Using auctions to allocate chores*

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Abstract

In this paper we present an application of the auction mechanisms to the allocation of a chore (a bad) to one of the bidders belonging to a given set $\mathscr{B}$. We also discuss an extension of such an application to the allocation of a set of chores among an initial set of bidders $\mathscr{B}$.
The paper aims at showing how the classic auction mechanism can be modified and adapted for the allocation of bads (chores) instead of the allocation of goods.
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1 Introduction

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The paper opens with some theoretical discussions of the characteristics and properties of some "classical" types of auctions then we present the basic motivations of the types of auction we propose. The following sections present the algorithm, the rules for the compensations, the strategies, the preferred compensation schemes and the possible extensions.
The paper closes with a section devoted to conclusions and future plans.

2 The theoretical background

In this section we present some theoretical considerations about a set of classical auction mechanisms as well as some basic considerations about the notion of chore and its main properties.
As to the auctions (Fragnelli (2005), Klemperer (1999), Milgrom (2004), Patrone (2006) and Wooldridge (2002)) we note how they are usually used for the allocation of goods so we are going to start with this case. A perspective that we fully disregard in this paper is how auctions can be used to get a fair division of goods (Brams and Taylor (1996)).
A **good** has a (not only monetary) **value** for both a seller and a buyer and this value may turn into the sum of money the seller gets from the buyer if the sale occurs. The seller is characterized by the minimum amount of money he is willing to accept for the good (m_s) and the buyer by the maximum amount of money he is willing to pay for the same good (m_b). It is easy to see that the sale occurs only if $m_s \leq m_b$ so that $m_b - m_s$ is the so called negotiation space.
We introduce at this point the main characteristics of the auctions so to define a not fully exhaustive set of classical auctions types for the exchange of goods.
Auctions (Klemperer (1999) and Wooldridge (2002), chapter 7) are characterized by a set of **factors** that can influence both the **protocol** and the **strategy** the agents use. Agents are the **auctioneer** and the **bidders**: the auctioneer tries to allocate a good to one of the bidders using an auction as an allocation mechanism.
Among the aforesaid factors we cite the **value** of the auctioned good that can be either **private** of each bidder, **common** to all the bidders or **correlated** if for each bidders it depends on the use the bidder is going to make with the good after having obtained it.
The other factors are how the winner is determined, whether the bids of the bidders are common knowledge among them or not and the number of rounds the bidders have for bidding.
The **winner** is the bidder who gets the auctioned good. In general the win-

ner is the bidder who bids the most and that can pay such sum (first-price auction) or a sum equal to the second highest bid (second-price auction). If the bids are common knowledge among the bidders we speak of **open cry** auctions otherwise we speak of **sealed-bid** auctions. As to the number of rounds if there is only one round for bidding we speak of **one shot** auction whereas if the auction is based on a succession of rounds (or it is **multi shot**) it can be **ascending** if the price starts low (possibly with a lower bound or reservation price) and rises up or **descending** if the price starts high and then descends up to a minimum value.
In the following subsections we are going to examine very briefly the following types of auctions: English auctions, Dutch auctions, First price auctions, Second price or Vickrey auctions. Of each type we describe the main features and state if bidders have an optimal strategy or not. We also devote a subsection to the definition of the concept of chore.
As to the auctioneer his goal is to maximize the revenue. It is possible to show that (Fragnelli (2005)):

1. in case of private evaluations we have $English\ auction \sim Dutch\ auction \sim First\ price\ auction \sim Second\ price\ auction$
2. in case of common evaluations we have $English\ auction \succ Second\ price\ auction \succ Dutch\ auction \sim First\ price\ auction$

where we use $\succ$ to denote a greater expected revenue and $\sim$ to denote the same expected revenue.
As to the bidders an optimal strategy (Fragnelli (2005)) is a strategy that guarantees a bidder the highest expected outcome. We comment on this for each type of auction we deal with. We note that the naming convention we use is not universally accepted since, for instance, in Klemperer (2002) page 181 what we call a sealed-bid auction is termed Dutch auction.

2.0.1 English auctions

In this case we have first-price, open cry ascending auctions where bidders make their public bids and the one who makes the current highest bid gets the auctioned good. The auctioneer starts from a low price (or reservation price that may be equal to 0) and the bidders begin offering higher and higher bids. The last offering bidder is the winner of the auction and the price he pays is the bid he made. We disregard many details and do not make any consideration about the so called winner's course or the over evaluation of the good from the winner, further details on Fragnelli (2005), Patrone (2006) and Wooldridge (2002). We only note that a dominant strategy is to bid a little more than the current bid and stop when the price reaches one bidder's evaluation of the auctioned good.

