# Additive Barter Models*

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**Abstract**

The paper presents some models involving a pair of actors that aim at bartering the goods from two privately owned pools of heterogeneous goods. In the models we discuss in the paper the barter can occur only once and can involve either a single good or a basket of goods from each actor/player. In the paper we examine both the basic symmetric model (one-to-one barter) as well as some other versions (one-to-many, many-to-one and many-to-many barters) none of which reproduces a symmetric situation. The paper presents the models, their structure and describe some possible strategies. It also presents a set of performance criteria and shows how the proposed models satisfy them.
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## 1 Introduction

The paper presents a family of models that involve a pair of actors/players that aim at bartering the goods from two privately owned pools of heterogeneous goods.

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In the models we discuss in the paper the barter is supposed to occur only once as an one shot process and can involve either a single good or a basket of goods from each actor/player. We examine both the basic symmetric model (one-to-one barter) and its extensions (one-to-many, many-to-one and many-to-many barters), none of which reproduces a symmetric situation.
The paper is structured as follows. It opens with the basic motivations of the proposed models and some definitions then it introduces some classical solutions of related problems. The next long section presents the various barter models, their structures and, for one of them, some possible strategies. The paper goes on with a description of the basic performance criteria for the evaluation of the proposed models and applies them to one of such models. The closing sections include a section devoted to some possible extensions, a section devoted to the conclusions and the plans for future work and research and an Appendix where we provide formal argumentations for some of the relations we have introduced and used in the paper.

# 2 The basic motivation

The basic motivation of the models we propose is the need to describe how an exchange of goods can happen without the intervention of any transferable utility such that represented by money or by any other numerary good. In this way the involved actors do not need to share anything, such as preferences or utilities as shared information, but the will to propose pool of goods that they present each other so to perform a barter.
All the barters are in kind and are essentially based on the following very simple basic scheme (see section 6.1): we have two actors that show each other the goods, each of them chooses one of the goods of the other and, if they both assent, they have a barter otherwise some rearrangement is needed and the process is repeated until either a barter occurs or both agree to give up.
The presence of more than two actors and the use of more complex schemes do not really modify greatly the above scheme since in any case the basic module is the one involving a pair of actors at a time. We note, indeed, how within this framework there is no numerary good so no auction like scheme is possible. Possible extensions will be examined briefly in section 9.
Lastly we underline the fact that, though we make some comments on the possible strategies (see for instance section 6.3) our approach will be more **descriptive** than **normative** since we are more interested in describing a framework that allows us to describe the actors' possible behaviors in various abstract settings than in giving (more or less detailed) recipes through which

players can attain their optimal outcomes.
Within this perspective it should be obvious why we do not explicitly describe fully detailed optimal strategies that the players can follow. Though it may seem strange we think that, given the purposes of the models, a purely normative approach would prove as too restrictive.

# 3 Goods as services

The key point of the proposed models is that each of the two players owns a set of items that enters it the barter process, $I$ for $A$ and $J$ for $B$.
In the paper we suppose both $I$ and $J$ contain goods or elements that have a positive value for both players. From this point of view a good may also be a service that one player is willing to perform on behalf of the other.
In this case, for instance, player $A$ asks to player $B$ for one of the available $B$'s services in exchange for one of the available $A$'s services that player $B$ asks to player $A$. Of course this occurs in the one-to-one barter case.
Another perspective that we disregard in the paper but that can easily be inferred from it sees the two sets $I$ and $J$ as containing chores or bads or items that have a negative value for both players.
In this latter case the two players try to allocate each other their chores so that a chore allocated from $A$ to $B$ can be seen as a service performed by $B$ on behalf of $A$ to solve a problem of $A$. In this way we can unify the two perspectives and consider the goods case as a general case.

# 4 Some definitions

With the term **barter** we mean, in this paper, an exchange of goods for other goods without any involvement of money or any other numerary good. It usually involves two players that act as peers in a peer-to-peer relationship. There may be variants such as more than two actors or not peer-to-peer relations and in section 9 we examine briefly only those of the former type. In the case of not peer-to-peer relations we think we are not in presence of a real barter mainly if one of the actors cannot refuse to accept the proposed barter.
As to the barter we note that we can have either a **one shot barter** or a repeated or **multi shot barter**.
In the **one shot case** the two actors execute the barter only once by using a potentially multi stage process that aims at a single exchange of goods and can involve a reduction of the sets of goods to be bartered.

In the **multi shot case** they repeatedly execute the barter process, every time either with a new set of goods or with a possibly partially renewed set of goods but usually excluding previously bartered goods.
In this paper we are going to examine only the one shot barter between the two actors so that there is no possibility of retaliation owing to repetitions of the barter.
In order to avoid interpersonal comparisons and the use of a common scale we let the two players show each other their goods and ask separately to each of them if he thinks the goods of the other are worth bartering. If both answer affirmatively we are sure that such interval exists otherwise we cannot be sure of its existence. Anyway the bartering process can go on, though with a lower possibility of a successful termination.
In this way we describe the absence of both a common market (as a place where goods have a common and exogenously fixed evaluation in monetary terms) or between the two players as well as the absence of any outer evaluator that can impose or even only suggest common evaluations according to a common numerary quantity to both players.
We introduce the following simplifications:

(1) the values of the goods the two actors want to barter cover, in the opinion of each player, two overlapping intervals so that a one shot barter is always possible (at least in theory);

(2) such goods and the associated values are chosen privately by each actor without any information on the goods and associated values of the other actor[1];

(3) such values are fixed once for all (at least during each barter) and cannot be changed as a function of the request from the other actor;

(4) such values must be truthfully revealed upon request from an independent third party after both requests have been made.

The last two assumptions have been made only to simplify the analysis and will be relaxed in future developments.

# 5 A brief analysis of some classical solutions

Before describing some classical solutions to at least partially related problems it is necessary to map our models of barter on an equivalent model

[1] Obviously each actor can make guesses about the nature and the values of the goods of the other actor and such guesses influence the composition of each set of goods to be bartered.

where the barter phase does not occur and is replaced by a selection procedure that can be compared, at least to some degree, with the other solutions we present in this section.
The mapping is obtained by using the following algorithm of merge, choice and separation.
The two players $A$ and $B$ have respectively $n$ and $m$ flags of two colors since $|I| = n$ and $|J| = m$. The first thing they do is to create a merged list of $n + m$ items from the lists of the contents of the two sets $I$ and $J$.
At this point each of the players has his own copy of the list so that he can put the flags on the items he wishes to obtain from the separation phase.
When the choice phase is over the two flagged lists are compared so to implement the separation phase.
We can have the following cases:

- no item has two flags on it so all the items have only one flag on them;
- some items have one flag on them, some items have two flags on them and some have no flags and them.

