A primer on Auction Theory

Lorenzo Cioni

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Dipartimento di Ingegneria della produzione, termoenergetica e modelli matematici (DIPTEM), Università di Genova
and
Dipartimento di Informatica, Università di Pisa

e-mail: cioni@diptem.unige.it,lcioni@di.unipi.it

Abstract

These notes constitute a primer on Auction Theory. They contain a brief description of some types of auctions, their main features and the strategies that the bidders can use in each case. These notes are heavily inspired (so to go very close to a plagiarism) by Krishna (2002). They also contain an Appendix where we present three non traditional auction types for the allocation of either a chore (an item with a negative value for both the auctioneer and the bidders) or a cost.

1 Introduction

These notes constitute a primer on Auction Theory and are heavily based on Krishna (2002). As such they are written in a rather informal style without any heavy use of mathematics but in the cases where this has been necessary. This primer structured as follows.
It opens with a short section where we firstly explain why we do use auctions then we present the main types of auctions, their equivalences and discuss some of their properties.
We go on with a certain number of sections that follow the structure of the book of Vijay Krishna (Krishna (2002)). Such sections are centered on single object auctions essentially in the symmetric model case (so in the case where the distribution and density functions are the same for all bidders) and close with a few notes on the linkage principle.

Auctions can be described as games of incomplete information (Gibbons (1992), Myerson (1991) and Osborne and Rubinstein (1994)) that involve a certain number of actors:

- an **auctioneer** A,
- a set of **bidders** B with $n = |B|$ members i, $i = 1, \dots, n$.

A game of **incomplete information** belongs to the family of the so called Bayesian games that are characterized by the concept of Bayesian Nash equilibrium. In such a game each player knows his own information (such as the payoff) but is uncertain about the same information of the other players. We have both static Bayesian games, where the moves are made simultaneously by all the players (as in a sealed bid auction), and dynamic Bayesian games, where the moves are made in succession by all the players (as in a open cry ascending auction). For further details we refer to Gibbons (1992).

2 Why do we use auctions

Auctions are used in all the cases where the auctioneer A is unsure about the values that the bidders $i \in B$ assign to the object that A wants to allocate or sell.
The main features of the auctions are that:

- they elicit information as bids denoting the willingness to pay for an object of the bidders;
- the outcome depends only on the submitted bids (that represent the received information)
- they are **universal** so that they may be used to sell any good;
- they are **anonymous** so that the outcome of an auction does not depend in any way on the identities of the bidders.

The auctions can be classified depending on the number of the auctioned objects and on the valuations of the bidders (for this issue see section 4). This means that we can have:

- auctions involving one single indivisible object with either **private values** (symmetrically and independently distributed) or **interdependent values** (and the use of signals);
- auctions involving multiple objects.

In this paper we are going to deal with auctions of the former type.

3 The main auction types

In this section we describe four classical or standard auction types:

(1) the open ascending price or English auction;

(2) the open descending price or Dutch auction;

(3) the first price sealed bid ($FPSB$) auction;

(4) the second price sealed bid ($SPSB$) auction.

The **English auction** has many variants depending on the ways through which the bidders signal their will to attend, go on with attending or abandon the auction.
In one variant the auctioneer starts calling out a low price and raises it in small increments as long as there are at least two interested bidders and the auction stops when only one bidder is left.
During the periods in which the price raises the bidders indicate clearly (for instance by raising up one hand) their interest in purchasing the auctioned good. When the price is too high a bidder may signal that he is no more interested by simply lowering his hand.
In this way the auction ends only when a single bidder remains signaling his interest.
The winning bidder:

- gets the auctioned good/object,
- pays a price equal to the price at which the second last bidder dropped out.

In a **Dutch auction** the auctioneer starts with a high price and then continuously lowers it until one o the bidders cries stop and blocks the auction. This bider gets the object and pays the sum at which he cried stop.
We note that in the former type we have a huge exchange of information among the bidders that may influence each other by either dropping out or going on bidding whereas this exchange of information is absent in the latter case where a bidder reveals his valuation only when he stops the auction itself.
In a **FPSB auction** the bidders independently and simultaneously[1] submit sealed bids and the object is assigned to the bidder who submitted the highest bid that pays such a sum (whence the name *first price*).

[1]This means that every submission is made by each bidder without knowing the submissions of the others and do not imply a prefect simultaneity.

In a **SPSB auction** the bidders independently and simultaneously submit sealed bids and the object is assigned to the bidder who submitted the highest bid that pays a sum equal to the second highest bid (whence the name *second price*).
We note how, in these cases, there is no exchange of information among the bidders that have very few possibilities to influence each other though this does not prevent the bidders from having valuations of the auctioned goods that are interdependent in some way (simply because the nature of the good is known in advance and may influence in similar ways the valuations of the bidders).
Other forms of auctions include:

- **hybrid Dutch-English auctions**;
- **third price auctions**;
- **all pay auctions**;
- **deadline auctions** (with a predetermined stopping time);
- **candle auctions** (with a random stopping time);

but many more may be conceived. Some of these will be examined in subsequent sections.
As a closing comment we note that:

- (1) and (2) are open auctions so that the bids are common knowledge during the execution of an auction of these types;
- (3) and (4) are sealed bid auctions so that the bids are not known during the execution of an auction of these types and are revealed (as the winning bid only) at the end of each auction.

4 Bidders valuations

The bidders may have different types of valuations of an object:

- private values,
- interdependent values,
- [pure] common values.

We speak of **private values** if each bidder knows the value of the object to himself at the time of bidding and this value is independent from the values that other bidders may attribute to that object and would remain unchanged even if such values would be known in some way. In this case the value of the auctioned object is given from his consumption or use.
We speak of **interdependent values** at the time of the bidding if the exact value of the object is unknown to the bidders but each of them knows privately a signal that is correlated in some way to the true value. If such signals were known by a bidder would cause him to change his estimation of the object. In his case the valuation of each bidder is influenced by the valuations of the other bidders as well as from the information at their disposal.
Last but not least we speak of [**pure**] **common values** (as a particular case of the interdependent values case) if the value of the auctioned object is unknown to the bidders (though each of them may have a private estimate of such a value based on private information or signals) at the time of the bidding but is the same for all the bidders when the auction is over.

5 Equivalences among the basic types

Beyond formal differences among the various types of auctions we can state some equivalences between the basic auction forms. We indeed can prove hat:

(a) Dutch auctions are strategically equivalent to $FPSB$ auctions;

(b) under certain hypotheses English auctions are strategically equivalent to $SPSB$ auctions.

With the term strategical equivalence of two games we mean that for every strategy in one game a player has a strategy in the other game that gives the same outcome. In other words two games are strategically equivalent if they have the same normal form but for duplicate strategies with the same outcome.
To prove (a) we can proceed as follows.
In a $FPSB$ auction the strategies available to the bidders are mappings form their valuations to their bids. On the other hand, in a Dutch auction the bidders may only know that one of them has agreed to buy at a certain price so to end the auction. In both cases the bidders have little information about the auction and the other bidders bids so that if a bidder bids a certain amount he can win or lose (in a $FPSB$ auction) and the same is true in a Dutch action if the bidder is willing to pay the same price and the good is

still available. In this way we have argued that at any strategy in $FPSB$ auction there corresponds an equivalent strategy in a Dutch auction so that the two types are equivalent.
To prove (b) we start by observing that when values are private (so that the valuations of the object from the bidders are independent from each other and do not vary) the English auction is strategically equivalent to a $SPSB$ auction.
Though in the English auction the bids are common knowledge among the bidders this knowledge is useless in case of private values. In this case therefore the strategy available to each bidder is to attend the auction until the current price reaches one's own valuation and, at that time, drop out in order to avoid a negative payoff (as a difference between the valuation of the good and the sum each bidder has to pay upon winning) but also not to drop out before (in order to avoid losing the auction and so a possible positive payoff).
The same holds in a $SPSB$ action where the best strategy for each bidder is to bid a sum equal to one's own valuation of the good (see further on).
In both types of auction, in the case of private values, the bidders have the same strategy: either to stay up until each own's value or to bid that value so that the two types are strategically equivalent.
We have therefore proved the following equivalences (among the strategies available to the bidders and not among the rules of the auctions) :

- Dutch auctions and $FPSB$ auctions,
- (under the constraint of private informations) English auctions and $SPSB$ auctions.

The second equivalence is valid only under the specified constraint since the English auction allows an exchange of information that may influence the behavior of the bidders and this exchange cannot take place in a $SPSB$ auction so that the two types are equivalent only when this exchange has no effect and so in the case of private values.

6 Some parameters

The auction formats are evaluated according to the following parameters:

- the **revenue for the seller** or the expected selling price for the auctioned object;

- the **efficiency** for the society as a whole so that an object is assigned to the bidder who, ex post[2], evaluates it at the most;

- the **simplicity** or the transparency of the rules of an auction and the easiness of implementing such rules;

- the existence of **collusion possibilities** among [subsets of] the bidders and [some of] the bidders with the auctioneer.

7 Single object private values auctions

7.1 Introduction

In this section we focus on auctions where a single object is auctioned and the values of the bidders are:

- private,

- independent,

- identically distributed.

Under these hypotheses we have the following equivalences:

- between Dutch auctions and $FPSB$ auctions,

- between English auctions and $SPSB$ auctions,

so that we can restrict our attention only to the sealed bid auctions both at the first and at the second price. In this way we have two auction formats and each of them defines a game of incomplete information among the bidders with Bayesian-Nash equilibria. In tall the cases where we have more than one equilibrium we may perform a selection on the basis of either the dominance or the perfection or the symmetry under the constraint to use the same criterion for all the formats so to be able to perform a comparison of the equilibrium outcomes in one format with the outcomes in another format.

7.2 The model

In the framework we have defined we have a single auctioned object, an auctioneer and N potential buyers (the bidders). Each bider i assigns a value X_i to the object with $X_i \in [0, \omega]$ for a suitable[3] $\omega > 0$. Such values are

[2]We are going to use the term **ex ante** to denote the time before the execution of the auction and **ex post** to denote a time after the execution of he auction when all the bids are known and the object is allocated according to the rules of the chosen auction form.

[3]With a little abuse in notation we may allow $\omega = +\infty$.

independently and identically distributed according to a common distribution function F that is supposed to be differentiable (and therefore continuous or atomless) with density $f = F'$ and full support (so that it is defined on $[0, \omega]$). Moreover we suppose that for every bidder we have $E[X_i] < +\infty$.
Under these hypotheses we have that each bidder i knows x_i or the current value of the random variable X_i and knows that the other random variables X_j (for every $j \neq i$) are identically and independently distributed according to F.
Moreover we are under the following hypotheses.

- the bidders are **risk neutral** and so each of them maximizes his expected payoff as the difference between two products evaluation and probability of winning and payment and probability of winning,
- the number of the bidders is common knowledge among the bidders,
- the cumulative distribution function F is common knowledge among the bidders,
- the distribution values are the same for all bidders that are therefore termed as symmetric bidders,
- the bidders have no budget constraints so they can pay up to their value whichever this may be.

In what follows we are going to deal with:

- $FPSB$ auctions where the highest bidder gets the object and pays his bid;
- $SPSB$ auctions where the highest bidder gets the object and pays the second highest bid.

In both cases the strategy for a bidder is a function:

$$\beta_i : [0, \omega] \longrightarrow \mathbb{R}_+ \tag{1}$$

that maps a bidder's value on his bid. The absence of budget constraints is mirrored by the structure of the range of the function ($\mathbb{R}_+$).
Under the hypothesis we have made of symmetric bidders we are going to look for a **symmetric equilibrium** or an equilibrium where all the bidders follow the same strategy.

7.3 $SPSB$ auctions

As we have already seen in the private values case $SPSB$ auctions are strategically equivalent to English auctions.
In a $SPSB$ auction if the bidder i has a valuation x_i and submits a bid b_i he obtains the following payoff:

$$\Pi_i = \begin{cases} x_i - max_{j \neq i} b_j \; if \; b_i > max_{j \neq i} b_j \\ 0 \; if \; b_i < max_{j \neq i} b_j \end{cases} \tag{2}$$

In case of a tie (or if $b_i = max_{j \neq i} b_j$) the object is allocated to one of the bidders with equal probability. We note how, however, that from the hypothesis of atomless distribution we have that a tie occurs with null probability.
In this context we have the following result.

Proposition 7.1 *In a $SPSB$ auction the truthful strategy:*

$$\beta^2(x) = x \tag{3}$$

*is a **weakly dominant strategy**.*
Proof
Since we are interested in a symmetric equilibrium we can focus on one of the bidder, say bidder 1. Now suppose that:

$$p_1 = max_{j \neq 1} b_j \tag{4}$$

is the highest competing bid. If 1 bids x_1 (his valuation) he may either win if $x_i > b_i$ or lose if $x_i < b_i$ or be indifferent if $x_i = b_i$. To analyze the convenience of a strategic bidding we examine the two possible cases:

- ***overbidding** if 1 bids $z_1 > x_1$;*
- ***underbidding** if 1 bids $z_1 < x_1$.*

If 1 underbids (so $z_1 < x_1$) we have the following cases:

- *$x_1 > z_1 > p_1$ so 1 wins with a profit $x_1 - p_1$ that is the same that he gets bidding x_1;*
- *$x_1 > p_1 > z_1$ so he loses an auction (with profit 0) that he could win by simply bidding x_1 (with a profit $x_1 - p_1$);*
- *$p_1 > x_1 > z_1$ he loses with a null profit.*

So underbidding never increases 1's profit and in some case may even decrease it.
If 1 overbids (so $z_1 > x_1$) we have the following cases:

- *$z_1 > x_1 > p_1$ and 1 wins with the same profits $x_1 - p_1$ that he would get bidding x_1;*

- *$z_1 > p_1 > x_1$ and 1 wins but gets a negative payoff $x_1 - p_1$ so he would have been better off by bidding x_1;*

- *$p_1 > z_1 > x_1$ and 1 loses in the same way as with the bid of x_1*

We have verified how overbidding is never positive and may be negative. From this analysis (and under the only assumption of private values) we see how truthful bidding is a weakly dominant strategy.

We now wish to evaluate what each bidder is expected to pay in equilibrium. We again use the symmetry condition and focus our attention on bidder 1 so to define:

$$Y_1 = Y_1^{(N-1)} \tag{5}$$

as the random variable that defines the highest value among the $N-1$ remaining bis $X_2, X_3, \ldots X_N$. If we denote with G the distribution function of Y_1 we get, for all x:

$$G(x) = P(Y_1 \leq x) = F(x)^{N-1} \tag{6}$$

where the last equality on the right depends on the hypotheses of:

- independence,

- identical distribution (according to the distribution F),

of the $N-1$ random variables X_j that have been represented through the corresponding ordered statistics Y_j.

In this way we can define the expected payment for a bidder with a value x that bids a sum equal to x (according to a truthful bidding strategy) as:

$$\begin{aligned} m^2(x) &= P[win]E[2nd\ highest\ bid \mid x\ is\ the\ highest\ bid] \\ &= P[win]E[2nd\ highest\ value \mid x\ is\ the\ highest\ value] \end{aligned} \tag{7}$$

or:

$$m^2(x) = G(x)E[Y_1|Y_1 < x] = \int_0^x yg(y)dy \tag{8}$$

where $g()$ is the density function corresponding to $G()$.

7.4 $FPSB$ auctions

In this case each bidder i submits a bid b_i having a valuation x_i that is a random signal over the interval $[0, \omega]$ so that the possible payoffs are:

$$\Pi_i = \begin{cases} x_i - b_i \, if \, b_i > max_{j \neq i} b_j \\ 0 \;\; if \, b_i < max_{j \neq i} b_j \end{cases} \tag{9}$$

Possible ties are resolved as we have seen in the case of $SPSB$ auctions.
In this case we can observe that:

- no bidder would bid more than his valuation otherwise by winning he would have a negative payoff;
- no bidder would bid an amount equal to his valuation so to have a null payoff;
- an increase of the bid would raise the probability of winning but, at the same time, would decrease the amount of the payoff.

As to the last point we note that the two effects of simultaneous increase and decrease balance off and to understand how this occurs we must evaluate the possible strategies of the bidders in the case of a symmetric equilibrium.
In order to obtain this we imagine that all the bidders $j \neq 1$ follow the strategy $\beta()$ we are defining and that we suppose to be:

- symmetric;
- increasing;
- differentiable;
- defined as $\beta(x) = b$;
- such that $\beta(0) = 0$ (see further on).

We therefore suppose that the bidder 1 does not follow the strategy β so that he receives a signal $X_1 = x$ and bids b. Our aim is to derive the optimal bid b as the bid that is the best reply to the other bidders strategy.
For this aim we use a direct approach:

(1) to determine the strategy;

(2) to prove it is optimal.

To solve point (1) we note that:

(a) it has no sense for bidder 1 to bid $b > \beta(\omega)$ since he would get the same outcome with a lower bid so we impose $b \leq \beta(\omega)$;

(b) if bidder 1 has a value $x = 0$ he cannot submit a positive bid since in case of victory of the auction (which can occur with a positive probability) he would have a loss equal to $0 - b$ (with $b = \beta(0)$) so we must have $\beta(0) = 0$.