2.0.2 Dutch auctions

Dutch auctions are open cry descending auctions where the price starts high and then descends up to a lower bound. At any moment any of the bidders can call stop and get the good for that current price. Winner's course can be present also in this case but in this case we have no optimal strategy.

2.0.3 First price auctions

In this case we have a sealed bid, one shot auction where the bidders submit a bid for the auctioned good. The bidder who makes the highest bid wins that good and pays his own bid. As a tentative dominant strategy we have that each bidder must bid a little less than his own evaluation of the good, how much less depends on the bids of the other bidders. There is no general rule and so in general there is no optimal strategy. The sure thing is that there is no worth in bidding more than one's own evaluation of the auctioned good.

2.0.4 Second price or Vickrey auctions

In this case we have a one-shot, sealed bid auction where the bidder who makes the highest bid wins the good but, for getting it, pays only the second highest bid. In this kind of auction every bidder's dominant strategy is bidding his true evaluation of the good. By bidding more, a bidder has higher probabilities to get the good but runs the risk of paying for it a price greater that his evaluation of the good. Bidding less a bidder has lower probabilities of winning the good and, if he wins, he must pay the same sum as if he had made a bid equal to his true evaluation.
This kind of auction makes it possible the so called antisocial behavior since a bidder can act spitefully and bid more than his true evaluation but less that the highest bid so to force the winner to pay a higher price. Of course this is a risky attitude and needs a strong knowledge of the other bidders' bids.

2.0.5 Other types of auctions

Other types of auctions include all the variations of first/second type auctions, so we can imagine n-th price auctions with $n > 2$, and all pay

auctions, a variation of first price auction where the bidder who bids more gets the goods but all the bidders pay the bid they made, and so on.
The treatment of all these other types of auction is outside the scope of the present paper. For further details see Klemperer (2002) and Wooldridge (2002), chapter 7.

2.0.6 The concept of chore

The other concept we introduce is the concept of **chore**. With this term we denote a difficult or disagreeable task. In this case the seller of a chore (the auctioneer) is willing to pay somebody else (a bidder or a server) to carry out the chore.
We note that the possible servers must have the possibility to refuse such a chore even if such possibility may have some cost. From its definition we see how the chore has a negative value for both the auctioneer and each bidder so that we can say that a chore is something that nobody wants.
We can say that each server is characterized by an evaluation of a chore under the form of either (1) a sum that he is willing to pay for not performing it or (2) a sum that he is willing to get for performing it.
The (1) parameter is at the core of the mechanism we propose from section 5 to section 9 whereas the (2) parameter is used in the mechanism we propose in section 11.

2.0.7 Modified auctions

We extend the auction mechanism so to have an auctioneer that proposes a chore to a set $\mathscr{B}$ of bidders through one of the following three mutually exclusive mechanisms, the first two of multi shots type and the latter of one shot type.

1. The auctioneer offers the chore and a sum of money m and raises the offer (up to an upper bound M) until when one of the bidders accepts it and gets both the chore and the money. The auction ends if either one of the bidders calls "stop" or if the auctioneer reaches M without none of the bidders calling "stop". In the latter case we have a void auction sale, though this is not in the best interest of the auctioneer. The auctioneer can avoid this by properly selecting the bidders that attend the auction.

2. The auctioneer offers the chore and fixes a starting sum of money L. The bidders start making lower and lower bids. The bidder who bid less gets the chore and the money. Of course the auctioneer has no

lower bound. Under the hypothesis that the bidders are not willing to pay for getting the chore we can suppose a lower bound $l = 0$. If this hypothesis is removed we can, at least theoretically, have $l = -\infty$. We can have a void auction sale if no bidders accepts the initial value L. The auctioneer can avoid this by fixing a high enough value L.

3. The auctioneer offers the chore and the bidders bid money for not getting it under the proviso that the one who bids less will get the chore whereas the bids of the others will be used (in a way to be specified) to form a monetary compensation for the loser. Also in this case it is possible to have a void auction sale, see section 5 for further details, though this is not in the best interest of the auctioneer.

