability** means that whenever two parties merge in a coalition (or a new party) they do not get fewer seats that those they get as separate entities.

We note, in passing, that **Quota method**[20] satisfies anonymity, 1., 2., 4. (only with regard to pairs of eligible parties) and 5. whereas **Largest remainders** method satisfies anonymity, 2. and 5.

5 The light stuff

Let us start with a very simple exercise. We suppose to have three voters and four alternatives that must be ranked so to define the best alternative

[19] This is a method of the majoritarian family where "winner takes all" so that in every uninominal district the candidate that receives a majority of votes is elected independently from the obtained percentage. Double ballot is articulated into two rounds (with or without threshold) so that the second occurs only if in the first no candidate gets an absolute majority of votes. If this occurs the most voted candidate is elected. In the case of Single transferable vote we have an iterative procedure where the less voted candidate is dropped and his votes are transferred to his next most preferred candidate still in competition until when one candidate reaches more that 50% of the votes cast.

[20] Quota method assigns the seats by evaluating a quota and rounding it up or down. Such a quota is evaluated as $q_i = \frac{S}{V} v_i$ where S is the total number of seats, V is the total number of votes and v_i is the number of votes of party i. Largest remainder method uses such a quota q_i to evaluate a remainder $r_i = q_i - \lfloor q - i \rfloor$ to assign the not yet directly assigned seats to the parties with the highest values of the remainder.

among the four or, at least, a total ordering on them.

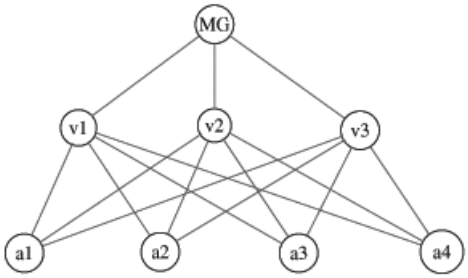

Figure 2: *Three voters and four alternatives*

The situation is shown in Figure 2. The Main Goal (i.e. the ranking of the alternatives) is labelled as MG whereas voters are labelled as $v1, v2, v3, v4$ and a similar convention holds also for the four alternatives. As a first step we evaluate the normalized vector of the weights of the voters with regard to MG. It is easy to see that imposing a full symmetry of the three voters we get a fully consistent 3×3 matrix with all elements at 1 to which it corresponds the eigenvalue $\lambda_{max} = 3$ and an eigenvector $L_1 = (1/3, 1/3, 1/3)$. This result is consistent also with our intuition of a fair evaluation tool since it seems obvious that the three voters have the same weight in the process. As the successive (and last in this case) step we have to evaluate three matrices 4×4 of the pairwise comparisons of the four alternatives, each matrix with regard to a single voter. For these evaluations we use the scale $1 \div 9$ we introduced in Section 2 and suppose that the four voters have respectively the following preferences on the alternatives[21]:

1. $a1 > a2 > a3 > a4$,
2. $a4 > a3 > a2 > a1$,
3. $a3 \sim a4 > a2 \sim a1$.

As to the three matrices we have those in Tables 1, 2 and 3.
By using the method of the $n-$th root of the product it is easy to evaluate the eigenvectors of such matrices, the corresponding eigenvalues and verify that every $C.R.$ falls below the threshold suggested by Saaty so that each matrix is consistent. Such eigenvalues form the matrix L_2 of Table 4. At this

[21] We denote with $>$ a binary relation of strict preference and with $\sim$ a binary relation of indifference. No rationality hypothesis is imposed on the voters and such relations are supposed to be endowed with classical properties.

v1	a1	a2	a3	a4
a1	1	2	5	7
a2	1/2	1	2	3
a3	1/5	1/2	1	2
a4	1/7	1/3	1/2	1

Table 1: *Pairwise comparisons with regard to* $v1$

v2	a1	a2	a3	a4
a1	1	2	1/2	1/4
a2	1/2	1	1/3	1/6
a3	2	3	1	1/3
a4	4	6	3	1

Table 2: *Pairwise comparisons with regard to* $v2$

point the normalized vector of the weights of the alternatives with respect to the main goal can be easily evaluated as:

$$W = L_2 L_1 \tag{12}$$

so to get:

$$W = (0,2559\,0, 1371\,0, 2571\,0, 3498) \tag{13}$$

From expression (13) we can easily deduce the following (and possibly counter intuitive) ordering on the alternatives:

$$a4 > a3 > a1 > a2 \tag{14}$$

At this point a question (at least) spontaneously arises: and now? We got a ranking, right. Can we use it as if it was an election outcome? Maybe. The main problem to face is the inconsistency issue. In the general case, indeed, we can have one or more inconsistent matrices: how can we deal with this? There is any threshold above which we should reject a ranking? Or should we consider it anyhow valid? Anyway be patient, we are going to give some more hints in Section 8.