It is easy to understand that bidder 1 wins if he submits the highest bid and therefore if:

$$max_{i \neq 1} \beta(X_i) < b \tag{10}$$

Since β is increasing we get:

$$\beta(max_{i \neq 1} X_i) < b \tag{11}$$

or, at last:

$$\beta(Y_1) < b \tag{12}$$

where Y_1 is, also in this case, the random variable that represents the highest of the $N - 1$ bids of the other bidders. In this way we have that bidder 1 wins the auction if relation (12) is satisfied or if (by using the fact that β is invertible since it is increasing):

$$Y_1 < \beta^{-1}(b) \tag{13}$$

with an expected payoff given by:

$$P(win) \times payoff = P(Y_1 < \beta^{-1}(b))(x - b) = G(\beta^{-1}(b))(x - b) \tag{14}$$

where G is the distribution of Y_1. If we denote with g the density function of G (that we suppose differentiable) and impose on the relation (14) a first order condition we get, by using the rule of the derivative of an inverse function:

$$\frac{g(\beta^{-1})}{\beta'(\beta^{-1})}(x - b) - G(\beta^{-1}) = 0 \tag{15}$$

At the symmetric equilibrium we impose it is $b = \beta(x)$ or or $\beta^{-1} = x$ so we can rewrite relation (15) as:

$$G(x)\beta'(x) + g(x)\beta(x) = xg(x) \tag{16}$$

or as:

$$\frac{d}{dx}[G(x)\beta(x)] = xg(x) \tag{17}$$

Integrating both members and using the initial condition $\beta(0) = 0$ we get:

$$\beta(x) = \frac{1}{G(x)} \int_0^x yg(y)dy = E[Y_1|Y_1 < x] \tag{18}$$

We have therefore defined a candidate function for the role of an equilibrium strategy.
To prove that this strategy is really a strategy of equilibrium (and so to prove point **(2)**) we must prove that if the $N-1$ bidders follow the strategy β it is optimal for bidder 1 (that with a value x bids $b = \beta(x)$) to follow the same strategy.
To prove this fact we can start by proving the following proposition.

Proposition 7.2 *The symmetric equilibrium strategies in a $FPSB$ auction are given by the following rule:*

$$\beta^1(x) = E[Y_1|Y_1 < x] \tag{19}$$

where Y_1 is the highest of $N-1$ independent and identically distributed values (the bids of the other bidders).
Proof
We suppose that all the bidders but bidder 1 follows the strategy $\beta^1 = \beta$ defined by relation (19) and prove that is optimal for the bidder 1 to follow the same strategy. To prove this we prove that by following any other strategy bidder 1 gets a lower payoff.
We note that from the fact that β is increasing and continuous we have that the bidder with the highest value submits the highest bid (below ($\beta(\omega)$)) and wins the auction. We therefore have that bidder 1 with value x bids $b \leq \beta(\omega)$ and want to derive his expected payoff.
If we denote with $z = \beta^{-1}(b)$ the value that corresponds to the equilibrium bid b with the strategy β so that $b = \beta(z)$ we have that the bidder 1 expected payoff if his value is x is:

$$\Pi(\beta(z), x) = G(z)(x - \beta(z)) = G(z)x - G(z)\beta(z) \tag{20}$$

If we use the last equality of the relation (18) we get:

$$\Pi(\beta(z), x) = G(z)x - G(z)E[Y_1|Y_1 < z] = G(z) - \int_0^z yg(y)dy \tag{21}$$

If we integrate by parts we get:

$$\Pi(\beta(z), x) = G(z)x - G(z)z + \int_0^z G(y)dy = G(x - z) + \int_0^z G(y)dy \tag{22}$$

In this way we have the two following expressions of the expected payoff:

(1) $\Pi(\beta(z), x) = G(x-z) + \int_0^z G(y)dy$ *if bidder* 1 *bids* $\beta(z)$,

(2) $\Pi(\beta(x), x) = \int_0^z G(y)dy$ *if bidder* 1 *bids the equilibrium strategy* $\beta(x)$.

By using such relations we can write the difference of the two payoffs as:

$$\Pi(\beta(x), x) - \Pi(\beta(z), x) = G(z)(z-x) + \int_z^x G(y)dy = G(z)(z-x) - \int_x^z G(y)dy \tag{23}$$

that we want to prove to be ≥ 0 *regardless of whether we have* $z \geq x$ *or* $z \leq x$ *so to prove that bidding* $\beta(x)$ *is better than bidding* $\beta(z)$. *In this way we prove that if the other* $N-1$ *bidders follow the strategy* β *the bidder* 1 *with a value* x *cannot be better off than by bidding* $\beta(x)$ *so that* β *is a* ***symmetric equilibrium strategy***.

If we consider:

$$G(z)(z-x) - \int_x^z G(y)dy \tag{24}$$

(with G *increasing) we have that the integral defines an area under a curve that is to be compared with the area of a "rectangle". We consider the two cases* $z \geq x$ *or* $z \leq x$.

In the case $z \geq x$ *by using the mean value theorem we know that we can find a suitable* $w \in [x, z]$ *such that:*

$$G(w)(z-x) = \int_x^z G(y)dy \tag{25}$$

so to get:

$$G(z)(z-x) - G(w)(z-x) \geq 0 \tag{26}$$

by the fact that G *is increasing and that* $w < z$.

In the case $z \leq x$ *we that the relation (24) can be rewritten as:*

$$G(z)(z-x) + \int_z^x G(y)dy \geq 0 \tag{27}$$

Again by using the mean value theorem we can find a suitable $w \in [z, x]$ *such that:*

$$G(w)(x-z) = \int_z^x G(y)dy \tag{28}$$

and, in this way, to get:

$$G(z)(z-x) + G(w)(x-z) \geq 0 \tag{29}$$

or:

$$-G(z)(x-z) + G(w)(x-z) \geq 0 \tag{30}$$

that is true since G *is increasing and* $w > z$.

In this way we have derived an equilibrium strategy that can be rewritten (by using the first equality of relation (18) and integrating by parts) as:

$$\beta^1(x) = x - \int_0^x \frac{G(y)}{G(x)} dy \tag{31}$$

so that the equilibrium bid of a bidder is lower than is valuation of the auctioned object.
In relation (31) the integral defines the reduction of the bid. From the fact that G is increasing and that $y \leq x$ we have that:

$$\frac{G(y)}{G(x)} \leq 1 \tag{32}$$

and, moreover, we can write (again using independence and identical distribution):

$$\frac{G(y)}{G(x)} = [\frac{F(y)}{F(x)}]^{N-1} \tag{33}$$

that tends to 0 as $N \to \infty$ so that the bid tends more and more to the valuation (truthful bidding) the higher is the number of the competing bidders.

Example 7.1 *If we suppose that the values are uniformly distributed in the interval* $[0, 1]$ *we get:*

$$\beta^1(x) = x - \int_0^x \frac{G(y)}{G(x)} dy = x - \int_0^x \frac{y^{N-1}}{x^{N-1}} dy = \frac{N-1}{N} x \tag{34}$$

with $F(x) = x$*,* $G(x) = F(x)^{N-1} = x^{N-1}$ *and* $G(y) = y^{N-1}$*.*
In this case we have that the equilibrium strategy is a constant fraction of the bidder's value. In the case of two bidders we have:

$$\beta^1(x) = \frac{x}{2} \tag{35}$$

so that none of the two bidders bid more than half of his valuation of the object.
The presence of an upper bound derives from the following inequality:

$$\beta^1(x) = E[Y_1|Y_1 < x] \leq E[Y_1] \tag{36}$$

In case of two bidders we have $Y_1 = X$*. If we add moreover the uniform distribution hypothesis over* $[0, 1]$ *we get* $E[X] = \frac{x}{2}$

7.5 Comparison of the revenues

Up to this point we have defined the symmetric equilibrium strategies for both a $FPSB$ auction and a $SPSB$ auction. We can now evaluate the revenue for the two auctions as the selling prices of the auctioned object.
In a $FPSB$ auction we have the following equilibrium strategy:

$$\beta^1(x) = E[Y_1|Y_1 < x] \tag{37}$$

so that the expected payment for the bidder with value x is:

$$m^1(x) = P[win] \times amount\ bid = P(Y_1 < x)\beta^1(x) = G(x)E[Y_1|Y_1 < x] \tag{38}$$

This quantity coincides with the expected payment in the case of a $SPSB$ auction so that we can write:

$$m^A(x) = P[win] \times amountbid = P(Y_1 < x)\beta^1(x) = G(x)E[Y_1|Y_1 < x] = \int_0^x yg(y)dy \tag{39}$$

where A stands for 1 in the case of a $FPSB$ auction and 2 in the case of a $SPSB$. This means that the expected revenues in the two auctions are the same and do not depend on the form of the auction. We now prove this assertion.
In order to do so we now can evaluate the ex-ante[4] expected payment of a bidder in one of the two auctions as:

$$E[m^A(x)] = \int_0^\omega m^A(x)f(x)dx = \int_0^\omega \int_0^x yg(y)dyf(x)dx \tag{40}$$

If we interchange the order of integration we get:

$$E[m^A(x)] = \int_0^\omega (\int_y^\omega f(x)dx)yg(y)dy = \int_0^\omega y[1 - F(y)]g(y)dy \tag{41}$$

The expected revenue for the auctioneer is N times the ex-ante expected payment of each bidder so that we can write:

$$E[R^A] = NE[m^A(x)] = N\int_0^\omega y[1 - F(y)]g(y)dy \tag{42}$$

where $g(y) = f^{N-1}(y)$. If $Y_2^{(N)}$ is the second highest value of N values we have:

$$f_2^{(N)} = N[1 - F(y)]f^{N-1}(y) \tag{43}$$

[4]In this case with this term we denote the time before the bidders get assigned their values by the move of a special player, the so called **nature**, as it occurs in the case of games with incomplete information.

and therefore we can write:

$$E[R^A] = \int_0^\omega y f_2^{(N)} dy = E[Y_2^{(N)}] \tag{44}$$

so that the expected revenue is the expected value of the second highest value of N values independently from the form of the auction. From this we have that the expected revenue of the seller in a $FPSB$ auction and in a $SPSB$ auction is the same. We therefore can state the following proposition.

Proposition 7.3 *Under the hypotheses of:*

- *independent,*
- *identically distributed,*

private values the expected revenue for the seller both in a $SPSB$ auction and in a $FPSB$ auction is the same.

Observation 7.1 *We note how this is true if we consider the expected values. In the case of uniform distributions and two bidders we have that in a $FPSB$ auction the equilibrium strategy is:*

$$\beta^1(x) = \frac{x}{2} \tag{45}$$

whereas in a $SPSB$ auction the equilibrium strategy is:

$$\beta^2(x) = x \tag{46}$$

If bidder 1 has a value x_1 and 2 has a value x_2 we have that if $x_1 > 1/2x_1 > x_2$ bidder 1 wins and pays $1/2x_1$ in a $FPSB$ auction but wins and pays only x_2 in a $SPSB$ auction so that the former is more profitable than the latter.
On the other hand if we have:

$$\frac{1}{2}x_2 < \frac{1}{2}x_1 < x_2 < x_1 \tag{47}$$

in a $SPSB$ auction (where bidders bid truthfully) wins 1 and pays x_2 whereas in a $FPSB$ auction (where both bidders follow β^1) wins 2 that pays $1/2x_1$ so that a $SPSB$ auction is more profitable than a $FPSB$ auction.

Observation 7.2 *We moreover note how the revenues in a $SPSB$ auction are more variable than those in a $FPSB$ auction. Indeed in the former case the prices may vary in the interval $[0, \omega]$ (owing to truthful bidding) whereas in the latter they can vary in the interval $[0, E[Y_1]]$.*
We moreover can state that for the seller a $SPSB$ auction is riskier than a $FPSB$ auction so a risk averse seller, in presence of risk neutral bidders (the hypothesis we have thought to valid up to now), prefers a $FPSB$ auction to a $SPSB$ auction.

7.6 Reserve price

Up to now we have examined two types of auction where the seller is going to to give away the auctioned item at every price he is able to collect from the bidders.
In this section we analyze the situation where a seller may fix a reserve price $r \geq 0$ so to not sell the good/item if the selling price is lower than r.
We wish to examine the effect of such a reserve price on the revenue for the seller so we consider both a $FPSB$ auction and a $SPSB$ auction where a reserve price is present and define a reserve price as the price that the winner has to pay in any case in order to get the auctioned object.

7.7 $SPSB$ auctions with a reserve price

In this case the seller fixes a reserve price $r > 0$ so that the price at which an object is sold can never be lower than r. This means that no bidder with a value $x < r$ can make a positive profit from attending the auction. We recall that we have:

$$profit = valuation - payment = x - r < 0\, if\; x < r \tag{48}$$

Notwithstanding the presence of the reserve price, however, the truthful bidding is again a weakly dominant strategy so the expected payment of a bidder with a valuation equal to r is $G(r)r$ (where $G(r)$ defines the probability of winning) whereas if a bidder has a valuation $x > r$ we get that the expected payment assumes the following form:

$$m^2(x) = r(G(r) + \int_r^x yg(y)dy \tag{49}$$

as a sum of a fixed expected payment up to r and an expected payment for the excess over r. We recall that the winer has, indeed, to pay the price r whenever the second highest bid is lower than r.

7.8 $FPSB$ auctions with a reserve price

In this case we fix $r > 0$ so that the price to be paid is at least r. This implies that no bidder with a valuation $x < r$ can make a positive profit by bidding b as defined by the equilibrium strategy since he would have a profit equal to $x - r < 0$ if $x < r$.
We now examine the other cases . If β^1 is a symmetric equilibrium strategy of a $FPSB$ auction with a reserve price $r > 0$ we have indeed to consider the two cases:

(1) $x = r$

(2) $x > r$

In the case (1) we have that a bidder with a value r can win only if all the other bidders have a value lower than r (from the fact that β is increasing) so that he can bid r that is the the sum that he should pay in case of victory (and thus having a null payoff).
In the case (2) it is possible to prove that the equilibrium strategy is:

$$\beta^1(x) = E[max\{Y_1, r\}|Y_1 < x] \tag{50}$$

where Y_1 is the highest of the other $N-1$ bids (that bidder 1 sees as random variables). We can rewrite relation (50) as:

$$\beta^1(x) = r\frac{G(r)}{G(x)} + \frac{1}{G(x)}\int_r^x yg(y)dy \tag{51}$$

where the first term on the right expresses the part due to the reservation price and the second term the excess over such reservation price.
In the case $x \geq r$ the expected payment from a bidder with a value x may therefore be expressed as:

$$m^1(x) = P[win]bid = G(x)\beta^1(x) = rG(r) + \int_r^x yg(y)dy \tag{52}$$

or as the sum of a fixed payment and an excess payment for $x > r$. We recall that $G(y)$ is the distribution of Y_1 and so denotes the probability for a generic bidder (by the symmetry hypothesis) of winning the auction. From the expression (52) we see how it is valid also in the cases $x = r$ and $x < r$.
From the relations (49) and (52) we therefore see how it is:

$$m^1(x) = m^2(x) \tag{53}$$

so that the expected payment is the same in both auction formats and the same holds also for the expected revenue.

7.9 The effect of the reserve price on the revenue

We now have to understand how the value of r affects the expected revenue of the seller in both types of auctions.
We have that a bidder with a value r is expected to pay $rG(r)$. In the general

case the ex-ante (or before the bidders get their values revealed) expected payment of a bidder can be expressed as:

$$E[m^A(x)] = \int_r^\omega m^A(x,r)f(x)dx \tag{54}$$

where with A we denote a $FPSB$ auction (and so it must be read as 1) or a $SPSB$ auction (and so it must be read as 2) depending on the case.
By using the expression of the expected payment (see for instance relation (52)) we can rewrite relation (54) as:

$$E[m^A(x)] = \int_r^\omega [rG(r) + \int_r^x yg(y)dy]f(x)dx \tag{55}$$

or as:

$$E[m^A(x)] = \int_r^\omega rG(r)f(x)dx + \int_r^\omega \int_r^x yg(y)dyf(x)dx \tag{56}$$

and so as:

$$E[m^A(x)] = rG(r)[1 - F(r)] + \int_r^\omega \int_r^x yg(y)dyf(x)dx \tag{57}$$

If we consider the double integral and we interchange the order of the integrations we get:

$$E[m^A(x)] = rG(r)[1 - F(r)] + \int_r^\omega y(\int_y^\omega f(x)dx)g(y)dy \tag{58}$$

ad, at last:

$$E[m^A(x)] = rG(r)[1 - F(r)] + \int_r^\omega y[1 - F(y)]g(y)dy \tag{59}$$

We now have to determine which is the optimal reserve price for the seller or the reserve price that maximizes his revenue.
To do so we suppose that the seller gives a value $x_0 \in [0, \omega]$ to the auctioned object as the value that he gets from the object if it is left unsold.
We obviously have $r \geq x_0$ (since if $r < x_0$ the seller would be better off by not selling the object) so the expected payoff for the seller can be expressed as:

$$\Pi_0 = expectedpayoff\ if\ sells\ +\ expected\,payoff\ if\ does\ not\ sell \tag{60}$$

or as:

$$\Pi_0 = NE[m^A(x,r)] + F(r)^N x_0 \tag{61}$$

where $F(r)^N$ is the probability that all the bidders bid less than r so that the auctioned item is left unsold.
By replacing the expression for $m^A(x,r)$ from (59) we get:

$$\Pi_0 = N[rG(r)[1-F(r)] + \int_r^\omega y[1-F(y)]g(y)dy] + F(r)^N x_0 \qquad (62)$$