In the first case the auctioneer has a maximum value M he is willing to pay for having somebody else carry out the chore otherwise he can either give up with the chore, choose a higher value of M or repeat the auction with a different (new or wider) set of bidders. This type of auction is a sort of **Dutch auction with negative prices** paid by the bidders to get the chore. We are going to examine it in some detail in section 11.
In the second case the bidders are influenced by the value of L that can act as a threshold since if it is too low none of them will be willing to bid. This case is as if the bidders start bidding from $-L$ and raise their bids up to $-l$ so that the one who bids the most gets the chore and pays that negative sum of money. In this case we have a sort of **English auction with negative bids** that we are not going to deal with in this paper.
The last case will be fully dealt with in the present paper, starting from section 5.

3 Performance and design criteria

In this section we introduce some **performance** and **design** criteria that can be applied to mechanism design (Klemperer (2002), Myerson (1991), Rapoprt (1989), Patrone (2006) and Wooldridge (2002)).
As to the **performance criteria** we use: guaranteed success, Pareto efficiency, individual rationality, stability, and simplicity.
We say that a mechanism **guarantees success** if its goal is guaranteed to be reached in a finite amount of time whereas one of its outcomes is **Pareto efficient** if there is no other outcome where one of the participants is better off while all the others are no worse off. Success requires termination (or the fact that any process based on a mechanism ends in a finite time) but in many cases we can have mechanisms that terminate without any guarantee

of success.**Individual rationality** means that following the rules of a mechanism is in the best interests of the participants. This is a key parameter since if it is absent potential participants have no incentive in participating. **Stability** means that a mechanism has incentives for participants to behave in a certain way whereas **simplicity** means that such a way is obvious to the participants themselves.
Our aim is to check if the auction mechanisms we propose satisfy or not those performance criteria and, if it is the case, why some of them are violated.
As to the **design criteria** (Klemperer (2002)) we cannot use the **possibility of collusions** or the **entry deterrence** or the **predation** or similar parameters that refer to the bidders with regard to the auctioneer since in the mechanism we propose (from section 5 on) bidders play against each other and any collusion (for instance) turns in a redistribution of money among the bidders themselves without any involvement (as to possible losses) of the auctioneer.
The only design criterion we can introduce involves the strategies that the auctioneer can adopt in fixing the fee (see section 5.2). Similar considerations hold for what concerns the profitability of the bidders to bid untruthfully (see section 8). For further and more targeted comments see section 9.
We end this section with some comments about **social welfare**. As to this point we note how we may define it either from an utilitarian point of view (as the sum of the welfare of the individuals) or from an egalitarian point of view (as the welfare of the worse off individual). In both cases what we want is to maximize such social welfare.

4 The framing situation

The mechanism we propose from section 5 to section 9 is inspired by the following situation.
We have a commissioning authority that wants to find a place where to implement a controversial plant such as an incinerator, a dumping ground, a heavy impact industrial plant or something like that. The essential feature is that the planned infrastructure is something that nobody wants but whose services, if the infrastructure is effectively implemented, may be used by a wide group of other authorities. From this perspective it could also be a commercial port or a marina or an airport. The discriminating criterion is that the object of the agreement causes problems mainly to the accepting authority but it has a use value for possibly that authority also and for a wider group of authorities that may include also the commissioning authority. We therefore explicitly disregard situations where an agreement among

a set of authorities is needed for building the infrastructure as it happens in cases such has railway lines, highways, shipways and the like.
We have therefore an authority that makes a request and another authority (to be selected in some way) that accepts to satisfy the request by essentially providing a portion of "its" territory.
The commissioning authority therefore can identify such an authority through an auction like mechanism that involves the selection of a certain number of potential contractors (on the base of technical and economical considerations) over which it has no binding authority but with which it tries to achieve an agreement.
Such an agreement may be achieved either directly through a negotiation (such as Contract Net, Wooldridge (2002) or the mechanism we propose in section 11) or indirectly through a "negative" approach: according to this approach the selected authorities must take part to an auction and bid so to avoid the auctioned chore.

5 Basic features

5.1 Introductory remarks

We have an **auctioneer** that wants to allocate a chore to one of the **bidders** of a set $\mathscr{B}$. The n members of $\mathscr{B}$ are indexed by a set $N = \{1, \ldots, n\}$. The bidders of $\mathscr{B}$ are identified by the auctioneer who is also free to identify the chore at will. For such selections the auctioneer can:

1. identify the heaviest or highest priority chore (among those that are present in a waiting list) for him to carry out;
2. identify a set of bidders whom he expects are willing to compete for not getting the chore but that have the capabilities to carry it out and
3. fix an exclusion fee at a value that prevents all bidders to pay it and do not take part to the auction.