In the former case the separation phase is easy since the global list is partitioned in two colored sublists. In this case if the two original sets $I$ and $J$ are unchanged there has been a void barter.
In the latter case we have both selected items (one flag), contested items (two flags, Brams and Taylor (1999)) and rejected items (no flag). The selected items are assigned to each player depending on the color of the flag. The others enter as input data in a settlement phase that can end either with an agreement (so that the barter effectively occurs) or without any agreement and so without any barter.
The structure of the settlement phase is based on the displacement of one flag at a time from one item to another that each player performs in turn. The structure is easier in the simplest case of the exchange of one item for one item and is more complex if multiple items exchanges are involved but anyway the settlement phase is guaranteed to end in a finite time with either a failure or a success.
The proposed equivalent algorithm is therefore characterized by a common pool of goods that the players select (in both the selection and settlement phases) by using private and possibly qualitative values that do not need to share any common quantitative scale. In order to perform a useful comparison, our starting point is Brams and Taylor (1996). In this book, the authors propose a lot of tools and algorithms for the allocation of goods for both divisible and indivisible cases. They start from $n = 2$ players and then extend their results to the general cases with $n > 2$. A common characteristic of

such models is that players aim at more or less fair sharing of a common pool of goods on which they state preferences that can be compared in some way, even on common cardinals scales.
Another good reference is Brams and Taylor (1999), where authors present various methods for the allocation of the goods from a single pool, starting with (strict and balanced) alternation methods to switch to divide-and-choose and to end with adjusted winner method.
Also all these methods are devised to allow more or less fair divisions between two players of the goods belonging to a common pool (though extensions to more than two players are provided for all the methods).
The **adjusted winner method** (Brams and Taylor (1999)) is a two-person point allocation procedure through which the two players can share out between themselves the goods of a common pool of by assigning to each good a certain number of points out of a fixed total (that is usually fixed at $T = 100$). Each good is assigned to the player who assigns to it the higher number of points. Some redistribution of the goods and the splitting of at least one divisible good may be needed to obtain a distribution that is as near to a fifty/fifty distribution as possible.
We remark how the **adjusted winner method** requires the use of a common cardinal scale among the players since it requires that each of them assigns to each good some points on $T = 100$ and that such points are compared (either directly or as ratios) so to determine to which player every good is assigned. The features of the method show how it is unsuitable in our context.
A short analysis of classical solutions for the division of goods can be found also in Fragnelli (2008a) again with regard to either one or more divisible goods or a pool of indivisible goods. Again the presence of a for the common evaluation of the goods from the players makes such tools inappropriate as solutions to our problem.
From the comments made in Fragnelli (2008a) about auctions, moreover, it is also evident how such tools are not suitable to solve our problem. In the case of auctions the bidders are supposed to compete for a common good whose evaluations are carried out on a common cardinal scale and the competition occurs with a single auctioneer who chooses the proper auction mechanism among the many that are available. Also in this case we have both a common good and a common scale of evaluations since the various bids must be compared among themselves to establish which is the winning bid according to the structure of the chosen auction mechanism.
Other solutions to division problems that can be found in the literature involve **market games** (Fragnelli (2008b) and Shubik (1959)), **assignment games** (Fragnelli (2008b)) and **cost games** (Fragnelli (2008b)).
In the **market games** each player has an initial endowment and a preference

relation on it. Each player has an utility function defined from such relation. Players aim at a redistribution of their initial endowments so to attain efficient redistributions. A redistribution is termed efficient if no player prefers any other distribution to this one. The main point here is the merging that assumes the use of common scales for the evaluation of the endowments.
In the **assignment games** players are subdivided in two groups: buyers and sellers. Every seller owns only one good (of which he knows the evaluation) and each buyer can buy one good (of which she knows the evaluation). Prices of the objects depend on these evaluations and on the ability to bargaining of the players. In these games players aim at obtaining their maximum gain with regard to each one's evaluation. Our models owe much to these games but for the fact that every player is both a buyer and a seller so that the gain each player obtains strictly depends on two simultaneous exchanges. Moreover we have no numerary good so there is no real possibility to sell or buy.
Last but not least, in the **cost games** we must define the division of the costs of a project among the many involved users so to take care of their roles and interests. From our perspective we could use this kind of games as a touchstone with the case of the barter of either bads or chores but for the presence of duality property. According to this duality property the dual of a cost game is a profit game. From this we derive that the players of the coalitions share the same worth function and this in contrast with what we want to describe: a pair of players each with his own evaluations of his and the other's good. It is, therefore, easy to see how also this family of games has nothing to do with the problem we aim at solving.

# 6 Barter models

We suppose the actor $A$ with his pool $I = \{i_1, \ldots, i_n\}$ of $n$ heterogeneous goods and the actor $B$ with her pool $J = \{j_1, \ldots, j_m\}$ of $m$ heterogeneous goods.
The sets $I$ and $J$ represent all the goods that both players are willing to barter on that occasion so that there is no "hidden good" that can be added at later stages. This is a design choice that qualifies the proposed models as models of **explicit barter**. If we imagine that the players have "hidden goods" that can be revealed and added to the sets at later stages we deal with what we may define an **implicit barter**. In the present paper we deal only with barters of the former type.
In this case $A$ assigns a **private** (i.e. known only by him) vector $v_A$ of $n$ values to his goods of the set $I$, one value $v(i)$ for each good $i \in I$.
Also $B$ assigns a private vector $v_B$ of $m$ values to her goods of the set $J$. These

vectors are fixed before the barter begins and cannot be modified during the barter. From these hypotheses, for any subset $K \subseteq I$, player $A$ once for all can evaluate, by using a property of additivity, the quantity:

$$v_A(K) = \sum_{i_k \in K} v_X(i_k) \tag{1}$$

A similar quantity may be independently evaluated by player $B$.
In a similar way we can define a private vector $s_A$ of $m$ values of the appraisals of the goods of $B$ from $A$ and a vector $s_B$ of $n$ values of the appraisals of the goods of $A$ from $B$. In this case $A$ can evaluate:

$$s_A(H) = \sum_{j_h \in H} s_X(j_h) \tag{2}$$

for any subset $H \subseteq J$. A similar quantity may be independently evaluated by player $B$.
These assignments reflect the basic hypotheses that $A$ can see the goods of $B$ but does not know $v_B$ (the values that $B$ assigns to her goods) and the same holds for $B$ with respect to $A$.
In this way we can define four types of barter:

1. **one-to-one** or one good for one good;
2. **one-to-many** or one good for a basket of goods;
3. **many-to-one** or a basket of goods for one good;
4. **many-to-many** or a basket of goods for a basket of goods.

The second and the third case are really two symmetric cases so they will be examined together in a single section. We are going to examine such types one after the other, starting with the one-to-one type.

## 6.1 One-to-one barter

Even in this simple type of barter there must be a pre-play agreement between the two actors that freely and independently agree that each other's goods are suitable for a one-to-one barter. The barter can occur either with **simultaneous** (or "blind") requests or with **sequential** requests.
In the case of **simultaneous requests**, at the moment of having a barter we can imagine that the two actors privately write the identifier of the desired good on a piece of paper and reveal such information at a fixed time after both choices have been made. In this case we have that $A$ requires $j \in J$ and $B$ requires $i \in I$ so that:

1. $A$ has a gain $s_A(j)$ but suffers a loss $v_A(i)$;
2. $B$ has a gain $s_B(i)$ but suffers a loss $v_B(j)$.