6 Some harder stuff

After that simple exercise, in this Section we have major aims. We use a set of properties that characterize the families of majoritarian and proportional methods to obtain a ranking of those properties and, depending of

v3	a1	a2	a3	a4
a1	1	1	1/5	1/5
a2	1	1	1/5	1/5
a3	5	5	1	1
a4	5	5	1	1

Table 3: *Pairwise comparisons with regard to* $v3$

$L_2 =$

W1	W2	W3
0,5488	0,1355	0,0833
0,2497	0,0782	0,0833
0,1269	0,2279	0,4167
0,0745	0,5583	0,4167

Table 4: *Matrix* L_2

this ranking, define the "preferred" method within each family. Of course what we present suffers some drawbacks but our intent is to introduce the hierarchic method and to show how it can be used to perform such tasks (see [dCMP$^+$99]).
Among the drawbacks we cite:

1. the rankings have been executed by a single person (or two at the most),
2. in many cases they have not been performed having a deep and sound knowledge and experience of the involved properties,
3. many of the rankings have been performed having in mind more the need to get consistent matrices than any deep comparison among the involved properties.

The first very simplified situation is illustrated in Figure 3 where we suppose to have (only) four voters who rank the six main properties that characterize proportional methods and so: **Anonymity** (A), **House Monotonicity** (HM), **Quota Satisfaction**(QS), **Population Monotonicity** (PM), **Consistency** (C) and **Stability** (S).
If we perform the pairwise rankings for each voter we can get the Tables 5, 6, 7 and 8.
We note that the four voters have respectively the following preference orderings:

1. $A > HM > QS > PM > C > S$

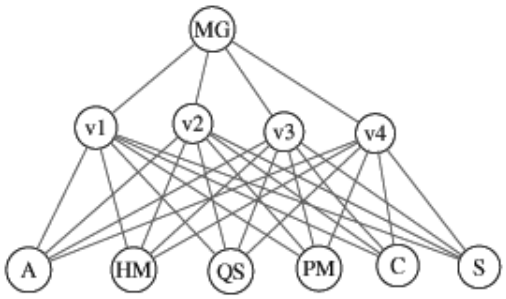

Figure 3: *Ranking properties of proportional methods*

v1	A	HM	QS	PM	C	S
A	1,00	2,00	3,00	4,00	6,00	9,00
HM	0,50	1,00	2,00	2,00	3,00	4,00
QS	0,33	0,50	1,00	1,00	2,00	2,00
PM	0,25	0,50	1,00	1,00	2,00	2,00
C	0,17	0,33	0,50	0,50	1,00	2,00
S	0,11	0,25	0,50	0,50	0,50	1,00

Table 5: *Pairwise comparisons with regard to* $v1$

2. $A \sim HM > QS \sim PM > C > S$
3. $S > C \sim PM > A > HM \sim QS$
4. $QS > HM \sim A > PM \sim C \sim S$

Also in this case, by using the method of the $n-$th root of the product, it is easy to evaluate the eigenvectors of such matrices, the corresponding eigenvalues and verify that every $C.R.$ falls below the threshold suggested by Saaty so that each matrix is consistent. Such eigenvalues form the matrix L_2 of Table 9. Also in this case, if we suppose that the four voters have the same weight with regard to the Main Goal (MG) and define the proper matrix at level 1, we get a matrix whose elements are all 1. Both by performing the calculations and by using fairness considerations it is easy to see that as eigenvector of level 1 we get $L_1 = (0,25\,0,25\,0,25\,0,25)$ so that as L_2L_1 we get the content of Table 10. In that table the first column contains the vector of the rankings of the properties with regard to the main goal, the second contains the listing of the mnemonics of the properties and the last their place in the classification. A close inspection of Table 10 and a comparison

v2	A	HM	QS	PM	C	S
A	1,00	1,00	3,00	3,00	5,00	7,00
HM	1,00	1,00	3,00	3,00	5,00	7,00
QS	0,33	0,33	1,00	1,00	5,00	7,00
PM	0,33	0,33	1,00	1,00	5,00	7,00
C	0,20	0,20	0,20	0,20	1,00	2,00
S	0,14	0,14	0,14	0,14	0,50	1,00