If we write relation (62) as:

$$\Pi_0 = N[rG(r)[1-F(r)] - \int_\omega^r y[1-F(y)]g(y)dy] + F(r)^N x_0 \qquad (63)$$

and evaluate $\frac{d\Pi_0}{dr}$ by using the rule:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \qquad (64)$$

we get:

$$\frac{d\Pi_0}{dr} = N[1-F(r)-rf(r)]G(r) + NG(r)f(r)x_0 \qquad (65)$$

(where $G(r) = F(r)^{N-1}$) if we note that at the second member of (63) we have a term that through the derivation contains $g(r)$ and cancels out with the derivative of the integral.
At this point relation (65) can be rewritten as:

$$\frac{d\Pi_0}{dr} = N[1-(r-x_0)\frac{f(r)}{1-F(r)}](1-F(r))G(r) \qquad (66)$$

If we recall the definition of **hazard rate** as:

$$\lambda(r) = \frac{f(r)}{1-F(r)} \qquad (67)$$

we get:

$$\frac{d\Pi_0}{dr} = N[1-(r-x_0)\lambda(r)](1-F(r))G(r) \qquad (68)$$

We now consider the two cases $x_0 > 0$ and $x_0 = 0$ (where x_0 is the value that the seller assigns to the good to be auctioned).
In the case $x_0 > 0$ we have that the derivative in $r = x_0$ is positive (as well as for $r < x_0$) so the function Π_0 is increasing and therefore may reach a maximum value only for $r > x_0$.
In the case $x_0 = 0$ we have:

$$\frac{d\Pi_0}{dr} = N[1-r\lambda(x)](1-F(r))G(r) \qquad (69)$$

and if $r = 0$ we have $\frac{d\Pi_0}{dr} = 0$ and, if $\lambda(r)$ is bounded, $\frac{d\Pi_0}{dr} >= 0$ if $r > 0$ so that Π_0 has a local minimum at $r = 0$ and this means that a small reserve price can cause an increase in the seller's revenue.
From these considerations we have that a revenue maximizing seller should always set a reserve price $r \geq x_0$.
We now wish a justification of this fact and so why a reserve price $r \geq x_0$ should cause an increase in the revenue for the seller.
In oder to do that we consider a $SPSB$ auction with two bidders and suppose that the seller has a value $x_0 = 0$.
In this case if $r > 0$ there is the risk that if $Y_2 < Y_1 < r$ (where Y_1 is the highest value among the two bidders) the object is left unsold with a loss. On the other hand it may be $Y_1 > r > Y_2$ (since in the case $Y_1 > Y_2 > r$ the value of r has no effect) so that the object is sold at r instead of Y_2 with a gain.
We can see how the probability of the first case is $F(r)^2$ (as the probability that both Y_1 and Y_2 are lower than r) so that the expected loss is at the most $rF(r)^2$ whereas the probability of the second case is[5] $2F(r)(1 - F(r))$ with an expected gain of $2rF(r)(1 - F(r))$ that, for small values of r, is greater than the expected loss.
This fact is called the **exclusion principle** and state that it is optimal for the seller to exclude some of the bidders and precisely those whose value is lower than r even if their valuation would be greater than x_0.
At this point if we impose a first order condition or relation (68) or:

$$\frac{d\Pi_0}{dr} = 0 \tag{70}$$

we get:

$$1 - (r - x_0)\lambda(r) = 0 \tag{71}$$

and so the optimal reserve price expressed as:

$$r^* - \frac{1}{\lambda(r^*)} = x_0 \tag{72}$$

If the hazard rate λ is increasing with r (and we recall $r > x_0$) we have that the derivative has the followings signs:

- for $r = r^*$ we have $1 - (r - x_0)\lambda(r) = 0$
- for $r < r^*$ we have $1 - (r - x_0)\lambda(r) > 0$

[5]We note that $F(r) = P(Y_2 < r)$ and $1 - F(r) = P(Y_1 > r)$ whereas the factor 2 counts the number of ways in which this can happen.

- for $r > r^*$ we have $1 - (r - x_0)\lambda(r) < 0$

so in $r = r^*$ we have a maximum of the function Π_0 and the condition is also sufficient. Another remarkable fact is that r^* does not depend on the number $N$of the bidders.

7.10 Other topics

We now make some comments on the following issues:

(1) the effect of the **entry fee**,

(2) the issue of **efficiency versus revenue**,

(3) the **commitment at not selling**.

As to the point (1) we note how an entry fee represents another way of excluding some potential bidders from the auction whereas the commitment at not selling is a way to make the fixing of a reserve price a credible threat. In the previous section we have seen how a positive reserve price r causes the bidders with a value $x < r$ to be excluded from the auction. The same effect can be obtained by the seller by fixing an entry fee f so to exclude from the auction buyers with low values.
With the term **fee** we denote a fixed and non refundable sum that the bidders must pay to the seller in order to be able to attend the auction and submit bids.
Since a reserve price excludes all the bidders with a value $x < r$ and we want to define an equivalent mechanism through the application of the fee we say that this is true if the fee is fixed as:

$$f = \int_0^r G(y)dy \tag{73}$$

and so at a level that is equal to the expected payoff in both a $FPSB$ auction and a $SPSB$ auction. From this we have that a bidder whose valuation is $x < r$ has no worth in paying f to attend the auction.
We have therefore seen how the same effect of a reserve price r can be obtained by fixing a proper value of the fee f and vice versa how the effect of f can be replicated through a properly fixed reserve price r.
As to the point (2) we note how both an entry fee and a reserve price raise the revenue of the seller (since we have proved this for the reserve price and we have shown how the two mechanisms are equivalent) but may have negative effects on the efficiency.

To see why this can occur let us suppose that $x_0 = 0$. In this case if $r = f = 0$ the auctioned object is always sold to the highest bidder and so (from the hypotheses on the bidding strategy β) to the bidder with the highest value. From this we therefore have that the auctions $FPSB$ and $SPSB$ allocate efficiently since the good is awarded to the bidder who evaluates it the most. On the other hand, if $r > 0$ there is the positive probability that the object is left unsold and this is a source of inefficiency that gives rise to a trade off between efficiency and revenue.
Last but not least, as to the point (3) we note that the commitment of not selling below the reserve price must be a credible threat in order to have some effect on the bidders since, if they not feel it as a credible threat, they can be tempted to wait for a further sale at a lower reserve price and this may increase the probability that the object is left unsold by reducing the demand. In this case the use of a secret reserve price may be of some help.

8 The Revenue Equivalence Principle

8.1 The definition

We have seen how the expected selling price is the same in a $FPSB$ auction and in a $SPSB$ auction so that a risk neutral seller is indifferent between the two auctions. In such auctions the bidders submit a bid (that measure the willingness to pay of each bidder) and such bids determine both who wins the auction and how much such a winner has to pay.
An auction is termed **standard** if its rules state that the object goes to the bidder who makes the highest bid so the foregoing auctions are examples of standard auctions as well as an **all pay auction** (where all the bidders pay their own bids including the losing bidders) or a **Third price sealed bid** ($TPSB$) auction where the price that is paid is the third highest bid.
We can therefore state that:

- given a standard auction mechanism A;
- given a symmetric equilibrium strategy β^A,
- if $m^A(x)$ is the equilibrium expected payment from a bider whose valuation is x,
- if the expected payment of a bidder with valuation/value 0 is 0,

then the expected payment function $m^A()$ does not depend on A so that the expected revenue in any standard auction is the same (from the relation we

have seen between the two quantities).
We formalize all this in the following proposition.

Proposition 8.1 (Revenue Equivalence Principle) *If, given a set of bidders, we have that:*

- *their values are independent and identically distributed,*

- *they are risk neutral,*

then any symmetric and increasing equilibrium strategy of any standard auction such that a bidder with value 0 *has a null expected payment yields the same expected revenue to the seller.*
Proof
We suppose to have a standard auction A *with a fixed symmetric equilibrium* β *and we denote with* $m^A(x)$ *the expected payment of the bidder with value* x *so that we have* $m^A(0) = 0$.
We use the symmetry hypothesis and consider bidder 1 *and suppose that all the other bidders follow the strategy* β.
We therefore have that with a value x *the equilibrium strategy would be* $b = \beta(x)$ *so that bidder* 1 *(since he deviates unilaterally) bids* $\beta(z)$ *instead of* $\beta(x)$.
Now we have that bidder 1 *wins the auction if* $\beta(z) > \beta(Y_1)$ *where we have that, as usual,* Y_1 *is the highest value of the other* $N-1$ *bidders. Since* β *is increasing such condition is equivalent to the condition* $z > Y_1$.
We now express the bidder 1 *expected payoff as:*

$$\Pi^A(z,x) = G(z)x - m^A(z) \tag{74}$$

as the difference between the expected gain and the expected payment (or loss) if $G(z) = F(z)^{N-1}$ *is the distribution of* Y_1 *under the usual hypotheses of independence and identical distribution.*
The key point that allow us to prove the principle of the equivalence of the revenues is that $m^A(z)$ *will result to depend on* β *and* z *but to be independent from* x.
If we impose a first order condition on $\Pi^A(z,x)$ *or:*

$$\frac{\partial \Pi^A(z,x)}{\partial z} = 0 \tag{75}$$

we get:

$$g(z)x - \frac{d}{dz}m^A(z) = 0 \tag{76}$$

At the equilibrium we have $z = x$ so that from (76) we get:

$$\frac{d}{dz} m^A(y) = g(y)y \tag{77}$$

and, integrating both sides with the initial condition $m^A(0) = 0$:

$$m^A(x) = \int_0^x g(y)ydy = G(x)E[Y_1 | Y_1 < x] \tag{78}$$

Since the right hand side does not depend on the form of the auction A we may derive that the expected payment (and therefore the expected revenue) is independent from A and so is the same in every standard auction.

Example 8.1 *If we consider the values as uniformly distributed on the interval $[0, 1]$ we get:*

- $F(x) = x$,
- $G(x) = x^{N-1}$ *if N is the number of the bidders,*
- $g(x) = (N-1)x^{N-2}$,

In this way for any standard auction such that $m^A(0) = 0$ we get as the payment of each bidder:

$$m^A(x) = \int_0^x y(N-1)y^{N-2}dy = \frac{N-1}{N} x^N \tag{79}$$

so that for the expected payment we get:

$$E[m^A(x)] = \int_0^1 m^A(x)f(x)dx = \int_0^1 \frac{N-1}{N} x^N dx = \frac{N-1}{N(N+1)} \tag{80}$$

since $f(x) = 1$. The ex-ante expected revenue for the seller is N times such expected payment so we get:

$$E[R^A(x)] = NE[m^A(x)] = \frac{N-1}{N+1} \tag{81}$$

We note that:

- $E[m^A(x)] \to 0$ *as* $N \to \infty$
- $E[R^A(x)] \to 1$ *as* $N \to \infty$

8.2 The use

The **Revenue Equivalence Principle** (REP) can be used in the two following cases, if its premises are satisfied so we can assume its conclusion is true:

- to derive the equilibrium bidding strategies in the case of other types of auctions;

- to derive the equilibrium strategy in the cases where the bidders are unsure about the numbers of rivals bidders they face.

To show its use in the former case we take into consideration these two types of auction:

- all pay auctions,

- third price auctions.

8.2.1 All pay auctions

We want to use REP to derive the equilibrium strategy in these types of auctions. We recall that in an all pay auction all the bidders submit a bid, the highest bidding bidder wins the auction and gets the auctioned object but all the bidders pay their submitted bid to the auctioneer. All pay auctions can be seen as models of lobbying activities where we have only one winner but all those who participated in the activity must pay their costs for making pressure.

Since we want to use REP to determine an equilibrium strategy in this case we must have its premises satisfied so that we can derive and use its conclusion.

We therefore suppose that:

- there is a symmetric, increasing equilibrium β such that $\beta(0) = 0$;

- the values are symmetric, independent and private,

- the bidders are risk neutral.

From these premises we derive that the expected payment of each bidder must be the same as:

$$m^A(x) = G(x)E[Y_1|Y_1 < x] = \int_0^x yg(y)dy \tag{82}$$

From the definition of this type of auction we have that the expected payment of a bidder with type (or value) x is his bid $\beta^{AP}(x)$ where β^{AP} is the symmetric, increasing equilibrium of an all pay auction.
We therefore must have that:

$$\beta^{AP}(x) = m^A(x) = \int_0^x yg(y)dy \tag{83}$$

We have now to verify that this is an equilibrium strategy and , in order to do so, we suppose that all the bidders but one follow such strategy, whereas this deviating bidder bids $\beta(z)$. In this way the expected payoff for a bidder with value x that bids $\beta(z)$ is given by the following expression:

$$G(z)x - \beta(z) = G(z)x - \int_0^z yg(y)dy \tag{84}$$

If we integrate by parts we get:

$$G(z)x - yG(y)|_o^z + \int_0^z G(y)dy = G(z)(x-z) + \int_0^z G(y)dy \tag{85}$$

The last expression on the right is the same as the expression of the payoff that it is obtained in the case of a $FPSB$ auction if a single bidder deviates and bids $\beta(z)$ instead of the equilibrium bid. As we have already seen such a quantity is maximized by posing $z = x$ so that also the deviating bidder is better off by following the same strategy that is therefore an equilibrium strategy.

8.2.2 Third price auctions

We want to use REP to derive an equilibrium strategy for this type of auctions. We suppose to have $N \geq 3$ or at least three bidders. In this case the rules of the auction are the following:

- the bidders submit a bid,
- the highest bidding bidder wins the auction and gets the object,
- he pays the third highest bid.

We again suppose that the hypothesis of the REP are satisfied so that the expected payment in this type of auctions can be expressed as:

$$m^3(x) = \int_0^x yg(h)dy \tag{86}$$

We now use the symmetry hypothesis and consider bidder 1 and suppose that he wins the auction with a value x so that $Y_1 < x$ and pays a price equal to $\beta^3(Y_2)$ where Y_2 is the second highest of the $N-1$ remaining $N-1$ bids (and so it is the third highest value among the values X_i).
At this point we can write an expression for the density of Y_2 conditioned on the fact that x is the winning value as:

$$f_2^{(N-1)}(y|Y_1 < x) = \frac{1}{F_1^{(N-1)}(x)}(N-1)(F(x) - F(y))f_1^{(N-2)}(y) \tag{87}$$

In relation (87) we have that:

- $(N-1)(F(x) - F(y))$ is the probability that $x > Y_1 > Y_2 = y$;
- $f_1^{(N-2)}$ is the density of the highest of $N-2$ values;
- $F_1^{(N-1)}(x)$ is the probability that x is the winning value.

At this point we can write in general terms the expected payment of a bidder in a third price auction as:

$$m^3(x) = F_1^{(N-1)}(x)E[\beta^3(Y_2)|Y_1 < x] = \int_0^x \beta^3(y)F_1^{(N-1)}(x)f_2^{(N-1)}(y|Y_1 < x)dy \tag{88}$$

If we replace the expression from (87) we get:

$$m^3(x) = \int_0^x \beta^3(y)(N-1)(F(x) - F(y)f_1^{(N-2)}(y)dy \tag{89}$$

If we use REP we can equate relation (89) with relation (86) so to get:

$$\int_0^x \beta^3(y)(N-1)(F(x) - F(y)f_1^{(N-2)}(y)dy = \int_0^x yg(h)dy \tag{90}$$

If we differentiate both members with respect to x but the first member within the integral sign we get, using the fact that $G(x) = F(x)^{N-1}$ and therefore $g(x) = (N-1)F(x)^{N-2}f(x)$:

$$(N-1)f(x)\int_0^x \beta^3(y)f_1^{(N-2)}(y)dy = xg(x) = x(N-1)f(x)F(x)^{N-2} \tag{91}$$

If we simplify the leftmost quantity with the rightmost we get:

$$\int_0^x \beta^3(y)f_1^{(N-2)}(y)dy = xF(x)^{N-2} \tag{92}$$

that can be rewritten as:

$$\int_0^x \beta^3(y) f_1^{(N-2)}(y)dy = xF_1^{(N-2)}(x) \tag{93}$$

by using the fact that $F(x)^{N-2} = F_1^{(N-2)}(x)$. If we differentiate again relation (93) with respect to x we get:

$$\beta^3(x) f_1^{(N-2)}(x) = x f_1^{(N-2)}(x) + F_1^{(N-2)}(x) \tag{94}$$

In this way we can derive at last β^3 as:

$$\beta^3(x) = x + \frac{F_1^{(N-2)}(x)}{f_1^{(N-2)}(x)} \tag{95}$$

We can further simplify relation (95) if we observe that, by independence and identical distribution:

- $F_1^{(N-2)}(x) = F(x)^{N-2}$
- $f_1^{(N-2)}(x) = (N-2)F(x)^{N-3} f(x)$

so that we can rewrite relation (95) as:

$$\beta^3(x) = x + \frac{F(x)}{(N-2)f(x)} \tag{96}$$

Since we want β^3 to be an equilibrium strategy it must be increasing. A necessary and sufficient condition for this to occur is that:

$$\frac{F(x)}{f(x)} \tag{97}$$

is increasing with x. This condition is equivalent to the following condition:

$$\frac{d\frac{F(x)}{f(x)}}{dx} > 0 \tag{98}$$

or at the condition that lnF is concave or that F is *log* concave. In other words we have that lnF is concave iff[6] $\frac{f}{F}$ is decreasing and so iff $\frac{F}{f}$ is increasing.

We note that this condition is satisfied if $F(x)$ is an uniform distribution whose density is a constant. We can therefore state the following proposition.

[6] We recall that iff stands for if and only if.

Proposition 8.2 *In presence of at least three bidders and if F (the common distribution) is log concave then a symmetric equilibrium strategy is a TP auction can be expressed as:*

$$\beta^3(x) = x + \frac{F(x)}{(N-2)f(x)} \tag{99}$$

so that the bid is greater than the bidder's valuation.

Example 8.2 *If the values are uniformly distributed on $[0, \omega]$ since we have:*

- $F(x) = \frac{x}{\omega}$ *(so that F is log concave),*
- $f(x) = \frac{1}{\omega}$,

we get:

$$\beta^3(x) = x + \frac{x}{(N-2)} \tag{100}$$

so that we verify that the equilibrium bid (or $\beta(x)$) exceeds the value x but tends to x as the number N of the bidders increases.