The two actors can, therefore, evaluate privately the two changes of value of their goods (that we may slightly improperly call **utilities**):

$$u_A(i,j) = s_A(j) - v_A(i) \tag{3}$$

$$u_B(i,j) = s_B(i) - v_B(j) \tag{4}$$

since all the necessary information is available to both actors after the two requests have been devised and revealed. In equation (3) and (4) we use differences and not ratios (see section 7) essentially because in this way we think to better describe the evaluation strategy of the players when they decide to accept or refuse a barter whereas, after the barter has occurred, the actors tend to use ratios to evaluate its fairness. Anyway it is easy to see how, for instance, from equation (3) and the test condition of rule (7) it is possible to derive equation (11) and vice versa. We also note how equations (3) and (4) can be replaced by more general expressions such as:

$$u_A(i,j) = f(s_A(j), v_A(i)) \tag{5}$$

and:

$$u_B(i,j) = g(s_B(j), v_B(i)) \tag{6}$$

on condition that the resulting values can be used in rules such as (7). We note that in equations (5) and (6) $f$ and $g$ are generic functions that are used by the players to evaluate their utilities as a function of each player's evaluations of the bartered goods. We only require that both functions satisfy the following conditions:

($c_1$) are increasing functions of either $s_A$ or $s_B$,

($c_2$) are decreasing functions of either $v_B$ or $v_A$.

Equations such (3) and (4) or (5) and (6) are privately evaluated by each player that only declares **acceptance** or **refusal** of the barter, declaration that can be verified to be true by an independent third party upon request. We note that a possible strategy for both players is to maximize the value they get from the barter (and so $s_A(j)$ and $s_B(i)$). Owing to the simultaneity of the requests this is not a guarantee for each player of maximizing his own utility since in equations (3) and (4) or (5) and (6) we have a loss due to what the other player asks for himself (and so $v_A(i)$ and $v_B(j)$) (see section

6.3).
The basic rule for $A$ is the following[2]:

$$\textbf{if}(u_A \geq 0)\ \textbf{then}\ accept_A\ \textbf{else}\ refuse_A \tag{7}$$

and a similar rule holds also for $B$.
We have therefore the following four cases:

1. both players accept, $accept_A$ and $accept_B$,
2. player $A$ refuses and $B$ accepts, $refuse_A$ and $accept_B$,
3. player $A$ accepts and $B$ refuses, $accept_A$ and $refuse_B$,
4. both players refuse, $refuse_A$ and $refuse_B$.

that we are going to describe in detail in section 6.2.
In the case of **sequential requests** we can imagine that there is a chance move (such as the toss of a fair coin) to choose who moves first and makes a public request. In this way both $A$ and $B$ have a probability of 0.5 to move first.
If $A$ moves first (the other case is symmetric) and requires $j \in J$, $B$ (since she knows her possible request $i \in I$) may evaluate her utility in advance using equation (4) or (6) whereas the same does not hold for $A$ that, when he makes the request, does not know the choice $i \in I$ of $B$ and so cannot evaluate $v_A(i)$. At this level $B$ can either explicitly refuse (if $u_B < 0$) or implicitly accept (if $u_B \geq 0$).
In the refuse case $B$ can only take the good $j$ off her set so that the process restarts with a new deliberation of the possibility of the barter and a new chance move.
In the accept case the implicit acceptance is revealed by the fact that also $B$ makes a request. In this case $B$ may be tempted to chose $i \in I$ so to evaluate:

$$max\ u_B(i,j) = max\ (s_B(i) - v_B(j)) = max\ s_B(i) \tag{8}$$

where the quantity $v_B(j)$ is fixed (since it depends on the already expressed choice of $A$) and cannot be modified by $B$.
Acting in this way, $B$ may harm $A$ by causing $u_A < 0$ and this would prevent the barter from occurring at this pass. Roughly speaking we can say that since $B$ choses after $A$ she can act accommodatingly or in an exploiting way: in the first case the probability that the barter occurs are higher than in the

[2] In the general case we have $u_A \geq \varepsilon$ with $\varepsilon > 0$ if there is a guaranteed minimum gain or with $\varepsilon < 0$ if there is an acceptable minimum loss.

second case. Anyway $B$ makes a request of $i \in I$ so that also $A$ can evaluate his utility through equation (3) or (6).
Now, using rules such as (7), we may have the cases we have already seen but except for the case of double refusal since the case where who choses as the second refuses is handled at a different stage of the algorithm (see section 6.2).
All this goes on until both accepts so the barter occurs or one of them empties his set of goods or both decide to give up since no barter is possible, how it will be clear from the description that we are going to make in section 6.2.

## 6.2 Formalization of the models

In this section we present a concise but fairly detailed listing of the two models of the one-to-one barter, starting from the case of **simultaneous or "blind" requests**.
In this case the algorithm is based on the following steps:

(1) both $A$ and $B$ show each other their goods;

(2) both players decide if the barter is [still] possible or not;

  (a) if it is not possible then go to step (6);
  (b) if it is possible then continue;

(3) both simultaneously perform their choice (so $A$ chooses $j \in J$ and $B$ chooses $i \in I$);

(4) when the choices have been made and revealed both $A$ and $B$ can make an evaluation (using equations (3) and (4) or equations (5) and (6)) and say if each accepts or refuses (using rules such as (7));

(5) we can have one of the following cases:

  (a) if ($accept_A$ and $accept_B$) then go to step (6);
  (b) if ($refuse_A$ and $accept_B$) then \\at $A$'s full discretion
    i. either $A$ executes $I = I \setminus \{i\}$ and if ($I \neq \emptyset$) then go to step (2) else go to step (6);
    ii. or $A$ only executes a new choice and then go to step (4);
  (c) if ($accept_A$ and $refuse_B$) then \\at $B$'s full discretion
    i. either $B$ executes $J = J \setminus \{j\}$ and if ($J \neq \emptyset$) then go to step (2) else go to step (6);

   ii. or $B$ only executes a new choice and then go to step (4);

(d) if ($refuse_A$ and $refuse_B$) then

   i. $A$ executes either $I = I \setminus \{i\}$ or a new choice; \\at $A$'s full discretion

   ii. $B$ executes either $J = J \setminus \{j\}$ or a new choice; \\at $B$'s full discretion

   iii. if (both $A$ and $B$ make a new choice) then go to (4);

   iv. if (only one of $A$ and $B$ makes a new choice and the reduced set of the other is not empty) then

      - if (the barter is still possible) then go to (4);
      - if (the barter is not possible) then go to (6);

   v. if (only one of $A$ and $B$ makes a new choice and the reduced set of the other is empty) then go to step (6);

   vi. if (both reduce each one's set and $I \neq \emptyset$ and $J \neq \emptyset$) then go to step (2) else go to step (6);

(6) end of the barter.

The solution we have adopted at point (5)($d$) is the most flexible since it mixes the two cases (5)($b$) and (5)($c$) and gives the two players the full spectrum of possibilities at the same time remaining simple enough to be understood and implemented by the players.
We remark how at the very beginning of the process we suppose that the barter is possible though this is not necessarily true at successive interactions.
We now give the description of the model with **sequential requests**. We denote the player who moves first as 1 (it can be either $A$ or $B$) and the player who moves second as 2 (it can be either $B$ or $A$) and for both we use male forms. With a similar convention we denote as $I_1$ the set of goods and $i_1$ a single good of player 1 whereas for player 2 we have respectively $I_2$ and $i_2$:

(1) both players show each other their goods;

(2) both players decide if the barter is [still] possible or not;

   (a) if it is not possible then go to step (10);

   (b) if it is possible then continue;

(3) there is a chance move to decide who moves first and makes a choice;

(4) 1 reveals his choice $i_2 \in I_2$;

(5) 2 can now perform an evaluation of all his possibilities;

(6) if 2 refuses he takes $i_2$ off his barter set then go to (2);

(7) if 2 accepts he can reveal his choice $i_1 \in I_1$;

(8) both 1 and 2 can make an evaluation (using equations such as (3) and (4) or equations (5) and (6)) and say if each accepts or refuses (using rules such as (7));

(9) we can have one of the following cases:

    (a) if ($accept_1$ and $accept_2$) then go to step (10);

    (b) if ($refuse_1$ and $accept_2$) then \\at 1's full discretion

        i. either 1 performs $I_1 = I_1 \setminus \{i_1\}$ and if ($I_1 \neq \emptyset$) then go to step (2) else go to step (10);

        ii. or 1 only performs and reveals a new choice and then go to step (8);

    (c) if ($accept_1$ and $refuse_2$) then \\at 2's full discretion

        i. either 2 performs $J_1 = J_1 \setminus \{j_1\}$ and if ($J_1 \neq \emptyset$) then go to step (2) else go to step (10);

        ii. or 2 only performs and reveals a new choice and then go to step (8);

(10) end of the barter.