Table 6: *Pairwise comparisons with regard to* $v2$

v3	A	HM	QS	PM	C	S
A	1,00	2,00	2,00	1,00	1,00	0,20
HM	0,50	1,00	1,00	0,50	0,50	0,14
QS	0,50	1,00	1,00	0,50	0,50	0,14
PM	1,00	2,00	2,00	1,00	1,00	0,33
C	1,00	2,00	2,00	1,00	1,00	0,33
S	5,00	7,00	7,00	3,00	3,00	1,00

Table 7: *Pairwise comparisons with regard to* $v3$

with the results of the table at page 83 of [dCMP$^+$99][22] allow us to assert that on the basis of our results the "best" proportional method (or, more correctly, the method the four voters prefer) is the **quota method**. We note indeed how the properties A, HM and QS summed up have a weight of $0,65$ or:

$$W_A + W_{HM} + W_{QS} = 0,65 \tag{15}$$

It is obvious that by changing the preferences of the voters (and also by augmenting their number) we surely get a different ordering, probably with ties, to which, again with regard to the cited table of of [dCMP$^+$99], can correspond either another "winner" or a pair of "winners" or no winner at all. Let us suppose, for instance, that only voter $v2$ changes his/her pairwise comparisons and takes those of Table 11 (associated to the following preference ordering $PM > A \sim C > S > HM > QS$). If we evaluate the new eigenvector $W2$ we get, obviously, a different vector that gives rise to a different second column of matrix L_2. This turns into a somewhat different final priority vector, see Table 12. A close inspection of Table 12 and a com-

[22]From that table we have that **largest remainders methods** satisfy A, QS and S; **divisor methods** satisfy A, HM, PM, C and S (but only in special cases) and **quota method** satisfies A, HM, QS, C (but only in special cases) and S.

v4	A	HM	QS	PM	C	S
A	1,00	1,00	0,33	2,00	2,00	2,00
HM	1,00	1,00	0,33	2,00	2,00	2,00
QS	3,00	3,00	1,00	7,00	7,00	7,00
PM	0,50	0,50	0,14	1,00	1,00	1,00
C	0,50	0,50	0,14	1,00	1,00	1,00
S	0,50	0,50	0,14	1,00	1,00	1,00

Table 8: *Pairwise comparisons with regard to v*4

$L_2 =$

W1	W2	W3	W4
0,43	0,31	0,13	0,15
0,22	0,31	0,07	0,15
0,12	0,15	0,07	0,48
0,11	0,15	0,14	0,07
0,07	0,05	0,14	0,07
0,05	0,03	0,47	0,07

Table 9: *Matrix* L_2

parison with the results of the table at page 83 of [dCMP$^+$99] (see footnote 22) allow us to assert that on the basis of our results the "best" proportional method (or, more correctly, the method the four voters prefer) are, in this case, the **largest remainder methods**: we note indeed that the properties A, QS and S weigh almost 60% over the total of the six properties.
Now we go through a similar exercise but with regard to the properties of majoritarian methods. In Figure 4 we again suppose to have (only) four voters who, in this case, rank the six main properties that characterize majoritarian methods and so: **Anonymity** (A), **Condorcet Winner** (CW), **Condorcet Loser**(CL), **Pareto Principle** (PP), **Weak Axiom of Revealed Preferences** ($WARP$) and **Path Independence** (PI).
In this case we give only the matrix L_2 of the eigenvectors and the vector of the priorities of the properties with regard to the main goal (and make some comments). We note that the four matrices that provide the eigenvectors of L_2 are based on the following preference orderings:

1. $A > CW > CL > PP > WARP > PI$
2. $CW \sim CL > PP > PI > A \sim WARP$
3. $PP > A > PI > WARP > CW \sim CL$

$L_2L_1 =$

W		
0,25	A	1
0,19	HM	3
0,21	QS	2
0,12	PM	5
0,08	C	6
0,16	S	4

Table 10: *Final ranking and classification*

v2	A	HM	QS	PM	C	S
A	1,00	3,00	6,00	0,50	1,00	3,00
HM	0,33	1,00	2,00	0,20	0,33	0,50
QS	0,17	0,50	1,00	0,14	0,14	0,20
PM	2,00	5,00	7,00	1,00	2,00	4,00
C	1,00	3,00	7,00	0,50	1,00	2,00
S	0,33	2,00	5,00	0,25	0,50	1,00

Table 11: *New pairwise comparisons with regard to* $v2$

4. $PP > PI > A \sim WARP > CW > CL$

The four matrices can be shown to be fully consistent according to the Saaty criterion. The final results are those of Table 13. The sixth column of that table is obtained by a matrix÷vector multiplication between the first four columns and the fifth column. From the values of the last column we can devise the following ordering:

$$PP > A > CW > CL > PI > WARP \tag{16}$$

with:

$$W_{PP} + W_A + W_{CL} = 0.64 \tag{17}$$

and:

$$W_{PP} + W_A + W_{WARP} = 0.57 \tag{18}$$

Result (16), in the light of the table at page 78 of [dCMP+99][23] and equations (17) and (18), can be a little bit tricky to interpret. By confronting all such informations we can say that:

[23] From that table we have that:

1. First-past-the-post method satisfies A, PP and $WARP$;
2. double ballot and single transferable vote satisfy A, CL and PP;

$L_2L_1 =$

W		
0,23	A	1
0,13	HM	5
0,18	QS	2
0,17	PM	4
0,12	C	6
0,18	S	2

Table 12: *Another final ranking and classification*

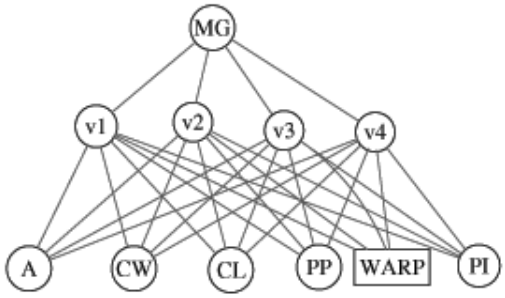

Figure 4: *Ranking properties of majoritarian methods*

1. both **Double ballot** and **Single transferable vote** methods satisfy A, CL and PP (and only these properties);

2. only **First-past-the-post** method satisfies A, PP and $WARP$.

By adding the corresponding elements of the priority vector up, all we can devise therefore is the following preference ordering:

$$Single\,transferable\,vote \sim Double\,ballot > First-past-the-post \quad (19)$$

and reach a final decision by using other criteria.

7 Two other attempts of ranking

In this Section we show two other attempts of performing a ranking of electoral systems. In the first simple exercise we consider high level properties such as **TRansitivity** (TR), **Universal Domain** (UD), **Binary Independence** (BI) and **Pareto condition** (P). Figure 5 shows the case of four

3. approval voting satisfies A, $WARP$ and PI.

W1	W2	W3	W4	W0	W	
0,49	0,05	0,21	0,17	0,25	**0,23**	**A**
0,16	0,35	0,06	0,05	0,25	0,15	CW
0,13	0,35	0,06	0,03	0,25	0,14	CL
0,11	0,12	0,43	0,42	0,25	**0,27**	**PP**
0,07	0,05	0,10	0,08		0,07	WARP
0,04	0,08	0,15	0,24		0,13	PI

Table 13: *Majoritarian methods: the vectors of the weights and the final ranking*

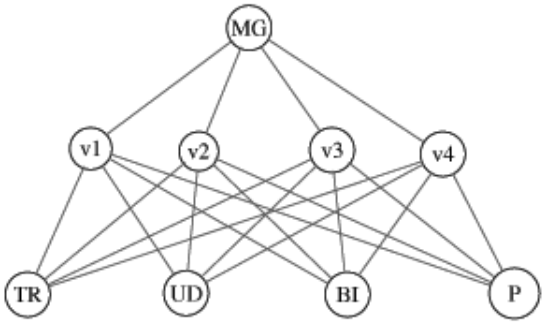

Figure 5: *Ranking electoral systems through ranking some basic properties*

voters that perform a ranking (trough pairwise comparisons) of these basic properties. The four matrices of the pairwise comparisons are those of Tables 14, 15, 16 and 17.
Such Tables are respectively based on the following preference orderings of the four voters:

1. $TR > UD \sim BI \sim P$
2. $P > TR > UD > BI$
3. $TR > P \sim UD > BI$
4. $UD > P > TR > BI$

By performing the proper calculations it is easy to verify that all such matrices are consistent and that the eigenvectors[24] are those of the first four

[24]It is obvious that in all the following cases:

v1	TR	UD	BI	P
TR	1,00	3,00	3,00	3,00
UD	0,33	1,00	1,00	1,00
BI	0,33	1,00	1,00	1,00
P	0,33	1,00	1,00	1,00

Table 14: *A first ranking of electoral systems, case of* $v1$

v2	TR	UD	BI	P
TR	1,00	1,00	2,00	0,33
UD	1,00	1,00	2,00	0,20
BI	0,50	0,50	1,00	0,14
P	3,00	5,00	7,00	1,00

Table 15: *A first ranking of electoral systems, case of* $v2$

columns of Table 18 whereas the fifth column represents the eigenvector of the matrix of the pairwise comparisons of the four voters with regard to the main goal. From both calculations and fairness considerations it is easy to see that such a vector has all components equal to $0,25$. The sixth column gives the global weights or priorities of the four properties with regard to the main goal. With our data we have $TR \sim P > UD > BI$. Such a ranking is satisfied, for instance, by the **Borda count** that does not satisfy binary independence. This does not mean of course that **Borda count** is the only method that satisfies our data but only that it is one of the methods that do that and, so, can be legitimately chosen. We note indeed that:

$$W_{TR} + W_{UD} + W_P = 0.90 \tag{20}$$

so that **Binary independence** can be surely neglected.
The second attempt involves a ranking between majoritarian methods M and proportional methods P if we consider them as two opposing families of methods. The basic idea is shown in Figure 6. In this case we suppose to have three actors (labelled as $ac1$, $ac2$ and $ac3$) that use four properties (labelled as $p1$, $p2$, $p3$ and $p4$) to obtain a ranking between the majoritarian method M and the proportional method P to see which is "better" on the

1. all eigenvectors are evaluated according to the method of the $n-$th root of the product,

2. all eigenvectors are normalized.

v3	TR	UD	BI	P
TR	1,00	3,00	5,00	3,00
UD	0,33	1,00	2,00	1,00
BI	0,20	0,50	1,00	0,50
P	0,33	1,00	2,00	1,00

Table 16: *A first ranking of electoral systems, case of* $v3$

v4	TR	UD	BI	P
TR	1,00	0,17	3,00	0,25
UD	6,00	1,00	9,00	2,00
BI	0,33	0,11	1,00	0,14
P	4,00	0,50	7,00	1,00

Table 17: *A first ranking of electoral systems, case of* $v4$

basis of the pairwise rankings of such properties. In this case we have a

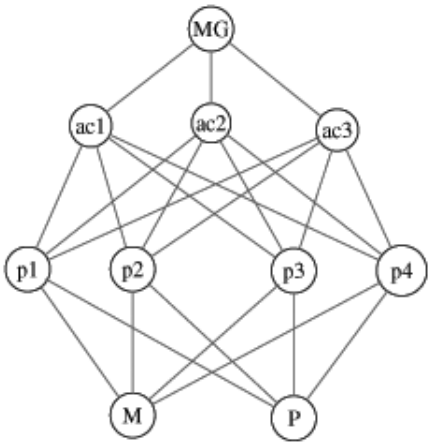

Figure 6: *Majoritarian or proportional? The basic dilemma*

rooted hierarchy where the leaves are at level 3 so we have to define the matrices for three layers and precisely:

1. 4 matrices 2×2 at level 3 to which there corresponds a matrix 2×4 of four eigenvectors L_3;

2. 3 matrices 4×4 at level 2 to which there corresponds a matrix 4×3 of three eigenvectors L_2;

W1	W2	W3	W4	W0	W	
0,50	0,17	0,53	0,10	0,25	0,32	TR
0,17	0,15	0,19	0,54	0,25	0,26	UD
0,17	0,08	0,10	0,04	0,25	0,10	BI
0,17	0,60	0,19	0,32	0,25	0,32	P

Table 18: *The vectors of the weights and the final ranking*

3. 1 matrix 3×3 at level 1 to which there corresponds a matrix 3×1 of one eigenvector L_1.