Observation 8.1 *We comment a little the reasons why $\beta^3(x) > x$.*
In a TP auction a bidder could either underbid (so to bid $b < x$) or overbid (so to bid $b > x$).
In the case $b < x$ the bidder may win having a gain equal to the case where he bids x but can lose having the possibility to win by bidding x and so having a loss. This means that by underbidding a bidder is never better off and may be worse off so underbidding is a dominated strategy.
We now prove that overbidding is not a dominated strategy so that a bidder is better off by using it.
We fix an equilibrium strategy, β, and suppose that all the bidders but bidder 1 follow it. Bidder 1 has a value x and bids $b > x$. If we have $\beta(Y_2) < x < \beta(Y_1) < b$ the bidder is better off since he wins an auction that he would have lost by bidding x. In this case he has a gain. On the other hand if we have $x < \beta(Y_2) < \beta(Y_1) < b$ by bidding b the bidder wins an auction that he would have preferred to lose since he has a loss.
To understand that overbidding is a good strategy we must compare the gain and the loss. If $b - x = \epsilon$ is small (or smaller than one) the gain is of the order ϵ^2 whereas the loss is of the order ϵ^3 so that the former is greater than the latter.

Observation 8.2 *If we compare the equilibrium bids in the $FPSB$, $SPSB$ and TP auctions (and under the hypothesis that the distribution F is log concave) we get:*

$$\beta^1(x) < \beta^2(x) = x < \beta^3(x) \tag{101}$$

8.2.3 Uncertainty on the number of the bidders

In the auction forms we have examined up to now we have supposed that.

- each bidder knows his value x_i;
- each bidder is uncertain about the values X_j of the other bidders[7]
- all the other aspects of the situation (such as the number of the other bidders and the nature of the common distribution of probability) are common knowledge among the bidders.

We now introduce an uncertainty on the number of the bidders that are involved in an auction. We note that this uncertainty may occur in the case of sealed bid auctions as well in the case of the Internet auctions (whose treatment is outside the scope of the present paper).
To do so we introduce the following sets:

- $\mathscr{N} = \{1, \dots, N\}$ as the set of the potential bidders,
- $\mathscr{A} \subset \mathscr{N}$ as the set of the actual bidders.

We maintain the hypothesis of the common distribution F and so of the symmetry of the bidders.
We perform the analysis by considering a bidder $i \in \mathscr{A}$ and denote with p_n the probability that any bidder of $\mathscr{A}$ assigns to the event $|\mathscr{A}| = n + 1$ or that he is facing n other bidders.
The process by which we pass from $\mathscr{N}$ to $\mathscr{A}$ is not important but for the fact that it is symmetric so that each bidder holds the same beliefs about how many other bidders he faces. This means that the probabilities p_n are common knowledge among the bidders and do not depend on the identity of a bidder.
We consider therefore a standard auction A and a symmetric, increasing equilibrium β. Since the bidders do not know exactly the number of the bidders each of them is facing we have that β does not depend on such number. We now want to evaluate the expected payoff of a bidder with a value x that bids $\beta(z)$ instead of the equilibrium bid $\beta(x)$.
The bidder we are considering wins the auction if the highest value of the other bidders (as drawn from the distribution F) is lower than z or if $Y_1^{(n)} < z$ where n is the actual number of the other bidders. This event occurs with a

[7]We recall that X is a random variable that takes values in a given interval whereas with x we denote a realization of such variable and so a known (at least from one bidder) value.

probability given by $G^{(n)} = F(z)^n$ that is derived by using the hypotheses of independence and identical distribution.
The overall probability that this bidder will win by bidding $\beta(z)$ is therefore expressed as:

$$G(z) = \sum_{n=0}^{N-1} p_n G^{(n)}(z) \tag{102}$$

with:

$$\sum_{n=0}^{N-1} p_n = 1 \; and \; p_n \geq 0 \tag{103}$$

In this case we have the following expected payment:

$$\Pi^A(z, x) = G(z)x - m^A(x) \tag{104}$$

as the difference between the expected gain and the expected payment if a bidder with value x bids $\beta(z)$.
In this way we have that REP holds also in the presence of uncertainty about the number of the bidders.
We now assume that the auctioned good is sold by using a $SPSB$ auction. Also in the presence of an uncertainty over the exact number of the bidders it is still a [weakly] dominant strategy to bid one's own value so that we have $\beta(x) = x$. In this case the expected payment of a bidder in $\mathscr{A}$ with the value x can be expressed as:

$$m^2(x) = \sum_{n=0}^{N-1} p_n G^{(n)}(x) E[Y_1^{(n)} | Y_1^{(n)} < x] \tag{105}$$

If we now consider a $FPSB$ auction with a symmetric increasing equilibrium β we have that a bidder with a value x that bids a sum $\beta(x)$ has an expected payment that can be expressed as:

$$m^1(x) = G(x)\beta(x) \tag{106}$$

as the product of the probability to win for the bid the bidder makes and that depend on his value. Since our aim is to obtain the unknown strategy β we can use REP and impose the equality:

$$m^2(x) = m^1(x) \tag{107}$$

or:

$$\sum_{n=0}^{N-1} p_n G^{(n)}(x) E[Y_1^{(n)} | Y_1^{(n)} < x] = G(x)\beta(x) \tag{108}$$

In this way we derive an expression for $\beta(x)$ as:

$$\beta(x) = \sum_{n=0}^{N-1} p_n \frac{G^{(n)}(x)}{G(x)} E[Y_1^{(n)} | Y_1^{(n)} < x] = \sum_{n=0}^{N-1} p_n \frac{G^{(n)}(x)}{G(x)} \beta^{(n)}(x) \qquad (109)$$

where $\beta^{(n)}(x)$ is the equilibrium bidding strategy in a $FPSB$ auction in which there are $n+1$ bidders for sure. In this way according to relation (109) the equilibrium strategy is the weighted average of the strategies that can be used when the number of the actual bidders is known for sure by all the bidders that attend the auction.

9 The extensions

9.1 Introduction

In order to examine the possible extensions we can introduce we recall the basic hypotheses on which the REP is based and also recall that the general framework is of private values. They are:

- independence of the values of the bidders,
- all the bidders are risk neutral so they try to maximize their expected payoff,
- all the bidders can pay up to their values and so have no budget constraints,
- all the bidders' values are distribute according to the same distribution F over the same support.

The extensions we introduce are relaxations of such hypotheses and tend to verify how REP is affected by each of them, if it keeps its validity or not.
In what follows we are going to introduce a relaxation at a time so that we drop one assumption at a time but keeping the others as valid and effective.

9.2 Risk averse bidders

If we relax the hypothesis of risk neutrality of the bidders and introduce[8] **risk avers bidders** the REP is no longer valid.

[8]We recall that the bidders (that can be seen as buyers) can be characterized as **risk averse**, **risk neutral** and **risk seeking** and the same classification is true also for the seller.

We have indeed that **risk neutrality** implies that the expected payoff can be expressed as the difference:

$$\textit{expected gain} \; - \; \textit{expected payment} \tag{110}$$

and so is both **linearly separable** and linear in the payment. This linearity is crucial in the derivation of the REP and since it is lost when the bidders are no more risk neutral this implies that REP is no more valid.
In case of **risk aversion** each bidder is supposed to have a von Neuman Morgernstern utility function:

$$u : \mathbb{R}_+ \longrightarrow \mathbb{R} \tag{111}$$

defined so to satisfy the following conditions:

- $u(0) = 0$,
- $u' > 0$,
- $u'' < 0$,

and each bidder is supposed to maximize his expected utility rater than his expected profit/payoff. For this purpose we may state the following proposition.

Proposition 9.1 *If the bidders are* ***risk averse*** *with the same utility function and their values are:*

- *private,*
- *independent,*
- *symmetric,*

we have that the expected revenue in a $FPSB$ auction is greater than the expected revenue in a $SPSB$ auction so that the REP(that states their equivalence) is violated.
Proof
We start by noting that risk aversion makes no difference in a $SPSB$ auction since it is still a [weakly] dominant strategy for a bidder to bid his own value or $\beta(x) = x$ so that the expected payment (or the expected price) is the same as in the case of risk neutral bidders.
In order to prove that the REP is violated we must consider the case of a $FPSB$ auction and prove that the expected payment is modified since it is either lower or higher so that the equality is violated and the REP is no longer valid.
We therefore consider a $FPSB$ auction where:

- *bidders are risk averse,*
- *the bidders have an utility function* u*,*
- *the equilibrium strategies are defined by an increasing differentiable function* $\gamma : [0,\omega] \longrightarrow \mathbb{R}_+$ *such that* $\gamma(0) = 0$.

Again we use the symmetry hypothesis and examine the individual deviation so we suppose that all the bidders but bidder 1 *follow the strategy* γ*. We therefore have that bidder* 1 *will never bid more than* $\gamma(\omega)$ *(since he wins for sure with a lower bid with a greater gain).*
Given a value x *the problem of each bidder is to chose* $z \in [0,\omega]$ *and bid* $\gamma(z)$ *so to maximize his expected utility and so to solve:*

$$max_z G(z)u(x - \gamma(z)) \tag{112}$$

We note that if u *were linearly separable we would get* $u(x) - u(\gamma(z))$ *and from the fact that* u *is increasing the problem of maximum over* u *would be equivalent on a problem of maximum over* $x - \gamma(z)$*.*
If we consider the problem (112) (where $G(z) = F^{N-1}(z)$ *is the distribution of the highest of* $N-1$ *values lower that* z *that is supposed to be the winning value) we can impose the first order condition (by evaluating the derivative with respect to* z*) so to get:*

$$g(z)u(x - \gamma(z)) - G(z)u'(x - \gamma(z))\gamma'(z) = 0 \tag{113}$$

In a symmetric equilibrium it must be optimal to chose $z = x$ *so we get:*

$$\frac{g(x)u(x - \gamma(x))}{\gamma'(x)} = G(x)u'(x - \gamma(x)) \tag{114}$$

or:

$$\gamma'(x) = \frac{u(x - \gamma(x))}{u'(x - \gamma(x))} \frac{g(x)}{G(x)} \tag{115}$$

We underline how our aim is to prove that $\gamma(x) > \beta(x)$ *so we derive a condition on* β *and* γ *as a condition on* β' *and* γ' *that gives a contradiction so that our hypothesis is false and we prove our assertion.*
We can indeed derive β' *and* γ' *and compare them so that from the second condition we derive* $\gamma(x) > \beta(x)$ *as we wished to prove.*
If bidders are risk neutral with an equilibrium strategy of $\beta()$ *we have* $u(x) = x$ *so that* $u'(x) = 1$ *and we can rewrite the relation (115) as:*

$$\beta'(z) = (x - \gamma(x)) \frac{g(x)}{G(x)} \tag{116}$$

If u is strictly concave and $u(0) = 0$ for all $y > 0$ we have:

$$\frac{u(y) - u(0)}{y - 0} < u'(y) \tag{117}$$

or:

$$\frac{u(y)}{u'(y)} > y \tag{118}$$

By using such relation in the relation (115) we get:

$$\gamma'(z) = \frac{u(x - \gamma(x))}{u'(x - \gamma(x))} \frac{g(x)}{G(x)} > (x - \gamma(x)) \frac{g(x)}{G(x)} = \beta'(x) \tag{119}$$

If (by absurd) we suppose $\beta(x) > \gamma(x)$ we get easily:

$$(x - \gamma(x)) \frac{g(x)}{G(x)} > (x - \beta(x)) \frac{g(x)}{G(x)} \tag{120}$$

so that $\gamma'(x) > \beta'(x)$. We have therefore proved that $\beta(x) > \gamma(x) \Rightarrow \gamma'(x) > \beta'(x)$ where β is the equilibrium strategy with risk neutral bidders and γ is the equilibrium strategy for risk averse bidders.
Since we have $\beta(0) = \gamma(0) = 0$ we should have $\beta(x) > \gamma(x) \Rightarrow \beta'(x) > \gamma'(x)$ so since the conclusion is false we have a contradiction and therefore the hypothesis that we have made is false and we have proved that $\gamma(x) > \beta(x)$.
In his way we have that in a $FPSB$ auction risk aversion causes an increase in the equilibrium bids . Since bids are increased also the expected revenue is increased. In a $SPSB$ auction, on the other hand, the expected revenue is unaffected by risk aversion so we have proved that the expected revenue in a $FPSB$ auction is greater than the expected revenue in a $SPSB$ auction so that REP does not hold anymore.

Observation 9.1 *The proof is based on the fact that we prove that $\gamma'(x) > \beta'(x)$. If on the other hand we suppose $\beta(x) > \gamma(x)$ since $\beta(0) = \gamma(0)$ we should have $\beta'(x) > \gamma'(x)$. Since we prove that by supposing $\beta(x) > \gamma(x)$ implies $\gamma'(x) > \beta'(x)$ we have a contradiction so that we can derive $\gamma(x) > \beta(x)$.*

We give now an intuitive justification of his fact.
In order to do so we consider bidder 1 and suppose that the strategies of the other bidders are fixed. Bidder 1 bids b with a value x. Now we imagine that the bidder 1 considers decreasing his bid from b to $b - \Delta$ so that:

- if he wins with $b - \Delta$ he has a gain of Δ;

- if he loses he has a null gain.

A risk averse bidder thinks that it is more important for him not to risk losing the auction than the small gain that he can make by slightly lowering his bid. This means that a risk averse bidder bids higher than a risk neutral bidder since he prefers winning by bidding more than possibly winning by saving a small Δ but with a higher probability of losing so that in this way this bidder has an insurance against the probability of losing by bidding $b - \Delta$.

Observation 9.2 *Standard auctions with risk neutral bidders are characterized by the fact that the payoffs functions are separable in money so they are quasi linear or linear in the payments. In this case each bidder maximizes his expected payoff that may be expressed as the difference between the expected value and the expected payment. This separation is crucial in the proof of the REP. Risk averse bidders, on the other hand, maximize the expected utility that depends on the difference between the value and the payment but, since the utility function is not linear but is concave, we have that the maximand is no longer linear in the the payments that the bidders make and this non linearity is a reason for the failure of the REP.*

9.3 Budget constraints

In some cases the bidders may face financial constraints so they are not always able to pay the amounts equal to their values. Such financial constraints may influence equilibrium behavior in both $FPSB$ auctions and $SPSB$ auctions so we have to examine how this may affect the REP.
Again we suppose that the values of the bidders are:

- symmetric,
- independent,
- private,

and that the auction form involves a single object and N potential buyers (the bidders) that bid for it.
Bidder i assigns a value X_i to the object and has a budget constraint W_i so that he is characterized by the pair (x_i, w_i) of the realizations of such random variables and cannot pay more than w_i since if he bids more than w_i and defaults he has to pay a small penalty.
We note that for each bidder the pair (X_i, W_i) is identically and independently distributed over $[0, 1] \times [0, 1]$ according to a common density f. We note how this is true among the bidders but for a single bidder the two values

may be correlated.
In this way we have that bidder i knows the realized pair (x_i, w_i) and knows that the other bidders' pairs are independently distributed according to f. We moreover underline how we have **risk neutral bidders** and our aim is to compare a $FPSB$ auction with a $SPSB$ auction. We recall that a risk neutral bidder has a payoff that is given by the difference between his valuation for the object and his bid so we can write:

$$payoff = \ x - \beta(x) \tag{121}$$

With x we denote the signal that is received by the bidder as his own type though he may decide to not use it and bid as if he has received a different signal z.
In this case the private information is bidimensional since the type of bidder i is (x_i, w_i). If we denote with superscript A a generic auction of either $FPSB$ or $SPSB$ type we have that a bidder's strategy is a function:

$$B^A : [0,1] \times [0,1] \longrightarrow \mathbb{R} \tag{122}$$

that defines the amount he bids depending on his value and his budget constraint.

9.3.1 $SPSB$ auctions

For this type of auctions we can state the following proposition.

Proposition 9.2 *In a $SPSB$ auction it is a dominant strategy for a bidder to bid according to the following strategy:*

$$B^2(x, w) = min\{x, w\} \tag{123}$$

Proof
To carry out the proof we consider separately the effects of:

- *the constraint on budget;*

- *the value.*

For what concerns the budget we have that bidding more than w is a dominated strategy since if the bidder wins the auction he must default and pay a penalty with a negative payoff. To be more correct we can state that if $b > w > Y_1$ the bidder would have won also by bidding w whereas if $b > Y_1 > w$ he wins but, since he cannot pay more than w, he has to renege so that he must pay the penalty having therefore a negative payoff.
In this way we have argued that it must be $b_i \leq w_i$.
For what concerns the value x we have two cases:

- $x_i \leq w_i$,

- $x_i > w_i$.