We note that the case (9.$c$) ($accept_1$ and $refuse_2$) can occur as a consequence of the case (9.$b$).

## 6.3 Possible strategies in the one-to-one barters

We now make some comments on the possible strategies that the players can adopt in the case of the algorithms we have shown in section 6.2.
In the case of **simultaneous requests** both players perform their choice without knowing the choice of the other. If they evaluate their utilities according to equations such as (3) and (4) their best strategy would seem to choose the good of the other that each value at the most.
In this case we have that:

$A$ requires $\hat{j} = argmax_{j \in J} s_A(j)$ and causes $B$ a loss that $A$ may only roughly estimate;

$B$ requires $\hat{i} = argmax_{i \in I} s_B(i)$ and causes $A$ a loss that $B$ may only roughly estimate.

Acting in this way each of them may have the other player to refuse the barter. As we have seen a refusal may turn into the withdrawal of a good from one of the sets $I$ or $J$. This fact is surely unfavorable for each player. Both players therefore have strong incentives to devise better strategies.
In what follows we introduce one possible strategy under the hypothesis the both players use a more slack rule than rule (7) so that acceptance or refusal are rather discretionary than linked to a condition satisfaction criterion.
We devise a strategy for player $A$ whereas for player $B$ we have two possibilities:

(1) $B$ follows a generic non systematic strategy,

(2) $B$ follows a similar strategy.

The strategy for $A$ is the following.
$A$ orders the set $J$ of $B$ in increasing order (from the lowest to the highest) according to the values he gives to its elements.
In the case **(1)** $B$ uses a generic strategy of selection whereas in the case (2) she uses an analogous strategy over the set $I$ of $A$.
The process of choice and request involves a certain number of pass until an agreement is reached either in a positive or in a negative sense. At the generic $l$-th pass (with $l = 1, \dots$) $A$ requires the current item of lower value $j_l \in J$ whereas $B$ chooses $i \in I$.
After the $l-$th choice from both $A$ and $B$ at pass $l$ we may have:

($a_1$) $A$ accepts so that everything depends on the decision taken by $B$,

($a_2$) $A$ refuses so that both goes at pass $l + 1$-th.

In this way $A$ (but a similar argument holds also for $B$) scans the vector $J$ from lower to higher values goods looking for the right opportunity to perform a barter and having as the last choice the remaining good of highest value. We recall indeed that at any pass both players may decide to prune their own sets of goods.
In the case of **sequential requests** we have that the two players make the choice one after the other according to an order that, at each step, depends on a random device. In this case, therefore, the players can adopt strategies similar to those we have seen for the simultaneous requests case but can try to exploit the advantage of being second mover.
Let us suppose we are at a generic step where $A$ moves as first and $B$ as

second. We consider $B$'s point of view but similar considerations hold also for $A$'s point of view. $B$ has ordered the goods of $I$ in increasing order of value. In this case we have:

- $A$ chooses $j \in J$ so that $B$ is able to evaluate $v_B(j)$
- $B$ can choose $i \in I$ so to get a high value of his utility $u_B(i, j) = g(s_B(i), v_B(j))$ but
- without hurting $A$ since in that case $A$ could refuse the barter.

We recall that a refusal may turn into the pruning of a set and so in a unfavorable outcome for the requesting player that had requested the pruned good. From these considerations we derive that the step-by-step strategy that we have seen in the simultaneous requests case can be profitably used also in this case.
Similar strategies can be conceived, with the proper modifications and adaptations, for the other three models of barter that we are going to describe in the next two sections.

## 6.4 One-to-many and many-to-one barters

In these two symmetric cases one of the two actors has the possibility to require one good whereas the other has the possibility to require a basket of goods (that can even contain a single good) and so any subset of the goods offered by the former. This kind of barter must be agreed on by both actors and can occur only if one of the two actor agrees to be offering a pool of "light" goods whereas the other agrees to be offering a pool of "heavy" goods.
The meaning of the terms "light" and "heavy" may depend on the application and must be agreed on during a pre-barter phase by the actors themselves. We remark how the adopted perspective (lack of any quantitative common scale) turns into qualitative evaluations of the goods so that they are termed **light** if they are assigned **qualitatively low values** whereas they are termed **heavy** if they are assigned **qualitatively high values**.
The aim of this preliminary phase is to give one of the two actors the possibility of asking for any set of goods whereas this same possibility is denied to the other. If there is no agreement during this phase, three possibilities are left: they may decide either to give up (so the bather process neither starts) or to switch to a one-to-one barter (see section 6.1) or to a many-to-many barter (see section 6.5).
If there is a pre-barter agreement we may have two symmetrical cases. In

this section we are going to examine only the "one-to-many" case. In this case we have:

1. $A$ owns "light" goods and may require only a single good $j \in J$,
2. $B$ owns "heavy" goods and may require (at her free choice) a subset $\hat{I}_0 \subseteq I$ of goods with $|\hat{I}_0| \leq n$,

and the two requests may be either simultaneous or sequential.
In the case of **simultaneous requests** both actors can evaluate their respective utilities, soon after the requests have been revealed, by using equivalent relations to (5) and (6) (or also to (3) and (4)):

1. $u_A(\hat{I}_0, j) = f(s_A(j), v_A(\hat{I}_0))$
2. $u_B(\hat{I}_0, j) = g(s_B(\hat{I}_0), v_B(j))$

where both players use equations like (1) and (2) and the additivity hypothesis.
Also in this case we can have the four cases we have seen in section 6.1. We note, however, how in this case if, for instance, $A$ refuses, using a rule such as (7), he can either repeat his request (with $B$ keeping fixed her request ) or can act as we are going to show in section 6.5. In the latter case indeed $A$ can partitions his goods in subsets that he is willing to barter, possibly updating these subsets at every refusal. Except for this fact the barter goes on as in the $one - to - one$ case with simultaneous requests.
In the case of **sequential requests** the procedure does not use a chance move to assign one of the two actors the right to move first but gives this right to the actor that owns the pool of "light" goods. After this first move the barter goes on as in the $one - to - many$ case with sequential requests but without any chance move and with the modification we have introduced for the case of the refusal (see section 6.5).