In this way we can evaluate the priorities vector W of the two alternatives M and P with respect to the root of the hierarchy (or the main goal) as a product of matrices:

$$W = L_3 L_2 L_1 \tag{21}$$

From considerations we have already made elsewhere in Sections 6 and 7 of this paper it is easy to see that $L_1 = (0,33\,0,33\,0,33)$. The hard part is the definition of the four properties. We can try with the followings properties (corresponding respectively to the pi of Figure 6)[25]:

1. **Electoral Participation** (EP) defined, roughly speaking, as the ratio between the number of vote cast and the difference between the total number of voters and the number of vote cast;

2. **Number of Political Parties** (NPP) defined through parameters that count both the number of parties that compete in a given election and their relative strength;

3. **Electoral Volatility** (EV) as a measure of the electoral fluxes among the competing parties from one electoral competition to the successive one;

4. **Government Stability** (GS) measured as a function of the longevity of the governments.

Once the actors have been singled out (as either voters or candidates), each of them must evaluate the matrix of the pairwise comparisons of the properties but, together with the others, must define the needed pairwise comparisons matrices of the alternatives with regard to each of the properties. Apart from

[25] We give only rough definitions of such properties. For further and more exact details see [dCMP+99].

this potential difficulty (that we are going to examine briefly in Section 8), our three actors are supposed to act respectively according to the following preference orderings:

1. $EP > NPP > GS \sim EV$
2. $EP > EV > NPP \sim GS$
3. $GS > NPP > EP > EV$

We note that the properties are considered from an abstract point of view as to their relevance with regard to a method without considerations such as "the higher is the better" or "the lower is the better". Of course each of the actors, performing a comparison, makes such considerations and the result may differ depending on the type of each actor. We can imagine that voters are more interested in EP and NPP whereas candidates are more interested in GS.
In what follows we give only a brief outline of the solution. In Table 19 we show the four matrices at level 3.

EP	M	P	NPP	M	P	EV	M	P	GS	M	P
M	1,00	0,33	M	1,00	0,50	M	1,00	0,20	M	1,00	4,00
P	3,00	1,00	P	2,00	1,00	P	5,00	1,0	P	0,25	1,00

Table 19: *Pairwise comparisons with regard to the properties*

It is easily seen how such matrices are fully consistent. Next we give the three matrices of the pairwise comparisons of the properties with regard to each actor. Such matrices of Tables 20, 21 and 22 are evaluated according to the aforesaid preference orderings.

ac1	EP	NPP	EV	GS
EP	1,00	5,00	7,00	7,00
NPP	0,20	1,00	2,00	2,00
EV	0,14	0,50	1,00	1,00
GS	0,14	0,50	1,00	1,00

Table 20: *Pairwise comparisons with regard to* $ac1$

It is easy to verify that these three matrices are fully consistent. At this point we have:

ac2	EP	NPP	EV	GS
EP	1,00	7,00	3,00	7,00
NPP	0,14	1,00	0,50	1,00
EV	0,33	2,00	1,00	2,00
GS	0,14	1,00	0,50	1,00

Table 21: *Pairwise comparisons with regard to ac*2

ac3	EP	NPP	EV	GS
EP	1,00	0,50	2,00	0,20
NPP	2,00	1,00	3,00	0,33
EV	0,50	0,33	1,00	0,11
GS	5,00	3,00	9,00	1,00

Table 22: *Pairwise comparisons with regard to ac*3

1. a matrix L_3 2×4 of the eigenvectors of the priorities of the alternatives with respect to the properties,

2. a matrix L_2 4×3 of the eigenvectors of the priorities of the properties with respect to the actors,

3. a matrix L_1 3×1 of the eigenvectors of the priorities of the actors with respect to the main goal[26].