In the former case it is as if the budget constraint were not present so that we have the same equilibrium strategy as in a unconstrained $SPSB$ *auction so that* $B^2(x,w) = x = min\{x,w\}$.
In the latter case we can have the following cases:

- $x_i > w_i > b_i > Y_1$ *so the bidder wins but would have won also by bidding* w_i,

- $x_i > w_i > Y_1 > b_i$ *the bider loses but could have won by bidding* w_i *so having an avoidable loss,*

- $x_i > b_i > w_i > Y_1$ *the bider wins but must renege so has to pay a penalty and suffers a loss.*

We have seen how bidding w_i *is never worse off and in some case it may be even better off so it is a dominant strategy.*

We can therefore define for every type $\{x, w\}$ a value x'' as:

$$x'' = min\{x, w\} \tag{124}$$

Since both x and w are supposed to assume values in the interval $[0, 1]$ we have that also x'' assumes values in the same interval. This allows us to define a bidder whose type is $\{x'', 1\}$. Such a bidder has no financial constraints by definition. Moreover from the fact that:

$$min\{x'', 1\} = x'' = min\{x, w\} \tag{125}$$

we have that:

$$B^2(x, w) = min\{x, w\} = x'' = min\{x'', 1\} = B^2(x'', 1) \tag{126}$$

or:

$$B^2(x, w) = B^2(x'', 1) \tag{127}$$

so that in a $SPSB$ auction the two types $\{x'', 1\}$ and $\{x, w\}$ behave in the same way and submit the same bids.
Once the strategy has been defined we can define as $m^2(x, w)$ the expected payment of a bidder (with value x and budget constraint w) in a[9] SP auction so that from equation (127) we can derive that:

$$m^2(x, w) = m^2(x'', 1) \tag{128}$$

[9]In what follows we are going to refer to a generic second price auction and identify it with the acronym SP.

so that the two bidders are equivalent also for what concerns the expected payment.
From this we can define the set of types of bidders who bid less than the type $\{x'', 1\}$ in a $SPSB$ auction as the following set:

$$L^2(x'') = \{(X, W \mid B^2(X, W) < B^2(x'', 1)\} \tag{129}$$

so that:

$$F^2(x'') = P(X'' < x'') = \int_{L^2(x'')} f(X, W) dx dw \tag{130}$$

is the probability that a bidder of type $(x'', 1)$ bids more than one of these bidders. In relation (130) F^2 is the distribution function of the random variable $X'' = min\{X, W\}$. A bidder of type $(x'', 1)$ (and so without budget constraints) will win the auction with a probability distribution defined as:

$$F^2(x'')^{N-1} = G^2(x'') \tag{131}$$

where again we have used both independence and identical distribution. In this way we can write the expected utility if a bidder of the type $(x'', 1)$ bids $B^2(z, 1)$ having a signal (or a value x'') as $G^2(z)x'' - m^2(z, 1)$. At the equilibrium (by definition of equilibrium) it is optimal to bid $B^2(x'', 1)$ if the true type is $(x'', 1)$ and in this way we get:

$$m^2(x'', 1) = \int_0^{x''} y g^2(y) dy \tag{132}$$

(where g^2 is the density function corresponding to G^2) so that the ex ante expected payment in a $SPSB$ auction with financial constraints is expressed as:

$$R^2 = \int_0^1 m^2(x'', 1) f^2(x'') dx'' = E[Y_2^{2(N)}] \tag{133}$$

where $Y_2^{2(N)}$ is the second highest of N draws from the distribution F^2. We get the rightmost equality in (133) if we replace the expression for $m^2(x'', 1)$ exchange the order of integration and use the expression for the density of $Y_2^{2(N)}$ that we already used in one of the foregoing sections so to derive the definition of the searched for expected value.

9.3.2 $FPSB$ auctions

In a $FPSB$ auction we can imagine that the equilibrium strategy is of the form:

$$B^1(x, w) = min\{\beta(x), w)\} \tag{134}$$

where β is a function that increases with x (the valuation for the bidder). In this case $\beta(x)$ is the bid if the evaluation (or the signal) is x.
We note that we must have $\beta(x) < x$ since in the case $\beta(x) \geq x$ the bidder with a value $x < w$ (where w represents the financial constraints) would have a negative payoff $x - \beta(x)$ and would be better off by bidding less.
In this case we assume that the searched for equilibrium exists so that we can define a value x' such that $\beta(x') = min\{\beta(x), w\}$ so to define, also in this case, the type of bidder $(x', 1)$ that is without any financial constraint.
Also in this case we have:

$$B^1(x, 1) = min\{\beta(x'), 1)\} = \beta(x') = min\{\beta(x), w)\} = B^1(x, w) \qquad (135)$$

and in this way we have $B^1(x, w) = B^1(x', 1)$ so that the two types $(x', 1)$ and (x, w) behave in the same way in the $FPSB$ auction.
If we define, as we have done in the case of the $SPSB$ auctions, the set:

$$L^1(x') = \{(X, W) \mid B^1(X, W) < B^1(x', 1)\} \qquad (136)$$

as the set of bidders who bid less than the bidder of type $(x', 1)$ we can derive also in this case that:

$$E[R^1] = E[Y_2^{2(N)}] \qquad (137)$$

where $Y_2^{2(N)}$ is the second highest of N draws from the distribution F^1.

9.3.3 Revenue comparison

The key point is that for all the valuations x we have $\beta(x) < x$. If we use such condition in the definitions (136) and (129) we get:

$$L^1(x') = \{(X, W)|B^1(X, W) < B^1(x', 1) = min\{\beta(x'), 1\} = \beta(x') < x'\} \qquad (138)$$

and:

$$L^2(x'') = \{(X, W)|B^2(X, W) < B^2(x'', 1) = min\{x'', 1\} = x''\} \qquad (139)$$

From these relations if we impose $x' = x'' = x$ we get $L^1(x) \subset L^2(x)$ so that from the definition of the distribution functions (see (130)) we get:

- $F^1(x) \leq F^2(x)$,
- $F^1(x) < F^2(x)$ if $x \in (0, 1)$,

so that F^1 stochastically dominates F^2 and therefore:

$$E[Y_2^{1(N)}] > E[Y_2^{2(N)}] \qquad (140)$$

Roughly speaking, relation (140) derives from the fact that the variable with distribution F^1 assumes higher values with higher probabilities that those of the variable with distribution F^2 whence such inequality. From these considerations we can derive the following proposition.

Proposition 9.3 *If we have bidders with financial constraints and:*

- *a $FPSB$ auction has a symmetric equilibrium $B^1(x,w) = min\{\beta(x), w\}$,*

- *a $SPSB$ auction has a symmetric equilibrium $B^2(x,w) = min\{x, w\}$,*

then the expected revenue in a $FPSB$ auction is greater than the expected revenue in a $SPSB$ auction.

This conclusion derives from the fact that budget constraints are less influent in a $FPSB$ auction than in a $SPSB$ auction.
We can try to compare two situations. Firstly we suppose to have a budget constraint W_i and then we suppose that the valuation of a bidder is given by $Z_i = min\{X_i, W_i\}$ so that he has no budget constraints. In this second situation we can apply REP so that a $FPSB$ auction and a $SPSB$ auction give the same expected revenue, be it R.
If we consider the first situation where we have a budget constraint we have that:

- in a $SPSB$ auction the value R is unchanged since the bidding strategy for the bidders is unchanged;

- in a $FPSB$ auction the revenue is greater than R since we compare a situation where the bidders have values $X_i \geq Z_i$ and budgets $W_i \geq Z_i$ with a situation where their values are Z_i and they have no budget constraints.

10 Some notes on auctions with interdependent values

10.1 Introductory remarks

Up to now we have supposed that the signals of the bidders are independent and independently distributed and in this way we made some hypotheses over the values and the information of the bidders. At this point we relax such hypotheses and introduce both interdependent values and correlated signals.

10.1.1 Interdependent values

We now relax the hypothesis of **private values** so that a bidder may modify the value he gives to an object depending on the signals he receives from the other bidders.
In this way we speak of **interdependent values** so that:

- each bidder has private information on the value of the object,
- such value may be represented through a random variable $X_i \in [0, \omega_i]$ that we term a **signal**.

The value V_i of the object for bidder i can therefore be expressed as:

$$V_i = v_i(X_1, X_2, \dots, X_N) \tag{141}$$

and so as depending on the signals of all the bidders. In (141) v_i represents the bidder i valuation and we assume it is:

- non decreasing in all X_j;
- twice continuously differentiable;
- strictly increasing in X_i.

The value V_i depends only on the signals of the bidders with no further uncertainty. We can however generalize and suppose that:

- $V_1, \dots, V_N$ are the unknown values to the bidders,
- $X_1, \dots, X_N$ are their signals,
- S is a signal known only to the seller,

so that we have:

$$v_i(x_1, \dots, x_N) = E[V_i | X_i = x_i \; i = 1, \dots, N] \tag{142}$$

as the expected value for bidder i conditional on the values x_i. In both cases we suppose that:

- $v_i(0, \dots, 0) = 0$,
- $E[V_i] < \infty$,
- bidders are risk neutral so each of them maximizes the difference between the value and the payment (if he wins) or $V_i - p_i$.

From this general scheme we may derive:

- the **private values model** if we impose $v_i(X_1, \ldots, X_N) = X_i$ for every bidder;

- the **pure common value model** if we impose $V = v(X_1, \ldots, X_N)$ for every bidder.

In the former case the valuation depends only on each bidder's signal whereas in the latter the bidders have the same valuation of the object. In he **pure common value case** we have that the bidders know only their signals so the ex post common value is ex ante unknown to the bidders.
In the case of **interdependent values** the decision problem is harder since the exact value of an object is unknown and depends on the signals of the other bidders. We have therefore that an a priori estimation of such a value may need to be revised owing to the information gathered during the auction. During the course of the auction indeed there may be events that convey information about the signals of the other bidders. Such events typically occur in open cry auctions and include:

- the number of the currently active bidders,

- the announcement that a bidder dropped out,

- the announcement that a bidder won the auction.

10.1.2 The winner's curse

In an auction we generally have both ex ante information (or prior to or during the execution of the auction) and ex post information (or after the end of the auction). If, thanks to the symmetry hypothesis, we focus on a particular bidder, say bidder 1, the ex ante information include:

- the private signal $X_1 = x$,

- an estimate of the value of the object $E[V|X_1 = x]$.

If the object is sold with a $FPSB$ auction we can try to understand what does it mean that bidder 1 is the winner by bidding a value that corresponds to the signal x.
If all the bidders are symmetric and follow the same strategy β this means that the highest of the $N-1$ signal is lower than x (since β is increasing) so that for bidder 1 we have:

$$E[V|X_1 = x, Y_1 < x] \tag{143}$$

(where the conditional event is the condition of being the winner since Y_1 is the highest of the other $N-1$ signals) as the estimate of the value upon being a winner and:

$$E[V|X_1 = x, Y_1 < x] \leq E[V|X_1 = x] \tag{144}$$

where $E[V|X_1 = x]$ is the same estimate without knowing to be the winner. The inequality holds since the latter conditioning events wider than the former.
The announcement of the victory causes a decrease in the estimated value and this may represent the bad news that give raise to the winner's curse as the possibility to have overestimated the value of an object.
The same phenomenon is present in a pure common value model where we have that the signals of the bidders can be expressed as:

$$X_i = V + \epsilon_i \tag{145}$$

In (145) the ϵ_i are independent and identically distributed random variables with $E[\epsilon_i] = 0$.
In this case we have:

$$E[X_i|V = v] = E[V + \epsilon_i|V = v] = E[v + \epsilon_i] = v \tag{146}$$

(since $E[\epsilon_i] = 0$) so that each signal is an unbiased estimator of the common value but for the largest of N such signals. This is caused by the fact that max is a convex function so that:

$$E[max\ X_i|V = v] > max\ E[X_i|V = v] = v \tag{147}$$

so that the expectation of the highest signal overestimates the value of the object. This implies the need for a bidder to shade their values below their initial estimates so to avoid the winner's curse.

10.1.3 English ad SP auctions

Our aim is to show how, in the current framework, an English auction is no more equivalent to a $SPSB$ auction.
We recall that an English auction is an open cry ascending auction where the bids of the bidders are common knowledge among them whereas the SP auction is a one shot sealed bid auction where this exchange of information cannot occur. In an English auction indeed the active bidders know the prices at which the other bidders drop out and so can make inferences about the values of such bidders and update their estimates of such values.
The exchanged information of an English auction may be irrelevant in two cases:

- if there are only two bidders since when one of them drops out the auction ends;

- if the bidders have private values since the information produced by the others have no meaning for each bidder that does not update his private value.

10.1.4 Affiliation

We suppose that the signals of the bidders are correlated so that the joint density function $f(\mathbf{X})$ of the signals $\mathbf{X}= \{X_1, \ldots, X_N\}$ cannot be expressed in general as:

$$f(\mathbf{X}) = \Pi_{i=1}^{N} f_i(X_i) \tag{148}$$

since we assume that the signals are positively affiliated as a strong form of positive correlation. We define affiliation in an informal way as follows: given a set of variables X_i if a subset of the X_i are all large this raises the probability that also the remaining X_j are large. From this informal definition we derive that:

- if the $Y_1, Y_2, \ldots, Y_N$ are the ordered statistics of the variables X_i and the X_i are affiliated also the Y_i are affiliated;

- if $G(\cdot|x)$ is the distribution function of Y_1 conditional on $X_1 = x$ and $g(\cdot|X)$ is the corresponding density function then, if $x' > x$ we have:

$$\frac{g(y|x')}{G(y|x')} \geq \frac{g(y|x)}{G(y|x)} \tag{149}$$

- if γ is an increasing function and $x' > x$ we have:

$$E[\gamma(Y_1)|X_1 = x'] \geq E[\gamma(Y_1)|X_1 = x] \tag{150}$$

10.2 The symmetric model

In this section we proceed as follows:

- we define symmetric equilibrium strategies,

- we compare the expected revenues,

for the following three types of auctions:

- English or open ascending;

- second price;

- first price.

We disregard Dutch or open descending auction that is still equivalent to a $FPSB$ auction.
We recall that in the case of independent private values the bidders are symmetric if their values are drawn from the same distribution. In the present case on the other hand we have:

- interdependent values,

- affiliated signals,

so we have to consider both the **symmetry of the valuations** v_i and the **symmetry of the distributions** of the signals.
We assume that all the signals X_i are drawn from the same interval $[0, \omega]$ and that the valuations of the bidders can be written as:

$$v_i(\mathbf{X}) = u(X_i, \mathbf{X}_{-i}) \tag{151}$$

for every bidder i. In this way we define the symmetry since the function u is the same for all bidders and is symmetric in the values $\mathbf{X}_{-i} = X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_N$ so that, with regard to a given bidder i, the signals of the other bidders can be exchanged without affecting the value of the function u. This means, for instance, that, in the case of three bidders, we have $u(x, y, z) = u(x, z, y)$.
At this point we assume that:

- the joint density function f of the signals is defined on $[0, \omega]^N$ and is a symmetric function of its arguments;

- the signals are affiliated.

We can define:

$$v(x, y) = E[V_1 | X_1 = x, Y_1 = y] \tag{152}$$

as the expected value for bidder 1 that receives a signal x if the highest signal among the other $N - 1$ bidder is y. In this case we have that:

- from the symmetry we derive that v is the same for all the bidders;

- from (150) we derive that v is a non decreasing function of both x and y;

- we assume that v is strictly increasing in x;

- we impose $v(0,0) = 0$ from the condition $v(\mathbf{0}) = 0$.

Once this new definition of symmetry has been introduced we can:

- use it to devise the symmetric equilibrium strategies in the three cases we listed at the offset of this section;

- use it to perform a comparison of the three formats by computing the expected revenue for each type of auction.

10.3 Second price auctions

We start by devising a symmetric equilibrium strategy in a $SPSB$ auction through the following proposition.

Proposition 10.1 *A symmetric equilibrium strategy in a SP auction is given by:*

$$\beta^2(x) = v(x,x) = E[V_1|X_1 = x, Y_1 = x] \tag{153}$$

if we consider bidder 1 that receives the signal x and the highest of the other values is x.
Proof
We suppose that all the bidders but bidder 1 follow $\beta = \beta^2$ whereas bidder 1 with a signal x bids $b = \beta(y)$ (as he had a value y) so to get a payoff equal to:

$$\Pi(b,x) = expected\, valuation\; -\; expected\, payment \tag{154}$$

or:

$$\Pi(b,x) = \int_0^y v(x,y)g(y|x)dy - \int_0^y \beta(y)g(y|x)dy \tag{155}$$

where $y = \beta^{-1}(b)$ and $g(\cdot|x)$ is the density of $Y_1 = max_{i\neq 1}X_i$ conditional on $X_1 = x$.
We can rewrite (155) as:

$$\Pi(b,x) = \int_0^{\beta^1(b)} (v(x,y) - v(y,y))g(y|x)dy \tag{156}$$

where we have used the equality $\beta(y) = v(y,y)$. If we impose a first order condition on Π with respect to b we have that the sign of the derivative depends on the sign of the term $(v(x,y) - v(y,y))$. To devise this sign we use the fact that v is increasing in the first argument so that:

- *if $x > y$ we have $v(x,y) - v(y,y) > 0$*

- *if $x < y$ we have $v(x,y) - v(y,y) < 0$*

so that $\Pi(b,x)$ *is maximized if we take* $\beta^{-1}(b) = x$ *or* $b = \beta(x)$. *In this way we have derived the first order condition by imposing the adoption of the strategy as the optimal strategy and then we have showed how this necessary condition is also sufficient.*

The proposition we have proved states that if the bidder 1 with a signal x bids $\beta(x)$ and the highest competing bid (and so the price) is $\beta(x)$ bidder 1 would just "break even". In this case if the highest competing bidder bids $\beta(x)$ then bidder 1 can infer $Y_1 = x$ as new information so that the expected value is:

$$E[V_1|X_1 = x, Y_1 = x] = v(x,x) = \beta(x) \tag{157}$$

In case of private values we have $v(x,x) = x$ and the equilibrium strategy is a dominant strategy. In the case of interdependent values where we have $v(x,y)$ the strategy $\beta^2(x) = v(x,x)$ is not dominant though it is unique in the case of:

- symmetric equilibria,
- with an increasing equilibrium strategy.

To illustrate this result we give an example where we have:

- a common value,
- bidders' signals are independently distributed conditional on that value.