## 6.5 Many-to-many barter

In this case $A$ may chose and require any subset $\hat{J}_0 \subseteq J$ with $1 \leq |\hat{J}_0| \leq m$ of the goods of $B$ whereas $B$ may chose and require any subset $\hat{I}_0 \subseteq I$ with $1 \leq |\hat{I}_0| \leq n$ of the goods of $A$ and the two requests may be either **simultaneous** or **sequential**.
Also this kind of barter must be agreed on by both actors in a pre-barter phase during which they both agree that in the course of the barter each of them can ask for a subset of the goods of the other player.
Since also in this case we can have either simultaneous or sequential requests

the algorithms are basically the same that in cases of one-to-one barter. The main differences are about:

(1) the use of the subsets,

(2) the way in which every case of refusal is managed.

As to the point **(1)** we note that in the algorithms we must replace single elements with subsets of the pool of goods so that the evaluations must be performed on such subsets by using equations (1) and (2) and so the additivity hypothesis.
As to the point **(2)** in the algorithms for the one-to-one barter the solution we adopted was the possible pruning of the set of the goods from the refusing actor (see the points 5 or 9 (*b*), (*c*) and (*d*) of the algorithms of section 6.2). This solution cannot be applied in the present case since this policy could empty one of the two initial pools or both in a few steps. To get a solution in this case we can devise an independent partitioning strategy of the two sets of goods from both actors $A$ and $B$.
In this case at the very start of the barter the two players show each other their sets of goods so to hide their preferences that are partially revealed only after each refusal. After every (possibly double) refusal the player who refuses (be it $A$) uses the procedure $partitioning_A(I)$ to split $I$ in labeled disjoint subsets so to make clear to $B$ which are the subsets of goods that he is inclined to barter at that stage. The case of $B$ is fully symmetric. We note that under the additivity hypothesis the sets $I$ and $J$ can be partitioned at will by their respective owner.
This solution is implemented by replacing all the occurrences of the assignment instructions $I = I \setminus \{i\}$ and $J = J \setminus \{j\}$ respectively with the following assignments:

$$I = partitioning_A(I) = \{I_i \mid \cup_i I_i = I \; I_i \cap I_j = \emptyset \, \forall \, i \neq j\} \tag{9}$$

$$J = partitioning_B(J) = \{J_i \mid \cup_i J_i = J \; J_i \cap J_j = \emptyset \, \forall \, i \neq j\} \tag{10}$$

so to replace a flat set with a set of disjoint labeled subsets.
In this case, referring to $A$, we have that if $A$ refuses the barter proposed by $B$ he can either repeat his request with $B$ keeping fixed her request or he can partition his set in subsets as collective goods that he is willing to barter with subsets of the goods of $B$. The case of $B$ is fully symmetrical.
We recall how the barter in this case may evolve as follows. At the very start the two players propose each other their sets of goods. Then we can have the following cases.

1. Both players make a request and both accept. In this easy case the barter is successful and ends.

2. Both players make a request but one accepts whereas the other refuses. The refusing player has the possibility to rearrange his set of goods. This rearrangement is a partitioning of the player's set of goods according to the rules (9) or (10) so that the other player, at the next step, knows which are the subsets that can enter successfully into a barter.

3. Both players make a request and both refuses. The rearrangement is performed by both players at the same time.

For further details we refer to section 6.2.

# 7 The basic criteria

In this section we introduce some basic criteria (from Brams and Taylor (1996), Brams and Taylor (1999) and Young (1994)) that allow us to frame the models we propose in a general context. Our aim is twofold: define "objective" criteria and use them to evaluate the goodness of the proposed models of barter.

Our main aim is to characterize **fair** barters. As a measure of fairness we refer to Brams and Taylor (1999) where a procedure is defined as fair if it satisfies the criteria of **envy-freeness**, **equitability** and [**Pareto**] **efficiency** so that each player can achieve a certain level of satisfaction by using a proper strategy. Since this goal is private of each player we say that the level of satisfaction of each player is fully independent from the level of satisfaction of the every other party/player.

In our context we have two players each possessing a pool of private heterogeneous goods and each aiming at a barter that possibly satisfies all the aforesaid criteria so to be fair.

Generally speaking, we say that an agreement turns into an allocation of the goods between the players that is **envy-free** if (Brams and Taylor (1996), Brams and Taylor (1999) and Young (1994)) none of the actors involved in that agreement would prefer somebody's else portion, how it derives to him from the agreement, to his own. If an agreement involves the sharing of benefits it is considered envy-free if none of the participants believes his share to be lower than somebody's else share, whereas if it involves the share of burdens or chores it is considered envy-free if none of the participants believes his share to be greater that somebody's else share. In other words a

procedure is envy-free if every player thinks to have received a portion that is at least tied for the biggest (goods or benefits) or for the lowest (burdens or chores).
If an allocation is envy-free then (Brams and Taylor (1999)) it is **proportional** (so that each of the $n$ players thinks to have received at least $1/n$ of the total value) but the converse is true only if $n = 2$ (as in our case).
As to **equitability** in the case of two players (and therefore in our case) we say (according to Brams and Taylor (1999)) that an allocation is equitable if each player thinks he has received a portion that is worth the same in one's evaluation as the other's portion in the other's evaluation. It is easy to see how equitability is generally hard to ascertain (Brams and Taylor (1996) and Brams and Taylor (1999)) since it involves inter personal comparisons of utilities. In our context we tried to side step the problem by using a definition that considers both utilities with respect to the same player.
Last but not least, as to **efficiency**, we say (according to Brams and Taylor (1999)) that an allocation is efficient if there is no other allocation where one of the players is better off and none of them is worse off. In general terms efficiency may be incompatible with envy-freeness but in the case of two players where we have compatibility.
Such criteria, in order to be used in our context of two players without either any common scale or any numerary good, must be adapted or must be redefined someway so to be in agreement either with the essence of their classical definitions or with intuition or with both. In what follows we are going to make use of a general notation that must be specialized in the single models we have already presented in the proper past sections.
We start with **envy-freeness**. If we denote with[3,4] $a_A(\cdot)$ and $l_A(\cdot)$ the values in $A$'s opinion and evaluation, respectively, of what $A$ obtains and loses from the barter (and with $a_B(\cdot)$ and $l_B(\cdot)$ the same quantities for player $B$) we say that the allocation deriving from a barter (or a barter tout court) is

[3] With $\cdot$ we denote a generic set of bartered goods. This set may contain also a single element.

[4] In the one-to-one barter model, for instance, we have that:

1. $a_A(\cdot) = s_A(j)$
2. $l_A(\cdot) = v_A(i)$
3. $a_A(\cdot) = s_B(i)$
4. $l_B(\cdot) = v_B(j)$

whereas in the other cases the single elements must be replaced by the properly defined subsets.

**envy-free** if we have for $A$:

$$\frac{a_A(\cdot)}{l_A(\cdot)} \geq 1 \tag{11}$$

and for $B$:

$$\frac{a_B(\cdot)}{l_B(\cdot)} \geq 1 \tag{12}$$

As we have already seen from section 6 on, if a barter actually occurs it is guaranteed to be envy-free. Relation (11) means that the value that $A$ assigns to what he gets from the barter is at least equal to the value that $A$ assigns to what he loses from the barter. We assign a similar meaning to relation (12) with regard to $B$.

Since, in our case of two players, we want to maintain the equivalence between proportionality and envy-freeness we must give a definition that mirrors the classical definition of proportionality and reflects this equivalence.