With all the ingredients at our disposal we can obtain the priorities of the two alternatives with respect to the main goal as:

$$W = L_3 L_2 L_1 \tag{22}$$

With some easy algebra we find:

$$W = (0,3970\,0,6029) \tag{23}$$

so to get:

$$P > M \tag{24}$$

[26]Again we have that the vectors of priorities of the three actors with respect to the main goal is $L_1 = (0,333\,0,333\,0,333)$.

8 The hierarchy: a real solution or a blind alley?

In the previous Sections we have introduced AHP and shown how we think it can be used in the arena of electoral systems. In Section 5 we have used it as a sort of voting system whereas in Sections 6 and 7 we have used it more as a meta-voting system or as a tool to obtain a ranking of pitted electoral systems.
In the former case (but similar considerations hold also in the latter case) indeed we set up a hierarchy to have three voters get the ranking of four alternatives and so a sort of "social choice function" of such alternatives. Are we sure in this way we got an electoral system that proves to be immune from the "contagion" of Arrow's Theorem and the other results we listed in section 3? Saaty ([Saa80]) is confident this is the case but this is quite obvious, he invented the method. A more neutral neutral source such as [Saa01] makes us almost as confident as Saaty himself. In [Saa01] the author shows how to overcome such theoretical limitations by using methods that do not miss useful information though performing pairwise comparisons between candidates. Our guess here is that the hierarchy (through the use of matrices) is what allows the preservation of such global information though, at each instant, only pairwise comparisons are performed.
So we are sure that the proposed method is a potential solution (at least from a theoretical point of view) to the problem of defining a "perfect" voting system.
If the method we proposed is a real solution, nevertheless, many open problems are yet present and beg for a solution. Here we list only the followings:

1. how can the system shown in Section 5 scale to be used with many more voters and alternatives?

2. how can be solved the problem of having actors evaluate the alternatives with respect to the properties (see Figure 6)? would this work also for many more actors and alternatives?

3. have we to care of any inconsistency? and how? is there any inconsistency threshold (beyond the value of $0,10$) above which we should declare any voting outcome as null and so the ranking/voting as to be repeated? If we are working with experts ranking policies or alternatives and any of them provides a heavy inconsistent matrix it is obviously possible to ask such an expert to be more accurate and revise his/her own judgements but what can we do in case of an election?

Alas, there is anyway yet the possibility that a more subtle and perverse version e. g. of Arrow's Theorem is lurking out there. In this case AHP would prove nothing more than another blind alley (at least for the search of a "perfect" voting system). Only works and research can tell us which is the case.

9 Promising and keeping

At this point, before the final remarks and the good intentions of the closing Section, we have to account for if we kept what we promised or not and, in this case, why.
Really we kept a lot of what we promised by introducing a powerful and flexible method and showing how it can be used to perform global ("social") rankings starting from individual judgements based on pairwise comparisons on a fixed ratio scale. Yet we did not keep something and basically the following points:

1. the problem of how experts or actors can rank alternatives with regard to properties or policies (see Figure 6);

2. the problem of fully taking into account the point of views and the goals of voters and candidates (and, why not?, the elected candidates);

3. the problem of fully framing our approach among the other proposed approaches (see, for instance, [dCMP$^+$99]) so to put in evidence its potential strengths and (almost surely present) weaknesses.

As to the first point we note that it involves the attainment of a consensus among the actors/voters/experts and this can happen essentially in two ways:

1. as a co-ordinate and co-operative simultaneous effort of all the actors/voters/experts,

2. as a two step process where **(a)** every actor/voter/expert produces all the pairwise rankings, including those of the others and **(b)** such rankings are merged in the appropriate global rankings.

We have therefore a wide set of open problems that cannot be solved only on a normative ground but that require a descriptive approach based on on the field experiments, that at the present we cannot execute, and this is the main reason we did not keep all the promises we made at the very start of the paper.

10 Concluding remarks and future plans

This paper presents a somewhat different approach to electoral systems. Our approach aims both at ranking electoral systems themselves and at defining a voting method for the ranking of alternatives. It is based on a hierarchically structuring of the voting system. In this way we define a rooted hierarchy. At the root we have the main goal whereas at the leaves we put the objects we want to rank through the hierarchy. The paper represents a starting point in these two directions: much work needs indeed to be done in the future both from a theoretical and from a practical/empirical point of view.

11 Thanks

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