In this way the auctioneer selects the potential members of $\mathscr{B}$ and defines both the exclusion fee and the chore to be auctioned. Such potential members may accept to pay the exclusion fee as a fee for being excluded from $\mathscr{B}$.

5.2 The role and meaning of the fee

Before stepping any further it is necessary to explain the role and meaning of the fee so to avoid any misunderstanding.

The auctioneer fixes a fee to allow the members of $\mathscr{B}$ (that have been selected against their will) to escape from the auction but, at the same time, his main goal is the allocation of a chore to one of such bidders so that the condition $\mathscr{B} = \emptyset$ (void auction) is not a good one for the auctioneer.
The auctioneer's strategy is to choose the n potential bidders and to fix a fee **f** so that some (say m) of the potential bidders can prefer to pay the fee but all the others (say $k = n - m > 1$) prefer to attend the auction and bid.
If this is the case the auctioneer has:

1. a sum $m \times f$ and can use it as a further compensation for the losing bidder;
2. a set of k bidders that attend the auction and form the set $\hat{\mathscr{B}}$.

The amount of the sum paid by the bidders who left the auction (and so the exact number of those bidders) is a private information of the auctioneer and, therefore, cannot be used by the remaining bidders to guide their strategic behavior.
If, anyway, all the potential bidders prefer to pay the fee so that $\hat{\mathscr{B}} = \emptyset$ the auction is void and the auctioneer must refund the sums he received since he cannot keep them for himself and there is no losing bidder to be compensated.
If the auctioneer chooses a null fee then the potential bidders can leave the auction for free and therefore it is not in the auctioneer's best interest to choose a null fee.
In is way we try to model the principle of individual rationality (Myerson (1991) and Wooldridge (2002)) within an auction mechanism where the attendance is not on a voluntary basis.

5.3 The basic structure

The basic structure of the game is the following. We have a **presentation phase** (of the chore to the bidders of the bidders $b_i \in \hat{\mathscr{B}}$), a **bidding phase** (where each of the b_i bids a sum x_i for not having the chore) and an **allocation phase** (where who bids less gets the chore).
The set $\hat{\mathscr{B}}$ contains at least two bidders. If it is empty the auctioneer can repeat the auction by defining a new set to be filtered with a fee payment mechanism. If it contains only one bidder no auction really occurs and the auctioneer compensates him with the revenue from the exclusion fees paid by the other bidders.
Without any loss of generality we suppose to have only one losing bidder and that such a sole bidder is bidder b_1 whereas all the other bidders are winning

bidders and are indexed by the set $N_{-1} = N \setminus \{1\}$.
Such basic structure must be enriched to take into consideration both the possibility of having monetary compensations for the losing bidder and some particular distributions of the various bids. We refer to section 10 for the case of more than one losing bidder.
Moreover we have to specify the role of the bids x_i within the model.
In the present paper we are interested mainly in auctions that are **one shot**, **sealed bid** and with **private values** expressed on a common scale as it is common practice in auctions where the bidders usually have money as a common numerary good.
We note how this type of auction is a sort of inverse first price auction where the chore replaces the good, who bids less gets it and receives a compensation for this. We also note how we cannot have a common value auction since every bidder values the chore differently from the others. We moreover note (see section 2) how all sealed bid auctions are one shot auctions and that we disregard open cry auctions since the mechanism we want to design is based on the fact that no bidders must be influenced by the bids made by the others (Klemperer (2002) and Wooldridge (2002)).
Since b_1 is the lone loser who gets the chore we surely have:

$$x_1 = min\{x_i \mid i \in N\} \tag{1}$$

where x_1 is b_1's willingness to pay for not having the chore and represents how much the chore is worth for him. We say that x_1 is the loss of b_1.
We can define, at this point, the following quantity:

$$X = \sum_{j \in N_{-1}} x_j \tag{2}$$

as the gain of the set of winning bidders where the single x_j are the sums that each b_j saved or, in a certain sense, gained. We note, indeed, that x_j is the sum that each bidders is willing to pay for not getting the chore but it is what each bidder gets for sure if he loses the auction and gets the chore.
At this point we have to decide how to use X, possibly as a way to evaluate how to compensate b_1 for his loss x_1. Before doing that we give the basic version of the algorithm with a single losing bidder.

6 The algorithm

The basic version of the algorithm is made of the following steps:

1. **a** presents the chore to the$b_i \in \hat{\mathscr{B}}$;

2. each b_i makes his bid x_i,

3. **a** collects the bids and reveals them once they have all been collected;

4. the bidder who bid less gets the chore;

5. the other bidders compensate him for this (see section 7) and the auctioneer gives him the total fee he received from the bidders of the set $\mathscr{B} \setminus \hat{\mathscr{B}}$ (those who gave up the auction).