For player $A$ we may define a barter as proportional if it satisfies the following condition:

$$\frac{a_A(\cdot)}{a_A(\cdot) + l_A(\cdot)} \geq \frac{l_A(\cdot)}{a_A(\cdot) + l_A(\cdot)} \tag{13}$$

so that the fractional value of what $A$ gets from the barter is at least equal to that of what he loses from it. We remark that $a_A(\cdot) + l_A(\cdot)$ represents the value that $A$ assigns to the bartered goods.

A similar condition holds also for $B$:

$$\frac{a_B(\cdot)}{a_B(\cdot) + l_B(\cdot)} \geq \frac{l_B(\cdot)}{a_B(\cdot) + l_B(\cdot)} \tag{14}$$

We say that a barter is proportional if both (13) and (14) hold.

It is easy to see how from equation (13) it is possible to derive equation (11) and vice versa. The same holds also for equations (14) and (12).

As to **equitability** we must adapt its definition to our framework in the following way. We need firstly some definitions. We define (with respect to the occurrence of the barter itself) $I$ and $I'$, respectively, as the ex-ante and ex-post sets of goods of $A$ and $J$ and $J'$, respectively, as the ex-ante and ex-post sets of goods of $B$. If $(i, j)$ denotes the bartered goods ($i$ from $A$ to $B$ and $j$ from $B$ to $A$) in a one-to-one barter, we have:

$$I' = I \setminus \{i\} \cup \{j\} \tag{15}$$

$$J' = J \setminus \{j\} \cup \{i\} \tag{16}$$

In the case of other kind of barters involving also subsets of goods we must appropriately replace single goods with subsets.

On the sets $I'$ and $J'$ we define, for the player $A$, the quantities that represent the values for $A$ himself, after the bater, of his goods and $B$'s goods, respectively, as $a_A(I')$ and $l_A(J')$. We therefore define a barter as **equitable** for $A$ if the fractional value of what he gets is at least equal to the fractional value he gives to what he loses from the barter or (see the Appendix):

$$\frac{a_A(j)}{a_A(I')} \geq \frac{l_A(i)}{a_A(I)} \tag{17}$$

On the other hand the barter is equitable for $B$ if, using the corresponding quantities we used in equation (17) but referred to player $B$, we have (also in this case see the Appendix):

$$\frac{a_B(i)}{a_B(J')} \geq \frac{l_B(j)}{a_B(J)} \tag{18}$$

If both relations hold we say that the barter is **equitable**. We remark that we are under an additivity hypothesis where the value of a set is given by the sum of the values of its elements so that the value that a player assigns to a set, such as $I'$ or $J'$, is the sum of the values that the player assigns to the elements of that set.
As to **efficiency** we say that a barter of the two goods $(i, j)$ (or of the one-to-one type) is efficient if there is not another pair of goods $(i', j')$ that gives at least to one player a better result without hurting the other.
For players $A$ and $B$ this means that there is no barter $(i', j')$ that satisfies the following inequalities:

$$\frac{a_A(j)}{l_A(i)} \leq \frac{a_A(j')}{l_A(i')} \tag{19}$$

$$\frac{a_B(i)}{l_B(j)} \leq \frac{a_B(i')}{l_B(j')} \tag{20}$$

with at least one of them satisfied with the $<$ relation.
In such relations the pairs $l_A(i)$, $a_A(j)$ and and $l_A(i')$, $a_A(j')$ are related to $A$ and are associated respectively to $(i, j)$ and to $(i', j')$. Similar quantities are defined also for player $B$.
We remark how we are under the hypothesis that at least one of the following inequalities hold:

1. $i' \neq i$
2. $j' \neq j$

Also in this case if the barter involves subsets of goods such relations must be modified by replacing single goods with properly defined subsets of goods. We note that if the barter is such that both players attain:

$$\frac{a_{A_{max}}}{l_{A_{min}}} \tag{21}$$

and

$$\frac{a_{B_{max}}}{l_{B_{min}}} \tag{22}$$

we are sure to have an **efficient barter** whereas if both attain:

$$\frac{a_{A_{min}}}{l_{A_{max}}} \tag{23}$$

and

$$\frac{a_{B_{min}}}{l_{B_{max}}} \tag{24}$$

we are sure that the barter is surely inefficient (see the Appendix). In (21) and (23) with respectively $a_{A_{max}}$ and $a_{A_{min}} \leq a_{A_{max}}$ we denote the maximum and minimum values that $A$ assigns to the goods he can get from the barter and with $l_{A_{max}} \geq l_{A_{min}}$ and $l_{A_{min}}$ we denote the maximum and minimum values that $A$ assigns to the goods he may lose from the barter. In (22) and (24) we have the same quantities assigned to the corresponding goods by player $B$.

We remark how conditions (21), (22), (23) and (24) are sufficient conditions of efficiency and do not represent effective strategies for each player since condition (21), for instance, has a quantity that depends on the choice of $A$ at the numerator but a quantity that depends on the choice of $B$ as denominator.

Last but not least, we note, from the equations (19) and (20), how **efficiency** of a barter cannot be always guaranteed and must be verified case by case.

# 8 Fairness of the proposed solutions

In this section we aim at verifying if the solutions we have proposed in the previous sections satisfy the criteria we stated in section 7 so that we can say whether they produce fair barters or not.

We start with **envy-freeness** in the one-to-one barter. In this case a barter occurs if and only if both $A$ and $B$ get a non negative utility form it or if both players think each of them gets no less than one looses. This turns, in the simplest case, in the following conditions (involving strictly positive quantities):

$(b_1)$ $s_A(j) - v_A(i) \geq 0$ or $\frac{s_A(j)}{v_A(i)} \geq 1$

$(b_2)$ $s_B(i) - v_B(j) \geq 0$ or $\frac{s_B(i)}{v_B(j)} \geq 1$

so that $(b_1)$ coincides with relation (11) and $(b_2)$ coincides with relation (12). In this way we can derive that if a barter occurs then it is guaranteed to be **envy-free** (and therefore **proportional**, since we have maintained the equivalence between the two concepts in the current case of two players).
In more complex settings things can be more tricky to prove but, following similar guidelines, it is possible to show that whenever a barter occurs it is guaranteed to be envy-free.
We recall that in every case where a set of goods is involved we can evaluate its worth by using the additivity hypothesis.
As to **equitability** (see relations (17) and (18)) we refer only to player $A$ since the case of $B$ is completely analogous. In this case we remark that (see also the Appendix):

$(eq_1)$ $a_A(j) < a_A(I')$

$(eq_2)$ $l_A(i) < a_A(I)$

From $(eq_1)$ and $(eq_2)$ we can easily derive $a_A(j)l_A(i) < a_A(I')a_A(I)$ or:

$$\frac{a_A(I')}{a_A(j)} > \frac{l_A(i)}{a_A(I)} \tag{25}$$

On the other hand from $(eq_1)$ it is possible to derive (see the Appendix):

$$\frac{a_A(I')}{a_A(j)} > \frac{a_A(j)}{a_A(I')} \tag{26}$$

If we compare relations (25) and (26) with relation (17) we can easily see that there may be possibilities to have an equitable barter for $A$ and, in a similar way, an equitable barter for $B$ so to get an **equitable barter**.
For $A$ this occurs if we get:

$$\frac{a_A(I')}{a_A(j)} > \frac{a_A(j)}{a_A(I')} > \frac{l_A(i)}{a_A(I)} \tag{27}$$

since the lrightmost inequality is equivalent to relation (17).
In order for this to happen we must have:

$$a_A(j)a_A(I) > a_A(I')l_A(i) \tag{28}$$

or (see also the Appendix):