Example 10.1 *We have three bidders with a common value* V *uniformly distributed over* $[0,1]$. *If the realization of* V *is* $V = v$ *the signals* X_i *are:*

- *uniformly,*
- *independently*

distributed over $[0,2v]$. *We have* $\mathbf{X} = \{X_1, X_2, X_3\}$ *and define* $Z = max\{X_1, X_2, X_3\}$. *In this case we have:*

$$f_{X_i}(x|V = v) = \frac{1}{2v} \tag{158}$$

over $[0,2v]$. *In this way the joint density of* $(V, \mathbf{X})$ *is the product of the three densities (by the independence) so that it is* $1/8v^3$ *over the set:*

$$\{(V, \mathbf{X})|\forall i\ X_i \leq 2V\} \tag{159}$$

From (159) we get:

$$V \geq \frac{1}{2}Z = \frac{1}{2}max\{X_1, X_2, X_3\} \tag{160}$$

so that we have:

$$f(x_1, x_2, x_3) = \int_{\frac{1}{2}z}^{1} f(x_1, x_2, x_3, V)dv = \int_{\frac{1}{2}z}^{1} \frac{1}{8v^3}dv = \frac{4-z^2}{16z^2} \tag{161}$$

We therefore get that the density of V conditional on $\mathbf{X} = \mathbf{x}$ is the same as the density of V conditional on $Z = z$ and the same holds also for the conditional expectations so we get:

$$f(v|\mathbf{X} = \mathbf{x}) = f(v|Z = z) = \frac{f(v, \mathbf{X} = \mathbf{x})}{f(\mathbf{X} = \mathbf{x})} = \frac{1}{8v^3}\frac{16z^2}{4-z^2} \tag{162}$$

over $[\frac{1}{2}z, 1]$. In this way we derive:

$$E[V|\mathbf{X} = x] = E[V|Z] = \int_{\frac{1}{2}z}^{1} vf(V|Z)dv = \frac{2z}{2+z} \tag{163}$$

The aim is to evaluate $v(x, y)$ since $\beta^2(x) = v(x, x)$. To do so we note that given the values X_1, X_2, X_3 if $Y_1 = max\{X_2, X_3\}$ we have $Z = max\{X_1, X_2, X_3\} = max\{X_1, Y_1\}$ so we get:

$$v(x, y) = E[V|X_1 = x, Y_1 = y] = E[V|Z = max\{x, y\}] = \frac{2max\{x, y\}}{2 + max\{x, y\}} \tag{164}$$

so that the searched for strategy is given by:

$$v(x, x) = \beta^2(x) = \frac{2x}{2+x} \tag{165}$$

since $max\{x, x\} = x$.

Observation 10.1 *Given $V \in [0, 1]$ the values X_i are independently and uniformly distributed over $[0, 2v]$ so that we can write:*

$$f(X_i|V = v) = \frac{1}{2v} \tag{166}$$

From the independence we can write:

$$f(X_1, X_2, X_3, V) = f(X_1, X_2, X_3|V)f(V) = f(X_1|V = v)f(X_1|V = v)f(X_1|V = v) = \frac{1}{8v^3} \tag{167}$$

from (166) and from the fact that $f(V) = 1$. *Relation (167) is true over the set:*

$$\{(V, \mathbf{X}) | \forall\, i\; X_i \leq\; 2V\} \tag{168}$$

The conditions that define such sets are equivalent to the following:

$$Z = max\; X_i \leq 2V \tag{169}$$

or:

$$V \geq \frac{1}{2} Z \tag{170}$$

We note that to evaluate $f(X_1, X_2, X_3)$ *we need to integrate* $f(X_1, X_2, X_3, V)$ *with respect to* V *over the interval* $[1, \frac{1}{2}Z]$ *and from this we derive the result of the example.*
Starting from the following equality:

$$f(v|\mathbf{X} = \mathbf{x}) = f(v|Z = z) \tag{171}$$

we can derive:

$$E[V|\mathbf{X} = \mathbf{x}] = E[V|Z = z] \tag{172}$$

10.4 English auctions

In a sealed bid auction there is no exchange of information during the auction so that the strategy of each bidder determines the amount he bids as a function of his private information. On the other hand, in an English (or open ascending) auction the various bidders can follow directly the bids of the others.
In this way each bidder knows:

- the prices at which some of the other bidders drop out,

- the number of the active bidders at each stage.

These information represent the relevant information in the case of the symmetric model whereas, for instance, the identities of both the active bidders and the bidders who dropped out have no relevance.
English auctions are represented by an open (commonly known to the bidders) ascending price but may have variable rules so it is necessary to specify precisely the rules we intend to follow in the formal definition of such type of auctions.
From our point of view we are going to adopt the following rules:

- an auctioneer starts with a null price and gradually raises it;

- the current price is common knowledge among the currently active bidders;

- the active bidders signal in some way (that is common knowledge among all the active bidders) their willingness to buy and so to attend the auction;

- bidders can drop out at any time (and so are no more active) and cannot reenter at a higher price;

- the auction ends when there is only one active bidder left that is therefore the winner of the auction.

In this case we define a symmetric equilibrium strategy as:

$$\beta = (\beta^N, \beta^{N-1}, \dots, \beta^2) \tag{173}$$

as composed by $N-1$ functions each defined as:

$$\beta^k \; [0,1] \times \mathbb{R}_+^{N-k} \longrightarrow \mathbb{R}_+ \tag{174}$$

for $1 < k \leq N$. The generic function:

$$\beta^k(x, p_{k+1}, \dots, p_N) \tag{175}$$

represents the price at which bidder 1 will drop out if:

- his signal is x,

- there are still k active bidders,

- the prices at which the other $N-k$ bidders dropped out were:

$$p_{k+1} \geq p_{k+2} \geq \cdots \geq p_N \tag{176}$$

For instance, if $k = 2$ (so there are only two active bidders) $N-2$ bidders already dropped out at the prices:

$$p_3 \geq p_4 \geq \cdots \geq p_N \tag{177}$$

On the other hand if $k = N$ all the bidders are active so that we have:

$$\beta^N(x) \tag{178}$$

We consider now the possible strategies of the bidders. At the offset of the auction all the bidders are active so we can write the following continuous and increasing function:

$$\beta^N(x) = u(x, x, \dots, x) \tag{179}$$

so that all the bidders have the same signal.
We suppose that bider N is the first to drop out at the price p_N with the unique[10] signal x_N such that:

$$\beta^N(x_N) = p_N \tag{180}$$

In this case the other $N-1$ bidders may infer the value of the signal for the dropping out bidder so that they can use the following strategy:

$$\beta^{N-1}(x, p_N) = u(x, x, \dots, x, x_N) \tag{181}$$

where β^{N-1} is again continuous and increasing. At this point we can proceed recursively for all k such that $2 \leq k < N$ and suppose that the bidders $N, N-1, \dots, k+1$ have dropped out at the prices:

$$p_{k+1} \geq p_{k+2} \geq \cdots \geq p_N \tag{182}$$

In this way the remaining k active bidders follow the strategy:

$$\beta^k(x, p_{k+1}, \dots, p_N) = u(x, x, \dots, x, x_{k+1}, \dots, x_N) \tag{183}$$

where:

$$\beta^{k+1}(x_{k+1}, p_{k+2}, \dots, p_N) = p_{k+1} \tag{184}$$

is the price at which the $k+1$-th bidder dropped out.
Our aim is to prove that such strategies represent an equilibrium of the English auction.
Before introducing and proving the corresponding proposition we make some comments.
If we are at the stage of the auction where bidders $k+1, k+2, \dots, N$ have dropped out we have k active bidders. Since the strategies are revealing we have that the prices at which each bidder dropped out has revealed to the others the corresponding signal so that the signals $x_{k+1}, x_{k+2}, \dots, x_N$ are common knowledge among the currently active bidders.
We now consider bidder 1 with a signal x and suppose that the other $k-1$ active bidders are following the strategy β^k so that bidder 1 has to evaluate

[10] Unicity derives from the fact that β is both continuous and increasing.

whether or not he should drop out at the price p. To do so bidder 1 can try to understand what could happen if he wins the auction at the price p or if, necessary and sufficient condition, the other $k-1$ active bidders drop out at p. In this case bidder 1 could be able to infer their signal y as the value such that:

$$\beta^k(y, p_{k+1}, \ldots, p_N) = p \tag{185}$$

so that the inferred value of the object would be:

$$u(x, y, \ldots, y, x_{k+1}, \ldots, x_N) \tag{186}$$

and should be greater than p for the bidder 1 to go on attending the auction. We note that in relation (186) we have $k-1$ times the value y (since $k-1$ bidders should drop out now for bidder 1 to win the auction) and $N-k$ signals x_{k+i} of the bidders who have already dropped out before the current stage.
In this way we have that the strategy for bidder 1 is to keep on with the auction until he wins the auction so that he just breaks even and the inferred value is equal to its price. We now give the following proposition.

Proposition 10.2 *In an English auction the symmetric equilibrium strategies are defined by the following relations:*

$$\beta^N(x) = u(x, x, \ldots, x) \tag{187}$$

and

$$\beta^k(x, p_{k+1}, \ldots, p_N) = u(x, x, \ldots, x, x_{k+1}, \ldots, x_N) \tag{188}$$

where in (187) all the N bidders have the same signal and in 188) the same signal is owned by the remaining k bidders.
Proof
We consider bidder 1 with a signal $X_1 = x$ and suppose that the other $N-1$ bidders follow the strategy β as defined by (187) and (188). We can denote with:

$$Y_1, Y_2, \ldots, Y_{N-1} \tag{189}$$

respectively the highest, the second highest and the smallest of the X_i for $i = 2, \ldots, N$ with realizations:

$$y_1, y_2, \ldots, y_{N-1} \tag{190}$$

Bidder 1 may either win the auction or lose it. In both cases we aim at showing that it is optimal also for bidder 1 to follow the same strategy followed by all the other bidders.

We suppose that the realizations (190) are such that bidder 1 *wins the auction if he too follows* β *so that his signal must satisfy the following condition:*

$$x > y_1 \tag{191}$$

If 1 *wins the auction he pays a price equal to the second highest bid or the price at which the bidder with a signal* y_1 *drops out so is payoff is given by:*

$$u(x, y_1, y_2, \ldots, y_{N-1}) - u(y_1, y_1, y_2, \ldots, y_{N-1}) \tag{192}$$

as the difference between the valuation and the paid price. The quantity expressed by difference (192) is strictly positive since u is increasing in all its arguments. In this way we conclude that since bidder 1 *cannot affect the price he pays and if he wins he has a positive payoff he cannot do no better than to follow* β*.*
The other possibility is that 1 *loses the auction even if he follows the strategy* β*. This may occur if* $x < y_1$*. In the eventuality of losing bidder* 1 *may either drop out or keep on.*
If he keeps on and wins he gets a payoff:

$$u(x, y_1, y_2, \ldots, y_{N-1}) - u(y_1, y_1, y_2, \ldots, y_{N-1}) < 0 \tag{193}$$

since u is increasing in all its arguments. In relation (193) we have the ex post value of the object.
We have proved also in this case that 1 *is better off by following the strategy* β*.*

The equilibrium strategy β does not depend on the distribution of the signals and so they form an ex post equilibrium. This means that:

- the strategy β is a Nash Equilibrium of the complete information game that results if the signals are completely known;

- the strategy β has a no regret feature so that the bidders have no reason to regret the outcome (both as winner and as losers) when the signals are revealed.

As to the second point we note how in a SP auction the bidders may suffer from a regret when the signals are revealed. This means that the equilibrium in a SP auction is not an ex post equilibrium so that at least one bidder would like to have behaved in a different way. With the term **regret** we denote indeed the fact that a bidder knows that he could have done better by behaving in a different way from that prescribed by the ex ante equilibrium. The English auctions have such no regret property since:

- if the signals are revealed to the winner he has no regret from winning,
- if the signals are revealed to the losers they have no regret from losing since if they were to win they would have had a negative payoff.

10.5 First Price auctions

We define also in this case the equilibrium bidding strategies by following classical guidelines and so by supposing that all the bidders but bidder 1 follow the strategy β that we want to define and that we suppose to be:

- increasing,
- differentiable.

Once the strategy has been defined we show how it is optimal also for bidder 1 to follow it.
Though bidder 1 does not follow the strategy β (that maps a value or a signal on the corresponding bid) we note that:

- he does not bid less than $\beta(0)$ since he would surely lose;
- he does not bid more than $\beta(\omega)$ since he could win the same by bidding less but so having a higher payoff.

We recall indeed that the signals X_i assume values over the common interval $[0, \omega]$.
In what follows we denote with $G(\cdot|x)$ the distribution of $Y_1 \equiv max_{i \neq 1} X_i$ conditional on $X_1 = x$ and with $g(\cdot|x)$ the conditional density function.
We therefore can define the expected payoff for bidder 1 in the case:

- he has a signal x;
- he bids as his signal were z and so bids $b = \beta(z)$ so that $z = \beta^{-1}(b)$.

The last condition, if the strategy is revealing, allows the inferring of a signal from the revealed bid. Under the foregoing hypotheses the expected payoff for bidder 1 can be expressed as:

$$\Pi(z, x) = \int_0^z (v(x, y) - \beta(z)) g(y|x) dy = \int_0^z v(x, y) g(y|x) dy - \int_0^z \beta(z) g(y|x) dy \tag{194}$$

or:

$$\Pi(z, x) = \int_0^z v(x, y) g(y|x) dy - \beta(z) \int_0^z g(y|x) dy = \int_0^z v(x, y) g(y|x) dy - \beta(z) G(z|x) \tag{195}$$

since $G(0|x) = 0$. If we impose over relation (195) a first order condition as:

$$\frac{\partial \Pi(z,x)}{\partial z} = 0 \tag{196}$$

we get:

$$v(x,z)g(z|x) - \beta(z)g(z|x) - \beta'(z)G(z|x) = 0 \tag{197}$$

or:

$$(v(x,z) - \beta(z))g(z|x) - \beta'(z)G(z|x) = 0 \tag{198}$$

From relation (198) we have that at symmetric equilibrium is optimal to have $z = x$ (though we have to prove such assertion) so we get:

$$\beta'(x) = (v(x,x) - \beta(x))\frac{g(x|x)}{G(x|x)} \tag{199}$$

Condition (199) is only a necessary condition. In order for it to become also a sufficient condition we must impose $v(x,x) - \beta(x) \geq 0$. If we had $v(x,x) - \beta(x) < 0$ a bid of 0 would be better. In this case we indeed have $\beta'(x) < 0$ would mean a negative expected payoff.
From the assumption $v(0,0) = 0$ we have $\beta(0) = 0$ that is the boundary condition of the differential equation (199).
The solution of the differential equation (199) together with the aforesaid boundary condition is a symmetric equilibrium strategy as it is stated in the following proposition.

Proposition 10.3 *In a FPSB auction a symmetric equilibrium strategy is expressed as:*

$$\beta^1(x) = \int_0^x v(y,y)dL(y|x) \tag{200}$$

where:

$$L(y|x) = exp(-\int_y^x \frac{g(t|t)}{G(t|t)}dt) \tag{201}$$

For the proof of this proposition we refer to Krishna (2002).

If the values are private we have $v(y,y) = y$ and if the signals are independent we have $G(\cdot|x) = G(\cdot)$ since G does not depend on x and the definition (201) boils down to the following:

$$L(y|x) = exp(-\int_y^x \frac{g(t)}{G(t)}dt) = exp(-\int_y^x \frac{1}{G(t)}dG(t)) = exp(-logG(t)|_y^x = \frac{G(y)}{G(x)} \tag{202}$$

In this way we have:

$$\beta^1(x) = \int_0^x \frac{v(y,y)}{G(x)} dG(y) = \frac{1}{G(x)} \int_0^x v(y,y)g(y)dy = E[Y_1|Y_1 < x] \quad (203)$$

so that the strategy (200) reduces to the equilibrium strategy in the case of private values.

10.6 Revenue comparison

In this section we examine the performance of the three auction types we have seen up to now (FP, SP and English auctions) through a comparison of the expected revenue as it results from the symmetric equilibrium for each type of strategy.
We can state that if the signals are affiliated we have:

$$Eng \succ SP \succ FP \quad (204)$$

where Eng denotes the English auction type and $\succ$ is the binary relation "outperforms" or "performs better than".
As a first step we have the following proposition.

Proposition 10.4 *In an English auction the expected revenue is at least as great as the expected revenue from a* SP *auction.*
Proof
In a SP *auction the equilibrium strategy is given by* $\beta^2(x) = v(x,x) = E[V_1|X_1 = x, Y_1 = x]$ *so that (if* $x > y$*) we can write the following chain of equalities/inequalities:*

$$\begin{aligned} v(y,y) &= E[u(X_1, Y_1, \dots Y_{N-1})|X_1 = y, Y_1 = y] \\ &= E[u(Y_1, Y_1, \dots Y_{N-1})|X_1 = y, Y_1 = y] \\ &\leq E[u(X_1, Y_1, \dots Y_{N-1})|X_1 = x, Y_1 = y] \end{aligned} \quad (205)$$

since u *is increasing in all its arguments and all the signals are affiliated.*
In this way we get $v(y,y) \leq v(x,y)$ *so that the expected revenue of a* SP *auction can be written as:*

$$\begin{aligned} & E[R^2] = E[\beta^2(Y_1)|X_1 > Y_1] \\ & = E[v(Y_1, Y_1)|X_1 > Y_1] \\ & \leq E[E[u(Y_1, Y_1, Y_2, \dots, Y_{N-1})|X_1 = x, Y_1 = y]|X_1 > Y_1] \\ & = E[u(Y_1, Y_1, Y_2, \dots, Y_{N-1})|X_1 > Y_1] \\ & = E[\beta^2(Y_1, Y_2, \dots, Y_{N-1})] \\ & = E[R^{Eng}] \end{aligned} \quad (206)$$

where R^2 is the revenue is a SP auction, β^2 is the strategy in an English auction when only two bidders remain and $\beta^2(Y_1, Y_2, \ldots, Y_{N-1})$ is the price paid by the winning bidder.

Observation 10.2 *We note that an English auction has a strictly greater revenue than a SP auction only if the values are interdependent and the signals are affiliated.*
If the values are private the revenue in the two cases is the same and the same holds if the signals are independent.