The algorithm is simple and linear, at least in this case, is supposed to end with only one losing bidder and differs from what we have seen in section 2 since:

1. the auctioneer has no revenue and no loss but only gets the chore allocated (a benefit, from his point of view, whose value does not influence in any way the auction since it is not known by the bidders);

2. the bidders are in competition among themselves in order to no get the chore;

3. one of the bidders loses the auction and gets the chore but is compensated by the all the other participants for his loss.

7 Compensations

As to the compensations they can involve indirectly the auctioneer and directly the winning bidders.
As to the auctioneer, he may manage the sum $m \times f$ to compensate the losing bidder on behalf of those who preferred to pay.
The auctioneer may have an incentive to be deceitful as to the amount of fees he received from the bidders who gave up and paid. To avoid this such sum should be "physically" handled by an authoritative independent third party that should collect the fees from the bidders and give them back if the auction is void.
As to the winning bidders we can devise the following two compensation schemes.

1. Every winning bidder pays to b_1 an amount proportional to his own bid:
$$p_j = \frac{x_j}{X} x_1 \tag{3}$$
for all $j \in N_{-1}$ (where X is defined in (2) and x_1 in (1)).

2. If there is a set of winning bidders $H \subseteq N_{-1}$ who bid the highest bid x_n (so that $x_n > x_j \forall j \notin H$) every member of H pays to b_1 the whole sum x_1.

When the auction is over the auctioneer can make use of a random device to choose which compensation scheme will be adopted for the current auction so that such scheme cannot be known for sure by each bidder b_j that only knows his expected payment or loss (see equation (2)):

$$E[j] = 0.5p_j + 0.5\pi_j x_1 < x_1 \tag{4}$$

since (in the worst case when $b_j \in H$):

$$0.5(\frac{x_j}{X} + \pi_j) < 0.5(1+1) = 1 \tag{5}$$

where $\pi_j \in [0,1]$ is a characteristic function that states when $j \in H$ so that we can define:

$$\pi_j = \begin{cases} 1\, if\, b_j \in H \\ 0\, otherwise \end{cases} \tag{6}$$

We note (see section 8) how the x_j are independent random variables uniformly distributed on the interval $[0, M]$ for a proper value $M > 0$ (Myerson (1991)).

8 Strategies

Before examining the strategies of each bidder b_i we define his private data that are:

1. a value m_i or the sum he is willing to pay for not getting the chore and the the sum he wants for getting it,

2. a value x_i he actually bids and that determines what he gets as a compensation (if he loses the auction) and that is actually common knowledge only when all the bids have been collected and revealed,

so that $x_i - m_i$ can be defined as the bidder's utility.
The following considerations hold under the compensation rules we have seen in section 7: if b_i wins the auction he has to pay x_1 or less whereas if he loses (so he is b_1) he gets x_1 or more. In both cases we can consider x_1 as the worst case.
We wish to prove that for every bidder b_i we have $x_i = m_i$ as the best strategy.

The intuition is the following. Making a bid x_i lower that m_i is not convenient to b_i since if he loses the auction and gets the chore he may get a low compensation, lower than his evaluation of the chore. On the other hand if he makes a bid higher than m_i he is more secure he will not lose the auction but he can run a winner's course like risk: he can be compelled to compensate the loser with a sum of money higher than his evaluation m_i of the chore (so it would have been better for him to get the chore). From this we conclude that each bidder should choose to bid a sum $x_i = m_i$. For a more formal proof of our claim we refer to the Appendix.

9 Performance and design criteria satisfaction

In this section we examine if the proposed mechanism satisfies the criteria we introduced in section 3. We start with the **performance criteria**. The mechanism **guarantees termination**, since it is a one shot auction, but does not guarantee success since, if the auctioneer badly fixes the fee, the auction can go void. Under the proviso the the fee is properly fixed the mechanism guarantees success since a losing bidder is surely identified and the chore is allocated.
As to **Pareto efficiency** we have that if the chore is allocated to one bidder that bid his own evaluation of the chore itself all the bidders are satisfied and there is no other solution in which one is better off and none is worse off so we have found a Pareto efficient solution.
As to **individual rationality** we tried to guarantee it through the mechanism of the fee as a compensation for the fact that the involvement in the auction does not occur on voluntary basis.
Stability and **simplicity** are both guaranteed by the fact the the best strategy for every bidder is to bid a sum equal to each bidder's evaluation of the chore, a very simple strategy that can be easily implemented by bidders with also a very bounded rationality.
As to the **design criteria** we have that the only parameter the auctioneer can control is the amount of fee **f** he asks to each bidder to let him leave the auction. We note that the amount of **f** is common knowledge among the bidders whereas the single values m_i are private information of each bidder. Other data of common knowledge among the bidders are:

1. if the auction is void the paid fees are refunded;

2. the paid fees are used to compensate the losing bidder.