$$a_A(j)a_A(i) = a_A(j)l_A(i)\alpha > a_A(I')l_A(i) \tag{29}$$

so that we need to find the minimum value $\alpha > 1$ such that:

$$\alpha a_A(j) > a_A(I') \tag{30}$$

holds. Instead than using ($eq_1$) we could have used ($eq_2$) so to derive the corresponding necessary value for $\beta$ (see the Appendix).
In this way, since we do not use at all the condition of envy-freeness, we establish an independence between the two concepts but for the fact that if a barter is not envy-free it does not occur so that it is not possible to evaluate its degree of equitability.
Last but not least we deal with the verification of the **efficiency** of a barter $(i, j)$ in the case of a one-to-one barter. In this case we must verify that there is not another barter $(i', j')$ such that the relations (19) and (20) hold.
Even if $A$ choses $\hat{j}$ (see section 6.3) $B$ could have chosen $i'$ such that $l_A(i') < l_A(i)$ so that relation (19) (with $j = \hat{j}$) would be verified implying that the current barter $(i, \hat{j})$ is not efficient.
Similar considerations hold also for $B$. From these considerations we derive that **efficiency** for both players can be verified only a posteriori. If it is violated we derive **inefficiency** from which both actors may derive a **regret** that could be (at least partially) compensated through repeated barters (see section 9).
Summing up, we can say that, in the case of one-to-one barter:

- envy-freeness is guaranteed every time a barter occurs,
- equitability may be guaranteed at every barter,
- efficiency must be verified a posteriori at every barter,

so that the **fairness** of a barter is a by-product of the barter process itself and is not a-priori guaranteed by its structure.
Similar considerations hold also for the other three models.

# 9 Extensions

The planned extensions include the possibility of (1) **repeated barter** involving (2) even **more than two** players and (3) the **relaxing of additivity**.

If we allow the execution of **repeated barters** we must introduce and manage the possibility of the retaliations between the players from one barter session to the following sessions and how the pool of goods are defined and/or modified between consecutive barter sessions. In the proposed algorithms (currently stateless) we can deal with the presence of the **retaliation** through **state variables** that account for past attitudes of the players (Axelrod (1985) and Axelrod (1997)).
If we allow the presence of **more than two actors** we must introduce the mechanisms for the execution of parallel and concurrent negotiations.
If, for instance, we have three actors $A$, $B$ and $C$ we can have (in the case of one-to-one barter with simultaneous requests) the following possibilities.

1. Circular one-to-one requests where, for instance, $A$ makes a request to $B$, $B$ to $C$ and $C$ to $A$.

2. One-to-many requests so that $A$ makes a request to $B$ and $C$, $B$ makes a request to $A$ and $C$ and $C$ makes a request to $B$ and $A$.

In the former case there can be no conflict/concurrence whereas in the latter it can occur that two actors ask the same item to the third causing a conflict that must be resolved some way.
In both cases we have:

1. the barter occurs if and only if every actor accepts what is proposed by the others;

2. if all actors refuse the others' proposals a rearrangement (that depends on the nature of the barter) of the respective pools occurs followed by a repetition of the barter;

3. in all the other cases the procedure must allow the refusing actors (two at the most) to repeat their request.

Obviously in all the other cases the interactions tend to be more and more complex. Analysis of such extensions can be carried out using the tools suggested in Myerson (1991), section 9.5 where *graphical cooperation structures* are introduced and used.
As a last extension we mention the **relaxing of additivity**. Additivity is undoubtedly a simplifying assumption and is based on the hypothesis of the relative independence of the goods that the actors want to barter among themselves. This hypothesis in many cases is not justified since functional links, for instance, make the goods acquire a value when and only when they are properly combined. In such cases the goods must be bartered as dynamically chosen subsets and cannot enter properly in a one-to-one barter. The

issue is very complex (so complex that Brams and Taylor (1996) and Brams and Taylor (1999) deal with it only marginally) and here we only make some basic comments and considerations and present a toy example.
We recall that player $A$ choses among the goods of $B$ and vice versa. What $A$ loses, owing to the choice performed by $B$, belongs to the set $I$ and is evaluated according to the values of $v_A$ and what he gets belongs to $J$ and is evaluated according to the values of $s_A$. Similar considerations hold also for player $B$.
Up to now we have supposed that $A$ evaluates subsets of the goods involved in the barter with additive rules and that the same holds also for $B$. From this point on we are going to consider both subadditivity and superadditivity for player $A$ but similar considerations hold also for player $B$.
We note that as to $s_A$ subadditivity (or the case where the value of the set is lower than the sum of the values of its composing elements) is meaningless since in this case $A$ would be better off by simply asking for a single good from $B$. On the other hand subadditivity on $v_A$ is highly implausible since there is no reason to believe that $A$ would bring to the barter goods that taken as sets are worth less than the single goods.
From these considerations we derive that:

(1) $A$ sees $J$ in a superadditive way by hypothesis,

(2) $A$ sees $I$ in a superadditive way as his worst case,

and similar considerations hold also for the player $B$.
As to (1) this means that $\forall K \subseteq J$:

$$s_A(K) \geq \sum_{j_k \in K} s_A(j_k) \tag{31}$$

$A$ is of course more interested in subsets $K \subseteq J$ such that:

$$s_A(K) > \sum_{j_k \in K} s_A(j_k) \tag{32}$$

We call the subsets for which relation (31) holds the **superadditive subsets** of $J$ and those for which relation (32) holds the **strictly superadditive subsets** of $J$.
As to (2) we recall that $I$ contains the goods that $A$ loses in the barter so that the condition:

$$v_A(H) \geq \sum_{i_h \in H} v_A(i_h) \tag{33}$$

(for $H \subseteq I$) represents a worst condition for $A$ with regard to the additive case in the evaluation of his utility in the one-to-many and many-to-many

| *A vs. B* | additive | superadditive |
|---|---|---|
| additive | one-to-one | one-to-many |
| superadditive | many-to-one | many-to-many |

Table 1: *Possible types for the ways in which each player evaluates their requested goods*

barter cases. At this point we have the cases of Table 1 where we show the possible typologies of the players with regard to the values $s_A$ for $A$ and $s_B$ for $B$.
From this perspective, the fact that $A$ is superadditive means that at least relation (31) holds and the same is true for $B$ if she is superadditive.
From that Table we se that if both players are superadditive they are more willing to agree on a many-to-many barter, if they are both additive they may prefer a one-to-one barter whereas if one is superadditive and the other is additive they may agree on either a many-to-one or a one-to-many barter depending on which is the superadditive player.
In the closing part of this section we are going to deal only with the **many-to-many** barter case with **simultaneous requests** where $A$ asks for the goods of the set $J_0 \subseteq J$ and loses the goods of the set $I_0 \subseteq I$ whereas $B$ asks for the goods of the set $I_0 \subseteq I$ and loses the goods of the set $J_0 \subseteq J$.
Also in this case the core of the algorithms (see sections 6.2 and 6.5) is composed by the four cases that may occur at each pass:

(a) both $A$ and $B$ accepts the proposed barter so that the process ends with a success;

(b) $A$ accepts but $B$ refuses;

(c) $A$ refuses whereas $B$ accepts;

(d) both $A$ and $B$ refuse.