We now consider the comparison between SP auctions and FP auctions. We can state the following proposition.

Proposition 10.5 *The expected revenue in a SP auction is at least equal to the expected revenue in a FP auction.*
Proof
We consider a bidder with a signal x so that:

- *in a FP auction his payment, conditional upon winning, is his bid $\beta^1(x)$;*
- *in a SP auction his payment, conditional upon winning, is given by $E[\beta^2(Y_1)|X_1 = x, Y_1 < x]$;*

where β^1 and β^2 are respectively the equilibrium strategies in a FP and in a SP auction. We want to prove that:

$$E[\beta^2(Y_1)|X_1 = x, Y_1 < x] \geq \beta^1(x) \tag{207}$$

For doing this we remark how in both auctions the probability that a bidder with a signal x wins the auction is the same and coincides with the probability that x is the highest signal among the signals of all the bidders. This remark is the key point of the proof for which we refer to Krishna (2002), pages 97 and 98.

From the preceding points we can state that in the symmetric model with:

- interdependent values,
- affiliated signals,

the three types of auctions we have seen can be ranked in terms of expected revenue as follows:

$$E[R^{Eng}] \geq E[R^2] \geq E[R^1] \tag{208}$$

11 A few words on the linkage principle

Up to now we have compared three auction formats through a computation of the expected revenues in the respective symmetric equilibria.
In a preceding section we have seen the **revenue equivalence principle** as a way to justify the equality of the expected revenues in FP and SP auctions.
We now introduce the revenue ranking principle or **linkage principle** as a way to justify the fact that the revenue in a SP auction is greater than the revenue in a FP auction.
Also in this case we consider a symmetric setting and assume a standard auction A where the highest bid wins the object and where we have a symmetric equilibrium β^A. In this case we denote as $W^A(z,x)$ the expected price that bidder 1 pays when he wins under the hypotheses that:

- he receives a signal x,
- he bids as if his signal were z so he bids $\beta^A(z)$.

In a FP auction a bidder pays what he bids so we have:

$$W^1(z,x) = \beta^1(z) \tag{209}$$

where β^1 is a symmetric equilibrium strategy of this type of auctions.
In a SP auction the winning bidder pays the second highest bid and so the following random value:

$$W^2(z,x) = E[\beta^2(Y_1)|X_1 = x, Y_1 < z] \tag{210}$$

where Y_1 is the second highest bid or is the highest of the $N-1$ remaining bids and β^2 is the symmetric equilibrium strategy in a SP auction.
We can now state the linkage principle as a proposition. Before we give some notation. With $W_i^A(z,x)$ where $i = 1,2$ we denote the partial derivative of W^A with respect to the $i-$th argument evaluated in (z,x).

Proposition 11.1 *If A and B are two auctions where the highest bidder wins and pays a positive amount and each auction has a symmetric and increasing equilibrium such that:*

(a) for all signals x we have $W_2^A(x,x) \geq W_2^B(x,x)$,

(b) $W^A(0,0) = 0 = W^B(0,0)$,

then the expected revenue from A is at least as large as the expected revenue from B.

Proof

We consider the auction A and suppose that all the bidders but bidder 1 follow the strategy β^A whereas bidder 1 with a signal x bids as if his signal were z and so bids $\beta^A(z)$. The probability for him to win are given by:

$$G(z|x) = P[Y_1 < z|X_1 = x] \tag{211}$$

Each bidder maximizes the difference between the expected gain and the expected payment that can be expressed as:

$$\int_0^z v(x,y)g(y|x)dy - G(z|z)W^A(z,x) \tag{212}$$

At the equilibrium it is optimal for the bidders to follow the equilibrium strategy and so to bid according to the received signal so that $z = x$. If therefore we impose a first order condition on (212) by differentiating with respect to z and we impose $z = x$ we get:

$$g(x|x)v(x,x) - g(x|x)W^A(x,x) - G(x|x)W_1^A(x,x) = 0 \tag{213}$$

that can be rewritten as:

$$W_1^A(x,x) = \frac{g(x|x)}{G(x|x)}v(x,x) - \frac{g(x|x)}{G(x|x)}W^A(x,x) \tag{214}$$

In a similar way for auction B we get:

$$W_1^B(x,x) = \frac{g(x|x)}{G(x|x)}v(x,x) - \frac{g(x|x)}{G(x|x)}W^B(x,x) \tag{215}$$

From equations (213) and (214) we can derive:

$$W_1^A(x,x) - W_1^B(x,x) = -\frac{g(x|x)}{G(x|x)}[W^A(x,x) - W^B(x,x)] \tag{216}$$

If we now define:

$$\Delta(x) = W^A(x,x) - W^B(x,x) \tag{217}$$

we get:

$$\Delta'(x) = [W_1^A(x,x) - W_1^B(x,x)] + [W_2^A(x,x) - W_2^B(x,x)] \tag{218}$$

or, by replacing the first term on the right side with the equivalent term form (216) and using definition (217):

$$\Delta'(x) = -\frac{g(x|x)}{G(x|x)}\Delta(x) + [W_2^A(x,x) - W_2^B(x,x)] \tag{219}$$

From the hypotheses we have:

(a) $\Delta(0) = W^A(0,0) - W^B(0,0) = 0$,

(b) $W_2^A(x,x) - W_2^B(x,x) \geq 0$.

Since $\Delta'(x) \geq 0$ we have $\Delta(x) \geq 0$ (by considering it as a function of x) whence the thesis.

Observation 11.1 *We recall that with the function $v(x,x) = E[V_1|X_1 = x, Y_1 = y]$ we denote the expectation of the value for bidder 1 when he receives the signal x and the highest signal among the other bidders is $Y_1 = y$. Such a function, owing to the symmetry, is the same for all bidders and is a non decreasing function of x and y.*

Proposition (11.1) allows us to rank alternative auction forms by comparing the statistical linkages between the signal of a bidder and the price he would pay upon winning so that the greater is the linkage the higher the expected price paid upon winning where the linkage is between a bidder's own information and how he perceives the other bidders will bid. This proposition does not make any assumption on the distribution of the signals but in the applications it is usually assumed that the signals are affiliated.

Example 11.1 (FP auctions versus SP auctions) *We now use the linkage principle to justify why a SP auction outperforms a FP auction as to the revenue.*
In a FP auction we have:

$$W^1(z,x) = \beta^1(z) \tag{220}$$

where we have that:

- *x is the received signal,*
- *z is the used signal,*
- *β^1 is the symmetric equilibrium strategy in this type of auctions.*

From equation (220) we derive, by performing a derivate with respect to x:

$$W_2^1(z,x) = 0 \tag{221}$$

for every x.
In a SP auction we have:

$$W^2(z,x) = E[\beta^2(Y_1)|X_1 = x, Y_1 < z] \tag{222}$$

where β^2 is the symmetric equilibrium strategy in this type of auctions.
From equation (220) and (222) we derive $W^1(0,0) = 0 = W^2(0,0)$ so to apply the linkage principle where A is the SP auction and B is the FP auction we must verify if the condition on the derivatives is satisfied or not.
If we consider relation (222) and apply the affiliation between X_1 and Y_1 we have that since β^2 is increasing and so is not decreasing E is a non decreasing function of X_1 and Y_1. From this we derive that:

$$W_2^2(z,x) \geq 0 = W_2^1(z,x) \tag{223}$$

With this we have proved that the hypotheses of the linkage principle are satisfied so that we can apply it and derive that the revenue in a SP auction is not lower than the revenue of a FP auction.
If the signals are independently distributed then $W^A(z,x)$ does not depend on x so that we have:

- $W_2^A(z,x) = 0$,
- $W_2^B(z,x) = 0$,

for any two auctions A and B. In this case from (219) we have that $\Delta(x)$ and $\Delta'(x)$ have opposite signs but, since $\Delta(0) = 0$, this is not possible so we must have $\Delta(x) = 0$ and therefore $W^A(x,x) = W^B(x,x)$ so the two auctions have the same revenue.

As we have already seen, the assumption of private values is unimportant for revenue equivalence since if the signals are independently distributed we have a revenue equivalence also with interdependent signals.
We now say something about the presence of **public information**.
With this wee mean the possibility that the seller may possess information that may be useful to the bidders. In these cases there is the problem of what the seller should do with this information and so if to keep it hidden or to reveal it and in this case if the revelation should occur in any case or only if it is favorable (so that the revelation occurs strategically).
In order to answer to these questions we have to modify the current model through the introduction of a further random variable S that denotes the information available to the seller. In this way the valuations of the bidders depend also on S and can be expressed as:

$$V_1 = v_i(S, X_1, X_2, \dots, X_N) \tag{224}$$

with $v_i(\mathbf{0}) = 0$. In the symmetric case we have:

$$v_i(S, \mathbf{X}) = u(S, X_i, \mathbf{X}_{-i}) \tag{225}$$

where u is any symmetric function of its last $N-1$ arguments. We assume that the variables S and X_i are affiliated and distributed according to a joint density function f that is a symmetric function of its last N arguments or of the signals of the bidders.
If the public information is not available the bidders do not know S before bidding so we have:

$$v(x, y) = E[V_1 | X_1 = x, Y_1 = y] \tag{226}$$

If, on the other hand, the seller reveals the public information in all the circumstances (and so in a non strategic way) the bidders know that $S = s$ before they bid so we have:

$$\hat{v}(s, x, y) = E[V_1 | S = s, X_1 = x, Y_1 = y] \tag{227}$$

with $\hat{v}(0, 0, 0) = 0$. Equation (227) defines the expectation of the value for the bidder 1 when:

- the public signal is s;

- the bidder receives a signal x;

- the highest signal among the other $N-1$ bidders is y.

From the symmetry hypothesis this function is the same for all the bidders and from affiliation it is an increasing function of its arguments. We note how it is:

$$\hat{v}(x, y) = E[\hat{v}(S, X_1, Y_1) | X_1 = x, Y_1 = y] \tag{228}$$

We now consider the effect of revealing the public information on the expected revenue of the seller in a FP auction.
In order to find a solution we can see the two situations:

- with public information,

- without public information,

as two different auctions. In this case, if the hypotheses of the linkage principle are satisfied, we can use it to see in which case the expected revenue is higher.
If the public information is available the strategy of a bidder depends on:

- the public information S,

- the signal of the bidder X.

We now can assume the existence of a symmetric equilibrium strategy $\hat{\beta}(S, X)$ increasing in both variables. The expected payment of a winning bidder when he gets the signal x but he bids as if his signal were z so he bids $\hat{\beta}(s, z)$ is:

$$\hat{W}^1(z, x) = E[\hat{\beta}(S, z)|X_1 = x] \tag{229}$$

From the fact that S and X_1 are affiliated and $\hat{\beta}$ is increasing (and so not decreasing) we have:

$$\hat{W}_2^1(z, x) \geq 0 \tag{230}$$

On the other hand when S is not available we have β^1 as the equilibrium strategy in a FP auction so that we have:

$$W^1(z, x) = \beta^1(z) \tag{231}$$

(since the winner pays his own bid) so that we have:

$$W_2^1(z, x) = 0 \tag{232}$$

From relations (230) and (232) we have:

$$\hat{W}_2^1(z, x) \geq W_2^1(z, x) \tag{233}$$

Since moreover we have $\hat{W}^1(0, 0) = W^1(0, 0)$ the hypotheses of the linkage principle are satisfied so that we can apply it and derive the conclusion that the expected revenue in a FP auction is higher when the public information is available than when it is not publicly available.
The linkage principle we have seen applies in cases where only the winning bidder pays a positive amount. We want to extend it to other auction types such as the "all pay" auctions where all the bidders pay their bid independently from being a winner or not.
In this case if we denote with $M^A(z, x)$ the expected payment of a bidder with a signal x who bids as if his signal were z in the auction mechanism A we have what follows.
In an "all pay" auction we have $M^{AP}(z, x) = \beta^{AP}(z)$ where $\beta^{AP}(z)$ is a symmetric and increasing equilibrium strategy. In an auction in which only the winner pays we have $M^A(z, x) = F_{Y_1}(z|x)W^A(z, x)$ so that in a FP auction we have $M^1(z, x) = F_{Y_1}(z|x)\beta^1(z)$ as the product of the probability of winning and the bid corresponding to a signal.
To deal with these cases we have the following proposition that represents an extension of the linkage principle.

Proposition 11.2 *If A and B are auctions were the highest bidder wins and each auction has a symmetric and increasing equilibrium such that:*

(1) $\forall x \; M_2^A(x,x) \geq M_2^B(z,x)$,

(2) $M^A(0,0) \geq M^B(0,0)$

the expected revenue of auction A is at least as large as the expected revenue of auction B.

We note that $M^A(z,x)$ can be seen as an unconditional expected payment whereas $W^A(z,x)$ is the expected payment conditional on winning.

12 Some additional and more informal notes

In this final section we use Klemperer (1999) to make some further and mainly qualitative comments on auction theory.
Auction theory is important for:

- practical,
- empirical,
- theoretical

reasons.
For **practical reasons** since the auctions have practical applications , for **empirical reasons** since they represent a test ground of economic theories and for **theoretical reasons** since they represent a tool for the development of general theories.
The **basic model** of auction theory is based on the hypotheses of a fixed set of symmetric and risk neutral bidders that bid independently one from the others and for a single object.
From this simple model we can define more complex models if we relax such assumptions singularly one at a time. With this we mean that we relax one of the hypotheses keeping the others as valid so that we can:

- introduce risk aversion,
- introduce correlation or affiliation among the information of the bidders,
- introduce asymmetries among the bidders.

We can also imagine that:

- the bidders may have a cost for entering the auction;

- the bidders may not know the exact number of the bidders each of them is facing;

- the bidders may collude among themselves;

- the bidders may have budget constraints.

12.1 The standard auction types

In this section we consider very briefly and within a rather informal framework the following types of auction:

(1) an ascending bid auction or open oral or English auction;

(2) a descending bid auction or Dutch auction;

(3) a $FPSB$ auction;

(4) a $SPSB$ auction.

In the case **(1)** the price is raised until a single bidder remains. That bidder wins the auction, gets the auctioned object and pays the final price of the auction or the price at which the penultimate bidder dropped out. The bidders gradually quit the auction while the price continuously raises and cannot reenter the auction at higher price nor they can make jump bid. This last feature derives from the fact that the bidders do not make themselves the offers but simply accept the offers made by the auctioneer. On the other hand if the bidders themselves make the price they could make jump bid if this is not explicitly forbidden by the rules of the auction.
In the case **(2)** the price starts high and lowers continuously until a bidder calls stop and get the object at that price.
In both these cases the information is public but in the latter it is of little use since when it is revealed the auction ends. This revelation has a great importance in the former case where the prices at which some of the bidders drop out give information to the remaining bidders on the value of the auctioned object.
In case **(3)** we have that the bidders:

- make independent sealed bids;

- the highest bidding bidder wins the auction;

- the highest bidding bidder pays a price equal to his bid.

From the last point we derive the fact that this type of auction is termed "first price".
The auctions of type **(4)** are identical to those of type **(3)** but for the fact that the paid price is the second highest bid whence the name "second price" we give to this type of auctions.

12.2 The basic model of auctions

In every auction model a basic feature is the presence of asymmetric information that turn in the adoption of the paradigm of games with incomplete information (Gibbons (1992)). In these cases each player is characterized by a type as a private information so that each knows his own type but has only a probabilistic assessment of the types of the other players/bidders. In these cases we can use the Bayesian-Nash equilibria where:

- the strategy of each player depends on his information;
- each player maximizes his expected payoff given the strategies of the other players and his beliefs (as probability distributions) about the information of other players.

Under these hypotheses we can have:

- a **private values model** where the value of the object is a private information of each bidder;
- a **interdependent values model** where the value of the object is unknown to each bidder at the time of the auction and may be affected by the information available to the other bidders;
- a **pure common values model** where the actual value is the same for all the bidders (ex post) but each bidder can have private information on the ex ante value of the object.

In the latter case the exchange of signals may have each bidder modify his valuation of the object in contrast with the private value case where this value is unaffected by such an exchange of the signals.
We can anyway define a general model where both the private value model and the pure common value model are seen as particular cases. In this general model we have that:

- each bidder receives a signal as his private information;

- each bidder's value is a general function of all the signals so that bidder i receives a signal t_i but is value is $v_i(t_1, \ldots, t_n)$ and so depends on the signals of all the other bidders (if they are known to him).

In this case we have a **private value model** if for each bidder i we have $v_i(t_1, \ldots, t_n) = v_i(t_i)$ and a **pure common value model** if we have $v_i(t_1, \ldots, t_n) = v_j(t_1, \ldots, t_n)$ for every set of signals $t_1, \ldots, t_n$ and for every pair of bidders i, j.

12.3 Bidding in the standard auctions

A standard auction is an auction of one of the types we have listed in section 12.1 or, in more general terms, it is an auction where the bidder who evaluates the most an object gets it.
If we consider the **descending auction** we have that each bidder must choose at which price to call himself out conditional on the object being still available. In this case that bidder is the winner of the auction and the price he pays is the highest price among the prices offered by the bidders.
We can easily see how such Dutch auction (DA) is strategically equivalent to a $FPSB$ auction so that the players' bidding strategies are the same in both cases. In this way we can state that $DA \equiv FPSB$ auction and that a DA can be said to be an open first price auction.
On the other hand if we consider the case of private values we have that:

- in an ascending or English auction for each bidder it is a dominant strategy to stay in the auction until the value reaches the bidder's value and then drop out;

- in this way the next to last bidder drops out when is value is reached and

- the bidder with the highest value (that would be the last one to drop out) wins he auction and pays a price corresponding to the value of the second highest bidder.