Which is the proper value is a guess of the auctioneer even if fixing it high may seem to be of no harm for him. A high fee is an incentive to each bidder for not paying it in the hope to be the only one that acts in this way and gets the total amount of the fees as a compensation. Since all the bidders have this incentive high values of the fee turn in none of the bidders paying them. This however does not represent a bad situation for the auctioneer that can find more easily a bidder who loses the auction and gets the chore. On the other hand, too low values of the fee may harm him since all the bidders can pay them so the auction runs the risk of being void.
Social welfare is worth some final comments. We must consider the situation before and after the auction. Firstly we note that if the auction is not void the welfare of the auctioneer can only increase since he succeeds in allocating a chore (at no cost) and so gets a benefit from the auction and suffers no loss of any kind. If, on the other hand, the auction is void the auctioneer fails in allocating the chore and may suffer the expenses needed to set up the auction mechanism. In this case he is worse off and so he has incentives to choose properly the bidders and in fixing properly the exclusion fee.
As to the bidders we can analyze the situation from either the perspectives of the single bidder or from that of the whole set of the bidders.
We can suppose that, before the auction starts, each b_i has a welfare measured as w_i and that every bidder is supposed to bid his true evaluation m_i of the chore. If we consider the single bidder we have[1]:

1. each of the m bidders who pay the fee $\mathbf{f}$ (lower than each bidder's m_i otherwise each of them would had attended the auction) sees his welfare becoming $w_i - f > w_i - m_i$;

2. each of the $n-1$ winning bidders is expected to pay (see equation (4) with $x_j = m_j$):

$$E[j] = 0.5\frac{m_j}{X}m_1 + 0.5\pi_j m_1 < 0.5(\frac{m_j}{X} + 1)m_1 \leq m_1 < m_j \quad (7)$$

(since $m_j \leq X$ and $m_1 < m_j$ by definition) so that their welfare becomes $w_j - E[j] > w_j - m_1$;

3. the losing bidder has an expected utility given by:

$$E[1] = mf + \sum_{i=2}^{k} E[i] - m_1 \quad (8)$$

[1]We recall that there are k potential bidders, m of them are supposed to pay the fee $\mathbf{f}$ whereas the remaining $n = k - m$ are supposed to bid.

From equations (7) and (8) we may derive the following two cases.

1. If $m = 0$ we have $E[1] = \sum_{i=2}^{n} E[i] - m_1$. If we use equation (7) we have:

$$E[1] = 0.5m_1 \sum_{i=2}^{n} (\frac{m_i}{X} + \pi_i) - m_1 = 0.5m_1 (\sum_{i=2}^{n} \frac{m_i}{X} + \sum_{i=2}^{n} \pi_i) - m_1 \quad (9)$$

   or (in the worst case when $|H| = 1$ and from equation (2)):

$$E[1] = 0.5m_1(1 + 1) - m_1 = 0 \quad (10)$$

   so that b_1 is no worse off.

2. If $m \geq 1$ we have $E[1] = mf + \sum_{i=2}^{n} E[i] - m_1 > 0$ (since $m_1 < f$ otherwise b_1 would have paid that fee) so b_1 is better off.

If we consider the complete set of bidders, from the equations (7) and (8), we have that those who pay the fee suffer a collective loss of mf, those who bid suffer a collective loss of $\sum_{i=2}^{k} E[i]$ and the losing bidder has an expected utility given by (8) so that the complete set of bidders is worse off by m_1 that, anyway, is the less they can lose since $m_1 < m_i$ for every $i \in [2, k]$.

10 Extensions

Up to now we have supposed to have only one losing bidder and only one chore to be auctioned. In this section we extend our approach to include both the possibility of having **more than one losing bidder** and the need to **allocate a set of chores** $\mathscr{C}$ in a balanced way to a set $\mathscr{B}$ of bidders, who actually attend the auction since they did not pay the exclusion fee.
If we have a set of losing bidders L with $1 < |L| \leq n$ we have the following possibilities:

1. the auctioneer uses a proper random device to select one of them so to be back to the lone loser case where all the other bidders are therefore winning bidders;

2. the auctioneer can set up an auction among the members of L (that cannot escape paying any fee) so to choose one of them.