In the symmetric cases $(b)$ and $(c)$ the accepting player keeps his request fixed while the refusing player has two possible mutually exclusive strategies:

- can repeat his choice;
- can partition (on the first refusal) or rearrange a partitioning (on successive refusals) his set of goods so that another round may occur.

In the case $(d)$ each player has both the repeater and the modifier strategies at his disposal.

The fact that a player rearranges in some way his goods through the definition of variable partitions interfere with the superadditive evaluations of the other player and this may cause both players agree that there is no possibility for the process to go on (see the step $(2)(a)$ of the simultaneous requests algorithm of section 6.2).
To make things more concrete we now make a toy example. We suppose to have a player $A$ with his set of goods $I = \{i_1, i_2, i_3, i_4, i_5\}$ and another player $B$ with her set of goods $J = \{j_1, j_2, j_3, j_4, j_5, j_6\}$.
We suppose that both $A$ and $B$ have [strictly] superadditive evaluations of the involved goods (so that relation (31) and possibly relation (32) hold) and both are interested in a many-to-many barter and agree to carry it on.
Such a barter may therefore involve not all the possible subsets of $I$ and $J$ but only some of them so that:

- $B$ can see $I$ as made of the following set of [possibly strictly] superadditive subsets without $A$ knowing this:

$$I = \{I_1, I_2, I_3, I_4\} = \{\{i_1, i_2\}, \{i_1, i_3, i_4\}, \{i_4, i_5\}, \{i_2, i_3, i_5\}\} \qquad (34)$$

- $A$ can see $J$ as made of the following set of [possibly strictly] superadditive subsets without $B$ knowing this:

$$\begin{aligned} J &= \{J_1, J_2, J_3, J_4, J_5\} = \qquad (35) \\ &\{\{j_1, j_2, j_3\}, \{j_1, j_3, j_4\}, \{j_4, j_5\}, \{j_3, j_5, j_6\}\}, \{j_2, j_3, j_5, j_6\}\} \end{aligned}$$

At the very start $A$ and $B$ agree on a many-to-many barter with, for instance, simultaneous requests. Both $A$ and $B$ make their requests and evaluate their utilities so to decide if each accepts or refuses.
If both players accept the barter ends with a success.
If only one player refuses, only the refusing player can either reiterate the request or partition his set of goods so to make clear to the other which subsets he is willing to barter. The players have the same possibilities also in the case of double refusal.
In this way the partitioning may conflict with the way in which each player sees the goods of the other so that either they are able to fix this mismatch, during the next phases, in order to attain the barter or both declare that no barter is possible and so the barter ends with a failure. For the fine grain structure of the algorithm we refer to the sections 6.2 and 6.5.
In our example we could have:

(1) $A$ asks for $J_1$ and $B$ asks for $I_2$ so the currently tentative barter is $(I_2, J_1)$;

(2) $A$ refuses and $B$ accepts;

(3) $A$ may change his request as $J_3$ so the currently tentative barter becomes $(I_2, J_3)$;

(4) $B$ refuses and partitions $J$ as $J = \{J', J''\} = \{\{j_1, j_2, j_4\}\{j_3, j_5, j_6\}\}$;

(5a) $A$ asks for $J'$ and both accepts so the barter occurs;

(5b) $A$ refuses so that both refuse and decide that no barter is possible;

(5c) $A$ may change his request as $J_5$ so the currently tentative barter becomes $(I_2, J_5)$;

(5d) $B$ refuses and partitions $J$ as $J = \{J', J''\} = \{\{j_1, j_3, j_4\}\{j_2, j_5, j_6\}\}$ and so on until either $(5a)$ or $(5b)$ occur.

We remark how at step (3) we have $s_A(J_1) = s_A(J_3)$ but at step (4) we have $v_B(J_1) < v_B(J_3)$ so that $A$ accepts but $B$ refuses.

# 10 Concluding remarks and future plans

In this paper we have introduced a family of barter models between two actors that execute a one shot barter through which they exchange, according to one among various mechanisms, the goods of two separate and privately owned pools. The various models have been introduced under the hypothesis of additivity according to which the value of a set is given by the sum of the values of its composing elements.
In the paper we presented the basic algorithms for the one-to-one barter, we showed the possible uses of the proposed models, we verified if some criteria of fairness are satisfied by the proposed models or not and we also introduced some extensions.
The main extension we presented is the relaxing of the additivity hypothesis with the adoption of superadditive sets where the value of a set is at least equal to the sum of the values of its elements. In this way we model functional relations among the goods that increase their joint values.
This is an introductory paper so a lot of formalization is still to be done for what concerns both the presented models, their extensions and the possible uses in concrete cases .
We need indeed to examine more formally the basic models of one shot barter; to improve the proposed algorithms; to examine the properties of such algorithms and their plausibility and, last but not least, to analyze and formalize the extensions we essentially only listed in section 9.

# Appendix

In this section we provide formal argumentations of some of the relations we have introduced and used in the previous sections.
As to the following relation:

$$\frac{a_A(j)}{a_A(I')} \geq \frac{l_A(i)}{a_A(I)} \tag{36}$$

we underline that in the one-to-one barter case we have $I' = I \setminus \{i\} \cup \{j\}$ so that the two sets have the same cardinality but different values for player $A$.
In this case we have, under the additivity hypothesis:

$$a_A(I') = a_A(I) - l_A(i) + a_A(j). \tag{37}$$

or:

$$v_A(I') = v_A(I) - v_A(i) + s_A(j). \tag{38}$$

From the structure of the sets $I'$ and $I$ we may also derive that:

(1app) $a_A(j) < a_A(I')$ since $j \in I'$ so that we can write $a_A(I') = \alpha a_A(j)$ with $\alpha > 1$;

(2app) $l_A(i) < a_A(I)$ since $i \in I$ so that we can write $a_A(I) = \beta l_A(i)$ with $\beta > 1$.

From the relation $a_A(j) < a_A(I')$ we can easily derive $a_A^2(j) < a_A^2(I')$ or:

$$\frac{a_A(j)}{a_A(I')} \leq \frac{a_A(I')}{a_A(j)} \tag{39}$$

In relation (39) we have used $\leq$ since if $x < y$ then $x \leq y$ but not vice versa.
Such relations have been used in the discussion of the equitability for player $A$.
Similar considerations hold also for the analogous relation of player $B$:

$$\frac{a_B(i)}{a_B(J')} \geq \frac{l_B(j)}{a_B(J)} \tag{40}$$

For what concerns relations (21), (23) for player $A$ we note that (since all the involved quantities assume only strictly positive values) we have:

- from $a_{A_{max}} \geq a_A(j)$ for each $j \in J$ and $l_A(i) \geq l_{A_{min}}$ for each $i \in I$ by multiplying side by side we get $a_{A_{max}} l_A(i) \geq a_A(j) l_{A_{min}}$ or:

$$\frac{a_{A_{max}}}{l_{A_{min}}} \geq \frac{a_A(j)}{l_A(i)} \tag{41}$$

- from $a_{A_{min}} \leq a_A(j)$ for each $j \in J$ and $l_A(i) \leq l_{A_{max}}$ for each $i \in I$ by multiplying side by side we get $a_{A_{min}} l_A(i) \leq a_A(j) l_{A_{max}}$ or:

$$\frac{a_{A_{min}}}{l_{A_{max}}} \leq \frac{a_A(j)}{l_A(i)} \tag{42}$$

With similar arguments we can justify relations (22) and (24) for player $B$.

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