In the latter case there is no guarantee that a single supplementary auction is sufficient to have a single losing bidder so it may be necessary to resort to a series of supplementary auctions. Every supplementary auction involves

only the bidders indexed by the current set L and this process goes on until when the auctioneer gets $|L| = 1$ or decides to resort to a random device to make the choice.
At any step it is indeed possible to use a random device to make a choice and to find the necessary lone losing bidder.
If the auctioneer wants to allocate a set of chores $\mathscr{C}$ he can order the chores of the set $\mathscr{C}$ according to his own evaluations and then proceed (in either ascending, descending or casual order) to allocate such chores in a series of rounds, each round for the allocation of exactly one chore to one bidder.
If $|\mathscr{C}| = c \leq n$ (with $n = |N|$) it is possible to use c executions to allocate at the most one chore to each bidder so that a bidder who gets a chore at execution $k \in \{1, \ldots, c\}$ exits the allocation process.
If $|\mathscr{C}| = c > n$ each of the bidders gets necessarily more than one chore. If it is possible to express c as $c = qn + r$ we have that every bidder gets q chores and r of them get an additional chore.
To get a fair allocation of the chores to the bidders the auctioneer can use the following algorithm:

1. he evaluates q and r such that $c = qn + r$;
2. he executes q rounds and at every round allocates n chores (using the algorithm we have seen for the case $c \leq n$), one for each bidder so that every bidder gets q chores;
3. the remaining r chores are allocated with one more round reserved to the r bidders who got the r lower total sums of chore values[2].

Things may differ if the bidders know the whole set of the chores $\mathscr{C}$ before the first round of the auction process or if they know the chores only when each of them is revealed by the auctioneer.
In the former case they can act strategically and, by ordering the chores according to private criteria, can try to get the most preferred chore among those that are available at step k.
In the latter case they can act only tactically and perform a choice only on the current auctioned chore with a regret on the past auctioned chores but not knowing the possible future chores, neither their type nor their number.

[2]A **chore value** for a bidder is the sum of all the losing bids he made in the auctions for the allocations of the q chores. If those bidders are more than r it is possible to use a random device to choose exactly r of them.

11 Reverse auction: paying more and more to allocate a chore

In this section we examine the first of the cases we listed in section 2.0.7 or the case where the auctioneer offers the chore and a sum of money and raises the offer (up to an upper bound M) until when one of the bidders accepts it and gets both the chore and the money.
The value M represents the maximum amount of money that the auctioneer is willing to pay to get the chore performed by one of the bidders. We note that the value M is a private information of the auctioneer and is not known by the bidders. This fact prevents the formation of consortia and the collusion among bidders (Klemperer (2002)) since M may be not high enough to be gainful for more than one bidder.
If x is the current offer of the auctioneer **a** we can define his utility as $M - x$. As to the bidders b_i, each of them has the minimum sum he is willing to accept m_i as his own private data so that $x - m_i$ may be seen as a measure of the utility of bidder b_i.
We note that, if we define the set:

$$F = \{i \mid m_i \leq M\} \tag{11}$$

as the feasible set, the problem may have a solution only if $F \neq \emptyset$.
In this case the algorithm is the following:

1. **a** starts the game with a starting offer $x = x_0 < M$;
2. bidders b_i may either accept (by calling "stop") or refuse;
3. if one b_i accepts the auction is over, go to 5 ;
4. if none accepts and $x < M$ then **a** rises the offer as $x = x + \delta$ with $0 < \delta < M - x$, go to 2 otherwise go to 5;
5. end.

At step 3 possible ties may be resolved with a random device. At this point we have to define the strategies of both **a** and the b_i. The auction we are describing is a sort of reversed Dutch auction where we have an increasing offer instead of a decreasing price and a chore instead of a good.
The best strategy for **a** is to use a very low value of x_0 (or $x_0 \simeq 0$ so to be sure to stay lower that the lowest m_i) and, at each step, to rise it of a small fraction δ with the rate of increment of δ decreasing the more x approaches M.

The bidder b_i's best strategy is to refuse any offer that is lower than m_i and to accept when $x = m_i$ since if he refuses that price he risks to lose the auction in favor of another bidder who accepts that offer.
We have moreover to consider what incentives a bidder may have to be insincere when defining the value m_i. Of course there is no reason for b_i to define a value of m_i lower than the real one (since he has no interest in accepting lower prices). He could be tempted to define a higher value $m'_i > m_i$ so losing the auction in favor of all the bidders who are willing to accept any offer within the range $m'_i - m_i$. This means that b_i may use a higher value of m_i only if he is sure that the private values of all the other bidders are higher. Since no bidder can be sure of this, each of them has a strong incentive to behave truthfully.
In this case, if $F \neq \emptyset$ (see equation (11)), the sum **a** expects to pay is equal to m_j where $j \in F$ is such that $m_j < m_i$ for all $i \neq j$, $i \in F$.
The algorithm in the present version can be used in all cases where the auctioneer wants to "sell a chore" to the "worst offering" or to have a chore carried out by somebody else by paying him the least sum of money.

12 Concluding remarks and future plans

In this paper we presented the use of classical tools such as auction mechanisms within an unconventional framework, the allocation of chores to a set of bidders.
We defined two modified types of auction, examined their properties and gave some hints about the contexts where each of them can be used.
Future plans include both a deeper theoretical examination of such properties (with a particular regard to the bidders' strategies and algorithm's extensions) and an examination of some practical applications in areas such as the localization of energy production plants, incinerators, garbage dumps and so on.

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Appendix

We give here a more formal proof of the claim we made in section 8.
If b_i bids $x_i < m_i$ he can:

1. lose the auction and get the chore so to obtain a compensation that is in the worst case lower than his evaluation of the chore;
2. win the auction so that, in the worst case, he has to pay to the loser b_1 a compensation x_1 lower than x_i.

If the bidder loses the auction he loses, in the worst case, $x_i - m_i$ (with an unknown probability p) whereas if he wins the auction he gains $x_i - x_1$ (if the losing bidder is b_1) with probability $(1 - p)$ so that the expected revenue for bidder b_i is:

$$p(x_i - m_i) + (1 - p)(x_i - x_1) \tag{12}$$

Given p it is easy to see how the best situation for b_i occurs when $x_i = m_i$.
If b_i bids $x_i > m_i$ he can, in the worst case:

1. lose the auction and get the chore so to obtain a compensation that is higher than his evaluation of the chore;

2. win the auction so that, in the worst case, he has to pay a compensation x_1 to the loser b_1, compensation lower than x_i but possibly greater than m_i.

We can evaluate the utility of bidder b_i as:

$$u_i(x, m) = \begin{cases} x_i - m_i \; if \; i = argmin_{j \in N} \, x_j \\ y \qquad \; if \; i \neq argmin_{j \in N} \, x_j \end{cases} \tag{13}$$

where m is the vector of the evaluations of the chore for the bidders and x is the vector of the current bids of the bidders whereas m_i and x_i (with $x_i > m_i$) are those values for bidder b_i.
If the former event occurs with an unknown probability p the latter (since the two events are a partition of the sure event) occurs with a probability $1 - p$ so that we can evaluate the expected revenue of b_i as:

$$p(x_i - m_i) + (1 - p)y \tag{14}$$

In equation (14) m_i if fixed for a given b_i and (Myerson (1991)) we can imagine the bids x_i as independent random variables uniformly distributed on the interval $[0, M]$ for a proper value of $M > 0$.
In equation (14) y represents the sum that b_i may gain or lose if he is one of the winning bidders so that, in the worst case, he has to pay x_1 to the lone loser b_1.
We have the following two cases:

1. if $x_1 \leq m_i$ then b_i gains $m_i - x_1$,

2. if $x_1 > m_i$ then b_i loses $m_i - x_1$.

and both cases concur (with the proper probability) in the evaluation of y.
From the aforesaid considerations we have that:

1. if $x_i \rightarrow M$ the probability p that b_i has to lose the auction tends to 0,

2. with an increasing probability b_i risks to get y (since $(1 - p) \longrightarrow 1$),

3. y is made of a positive component upperly bounded by m_i and a negative component with a lower bound of $m_i - M$,

4. the former component is associated to a probability m_i/M and the latter to $(x_i - m_i)/M$,

5. since all the bidders tend to behave in a similar way and so tend to bid high values of x_j also x_1 tends to grow so that it is more and more probable for b_i to pay a high fee x_1 with a high probability.

We can conclude that using high bids is wrong and that the best strategy is to bid m_i. In this way b_i sets to 0 his probability to win but pay a fee higher than his evaluation of the chore.