In this way, under the hypothesis of private values, an English auction is strategically equivalent to a $SPSB$ auction. We note that in a $SPSB$ auction an optimal strategy for a bidder is to bid one's own value for the object independently from the strategies of the other bidders. This means that **truth telling** is a dominant strategy so that the bidder with the highest value wins the auction and pays a price equal to the value of the second highest bidder (since all the bidders use the strategy of truthful bidding).
Let us now verify that truthful bidding is a weakly dominant strategy. For

the bidder i we assume a value v and suppose that the other highest bid is w. We can have two cases:

- **underbid** so that i bids $v - x$,
- **overbid** so that i bids $v + x$,

for some arbitrary value $x > 0$.
In the **underbid** case we can have the following cases:

- $v > v - x > w$ so that i wins and pays w as in the case he bids v;
- $v > w > v - x$ so i loses but could have won (by bidding v) with a surplus of $v - w$;
- $w > v > v - x$ so i loses as in the case he bids v.

In the **overbid** case we can have the following cases:

- $v + x > v > w$ so that i wins and pays w as in the case he bids v;
- $v + x > w > v$ so i wins but with a loss $v - w$;
- $w > v + x > v$ so i loses as in the case he bids v.

From these considerations we easily see how:

- bidding $v - x$ never causes a gain but may cause a loss;
- bidding $v + x$ never causes a gain and can even cause a loss;

so that the best strategy is bidding v.
In the case of private values (or in presence of only two bidders) we therefore have that *English auction≡SPSB auction* so that an English auction can be termed a second price open auction.
If, on the other hand, values are not private we have that in an English auction the players gain information on the values at which some of the other bidders drop out. This exchange of information cannot occur in a $SPSB$ auction so that the two formats cannot be equivalent in the cases where the values are either common or interdependent.
A key feature in auctions with common values components is the so called **winner's curse**. With this term we denote the fact that each bidder must recognize that in a symmetric equilibrium he wins only when he has the highest signal. From this we have that in presence of common values components (and so in presence of relations among the signals) if a bidder disregard the information coming from the other bidders' behavior he may pay more on

average than the proper value of the object.
From these premises we therefore can use the equivalences we have proved and speak of:

- first price auctions as comprising both $FPSB$ auctions and Dutch auctions;

- second price auctions as comprising both $SPSB$ auctions and ascending or English auctions.

We state that a model where the value for a bidder depends on some extent from the signals of the other bidders may be defined as a **common value model** so we have:

- **common values** or interdependent values;

- **pure common values** if the bidders' actual values are identical functions of the signals of the other bidders;

- **private values** if the actual value for a bidder depends only on [the realization of] his signal.

We underline how up to now we have dealt with the so called **normal auctions**. In these cases the auctioneer is a seller of an object whereas the bidders are his buyers so that:

- the object is transferred form the seller to one of the buyers:

- a sum of money is transferred from [one of] the buyers to the seller.

In this case the highest offering bidder wins the auction and gets the object.
Another type of auctions is represented by the **procurement auctions** where the auctioneer is a buyer whereas the bidders are the sellers and:

- a sum of money is transferred from the buyer to one of the sellers

- the object is transferred from one of the sellers to the buyer.

In this case the lowest offering seller/bider is the winner of the auction. In these notes, if not stated otherwise, we deal with normal auctions.
We note how we could reduce a procurement auction to a normal auction by reversing the flows and this requires that we speak of an exchange of negative money from the bidders to the auctioneer and an exchange of a "negative" object from the auctioneer to the bidders.

12.4 The basic result

If we have a certain number of risk neutral bidders with:

- privately known signals
- independently drawn from a common, strictly increasing and atomless (or continuous) distribution,

then any auction mechanism where:

- the object is allocated to the bidder with the highest signal,
- the bidder with the lowest feasible signal expects a null surplus,

gives the same expected revenue to the auctioneer since each bidder makes the same expected payment as a function of his private signal.
This result applies:

- in the private value model where the value of a bidder depends only on his signal;
- in the common value model if the bidders have independent signal.

On the ground of this result the four types of auctions we have seen in the foregoing sections yield the same expected revenue under the stated conditions and the same is true for other types of auctions such as the "all pay auctions".
In an **all pay auction**:

- the bidders submit sealed bids at once;
- the highest bidder wins the auction and gets the object;
- all the bidders pay their bid.

We note that "all pay auctions" are models of situations where all the involved players suffer a cost for carrying out an action but only one player (the winner) benefits (wins) from the positive effects of such actions. Typical examples are:

- lobbying competition;
- queues;
- legal battles;
- war of attrition.

The general result we have enunciated in the foregoing paragraphs is known as the **Revenue Equivalence Theorem** or **Principle** and to denote it we use the acronym REP.

12.5 The attitude towards risk

Up to now we have generally supposed to deal with **risk neutral bidders** so that they maximize their expected payoff.
Within the framework of normal auctions, we now widen our perspective and consider both the types of the auctioneer/seller A and the types of the bidders/buyers B. Such types are:

- risk averse ra,
- risk neutral rn,

and allow us to define the Table 1.

A vs B	rn	ra
rn	(0)	(1)
ra	(2)	(3)

Table 1: *The various attitudes towards risk*

The case **(0)** is a classical case where both the auctioneer and the bidders are risk neutral. If we start with considering the risk averse attitude of the bidders we can state that:

- in a SP auction risk aversion has no effect since it is still optimal strategy for the bidders to use truthful bidding;
- in a FP auction a slight increase in the bid increases slightly the probability of winning with a parallel reduction of the value of winning but this is desirable for a risk averse bidder if the current biding level were optimal for a risk neutral bidder.

From this we have that risk averse bidders bid more aggressively in FP auctions.
Summarizing we have:

- for risk neutral bidders SP and FP auctions are revenue equivalent;
- a risk neutral seller with risk averse bidders (case **(1)**) prefers a FP auction to a SP auction.

In the case **(2)** where a risk averse auctioneer faces risk neutral bidders (so that REP holds) we can state that in a SP auction the winner pays a price set up by the runner up and, by REP, must bid the expected value of this price if a FP auction. This means that, conditional on the winner's information:

- the price is fixed in a FP auction,

- the price is random but with the same mean in a SP auction,

so, if we disregard the winner's information, the price is riskier in a SP auction and therefore a risk averse seller has the following ranking:

$$FP \succ SP \succ ascending/English \tag{234}$$

where $\succ$ denotes a binary relation of strict preference.
Before going on we try to fix what we are doing. We started from a **basic auction format** that is grounded on a set of basic hypotheses and, in order to introduce and evaluate some possible extensions, we relax one hypothesis at a time but keeping all the remaining hypotheses valid.
We started with **risk neutrality** and relaxed it to introduce **risk aversion** (case **(3)**). We now consider the hypothesis that the private information of each bidder is independent from the private information of other bidders and relax it so to admit correlated or affiliated information.
With the term **affiliated information** we denote the fact that the signals of the bidders are affiliated so that a high value of one bidder's signal makes high values of the other bidders' signals more likely.
In case of affiliated signals we can state that:

$$English\,auction \succ SPSB \succ FPSB \tag{235}$$

where $\succ$ denotes a binary relation of "higher expected prices".

12.6 Correlation and affiliation

At this point we relax the hypothesis that each bidder's private information is independent from the information of the other bidders and, on the other hand (since we relax an hypothesis at a time), we revert to the assumptions that the bidders are risk neutral.
We therefore assume that the private information of the bidders is affiliated. In this case we have:

$$English\,auction \succ SPSB \succ FPSB \tag{236}$$

where $\succ$ denotes a binary relation of "higher expected prices".
This ranking derives from the fact that the surplus (the difference between the value and the paid price) of the winning bidder is due to his private information so the more the price depends on the information of the other bidders the more closely the price is related to the winner's information owing to the affiliation among such information.
We recall that:

- in an ascending auction with common value elements the price depends on the information of all the other bidders;

- in a $SPSB$ auction the price depends on the information of only one bidder.

As to the auctioneer side we note that if the auctioneer has access to private information he is better off by revealing it honestly. This is due to the linkage principle that states that the expected revenue is raised by linking the winner's payment to information that is affiliated with the winner's information.

12.7 Asymmetries

So far we have considered some of the basic hypotheses of the REP and we have relaxed one of them at a time so:

- we have relaxed risk neutrality as risk aversion,

- we have relaxed independent private information as correlated or affiliated information.

Another assumption is that the private values or signals of the bidders are drawn from a common distribution (the so called symmetry assumptions). We can now relax this assumption while keeping the others as valid.
We recall that the symmetry involves both the same shape and the same support so the possible asymmetries include:

- distributions with the same shape but with different support,

- distributions with [almost] the same support but with different shapes,

- almost common values.

12.8 Collusions

With the term collusion we denote an agreement that the bidders subscribe before the auction occurs so to the reduce the payment to the auctioneer. If the plan succeeds the object is allocated to one of the bidders at a lower price than in the case where the auction were conducted normally without any ex ante agreement. Once the object has been obtained at a lower price can be resold and the revenue be divided among the colluding bidders.
Here we only note that a collusive agreement may be easily subscribed in a $SPSB$ auction where:

- the designated winner bids a very high sum,

- all the other bidders bid 0,

- no bidder has an incentive for deviating unilaterally from this agreement.

In this case the object is allocated at a 0 price.
In a $FPSB$ auction, on the other hand, a possible collusive agreement could be the following: the designated winner may bid a small amount whereas all the others bid 0. Since such amount is known to all the other bidders the agreement is fragile. In this case indeed the bidders that should bid 0 have strong incentives to deviate and bid slightly more than the agreed on sum for the designated winning bidder in order to secure the object with a high surplus.

12.9 Other types of auctions

So far in these notes we have dealt with types of auctions involving the sale of a single indivisible object.
Other auction formats include the sale of multiple objects in cases where:

- bidders demand one object each,

- bidders demand more than one object each.

In the case of sales of multiple objects (that can be either homogeneous or heterogeneous) we can have the following cases:

- simultaneous auctions so that the objects are sold simultaneously,

- sequential auctions so the objects are sold sequentially and no buyer is interested in more than one object

We recall that in a standard auction the seller/auctioneer controls the mechanism whereas the buyers/bidders submit bids. A possible variant is represented by the **double auctions** where the buyers and the sellers are treated symmetrically and the buyers submits bids whereas the sellers submit asks.

12.10 Budget constraints

If the bidders face budget constraints the REP may fail. A budget constraint is an upper bound on the capability of a bidder to pay so if he binds himself to paying more than this constraint he must default and pay a penalty. To see why REP may fail we suppose it holds in the case where the bidders:

- have private independent values v_i,

- bidder i has a budget constraint b_i.

We underline how the b_i are independently drawn from a strictly increasing atomless (or continuous) distribution so also the x_i (see further on) correspond to independent draws from a strictly increasing atomless distribution. In a SP auction where truthful bidding is a dominant strategy bidder i bids as if he had a value $x_i = min(b_i, v_i)$ and no budget constraint.
By using REP therefore we have that the expected revenue is the same as the revenue in a FP auction where the bidders have value x_i and no budget constraint or in a FP auction where the bidders have values x_i and budget constraints x_i. This is an expected revenue lower than the expected revenue from a FP auction where the bidders have values $v_i \geq x_i$ and budget constraints $b_i \geq x_i$ so FP auctions are more profitable than SP auctions and REP (that states that the two formats are equivalent as profitability) fails to hold.

12.11 The Revenue Equivalence Principle

We now present the **Revenue Equivalence Principle** and give a proof from Klemperer (1999).
For these purposes we consider a model where the values are independent ad private and the bidders compete for a single object. In this model bidder i values the object v_i and it is common knowledge that each v_i is independently drawn from the same distribution $F(v)$ on the interval $[\underline{v}, \bar{v}]$ so that:

- $F(\underline{v}) = 0$,

- $F(\bar{v}) = 1$,

- the corresponding density function is $f(v)$,

- all the bidders are risk neutral.

Now we consider a generic mechanism for allocating the object to one of the bidders and denote with $S_i(v)$ the expected surplus that the bidder i obtains in equilibrium when his value is v. Within the current framework we have:

$$S_i(v) = vP_i(v) - E \tag{237}$$

where $P_i(v)$ is the probability for bidder i of winning the auction and so of getting the object at the equilibrium with a value v and E is the expected value of the payment by bidder i when his value is v.

We can compare $S_i(v)$ with the surplus that i with a value v deviates so to follow the strategy that the type $\tilde{v}$ is supposed to follow at the equilibrium. In this latter case i has the same surplus plus an additional surplus due to the difference between v and $\tilde{v}$ times the probability of winning by following the strategy of $\tilde{v}$. At the equilibrium we impose that type v must prefer not to deviate so we impose the following inequality:

$$S_i(v) \geq S_(\tilde{v}) + (v - \tilde{v})P_i(\tilde{v}) \tag{238}$$

If $\tilde{v} = v + dv$ so that $v - \tilde{v} = -dv$ we get the following equilibrium condition:

$$S_i(v) \geq S_(v + dv) - dvP_i(v + dv) \tag{239}$$

that expresses the preference of bidder i for type v over $v + dv$. If we impose the symmetrical preference for $v + dv$ over v we get:

$$S_i(v + dv) \geq S_(v) + dvP_i(v) \tag{240}$$

If we rearrange relations (239) and (240) we get:

$$P_i(v + dv) \geq \frac{S_i(v + dv) - S_i(v)}{dv} \geq P_i(v) \tag{241}$$

At this point if we take the limit as $dv \longrightarrow 0$ we get:

$$\frac{dS_i(v)}{dv} = P_i(v) \tag{242}$$

The differential equation (241) can be easily solved so to obtain:

$$S_i(v) = S_i(\underline{v}) + \int_{\underline{v}}^{v} P_i(x)dx \tag{243}$$

Relation (243) allows us to say that at any type $\hat{v}$ the slope of the surplus function is $P_i(\hat{v})$ so that if we know $S_i(\underline{v})$ we now the whole graph.
At this point we can consider two generic mechanisms with:

- the same $S_i(\underline{v})$,
- the same $P_i(v)$,

for every bidder i and every value v so that the have the same $S_i(v)$. From this we have that any type v of the bidder i makes the same expected payment in both mechanisms since $S_i(v) = vP_i(v) - E$ and the bidders are risk neutral. This means that the bidder i expected payment averaged across the possible types is the same for both mechanisms. Since this holds for all the bidders we have that the two mechanisms have the same expected revenues for the auctioneer.
We have:

- any mechanism that gives the auctioned object to the bidder with the highest value at the equilibrium (as all the standard auctions do) is characterized by $P_i(v) = F(v)^{n-1}$ since a bidder wins the auction only if all the others have lower values and in this way we evaluate the probability that this event occurs;

- many mechanisms, among which we include the standard auctions, give to the lowest feasible type no chance of surplus so that $S_i(\underline{v}) = 0$.

Under these hypotheses all these mechanisms yield the same expected payment by each bidder and yield the same expected revenue for the auctioneer. Since we have never used the hypothesis that the auctioned object is a single object we can extend the REP to k indivisible objects with the constraint of one object to each winning bidder. We can therefore state the following proposition.

Proposition 12.1 *In the cases where we have:*

- *n risk neutral potential buyers*

- *each with a privately known value independently drawn from a common distribution $F(v)$ that is strictly increasing and atomless (or continuous) over $[\underline{v}, \bar{v}]$ and*

- *no buyer wants more than one of the available k identical and indivisible objects*

then any auction mechanism where:

(1) the objects always go to the k buyers/bidders with the highest values,

(2) any bidder with value $\underline{v}$ expects 0 surplus,

yields the same expected revenue and results in a buyer with a value v making the same expected payment.

The result can be extended to the common and/or private value case in which each buyer i receives a signal t_i drawn from the interval $[\underline{t}, \bar{t}]$ and the value of each bidder $V_i(t_1, \ldots, t_n)$ depends on all the signals.
The REP main use is in the determination of the bidding strategies of one type of auction, that satisfies its hypotheses, through the comparison with another type of auction of which the strategy is known.
In an **ascending auction**the expected payment for a bidder with type v is given by:

$$P_i(v)E[Y_1|Y_1 < v] \tag{244}$$

where $P_i(v)$ is the probability of winning the auction and Y_1 is the highest of the remaining $n-1$ values. In this case we have:

(1) $G(v) = F(v)^{n-1}$ as the distribution of Y_1,

(2) $g(v) = (n-1)F(v)^{n-2}f(v)$ as the corresponding density function.

We can write:

$$E[Y_1|Y_1 < v] = \frac{\int_{\underline{v}}^{v} x(n-1)f(x)F(x)^{n-2}dx}{\int_{\underline{v}}^{v} (n-1)f(x)F(x)^{n-2}dx} \tag{245}$$

or, using integration by parts over the denominator:

$$E[Y_1|Y_1 < v] = \frac{xF(x)^{n-1}|_{\underline{v}}^{v} - \int_{\underline{v}}^{v} F(x)^{n-1}dx}{F(v)^{n-1}} \tag{246}$$

and, at last:

$$E[Y_1|Y_1 < v] = \frac{vF(v)^{n-1} - \int_{\underline{v}}^{v} F(x)^{n-1}dx}{F(v)^{n-1}} = v - \frac{\int_{\underline{v}}^{v} F(x)^{n-1}dx}{F(v)^{n-1}} \tag{247}$$

At this point we can write or an ascending auction:

$$P_i(v)E[Y_1|Y_1 < v] = P_i(v)[v - \frac{\int_{\underline{v}}^{v} F(x)^{n-1}dx}{F(v)^{n-1}}] = P_i(v)b(v) \tag{248}$$

where we have that:

- the last term on th right is the expected payment in a $FPSB$ auction where the winning bidder pays his own bid,

- the last equality on the right is imposed by REP that imposes the equality between the two expected payment so that we can derive the bidding strategy in a $FPSB$ auction as:

$$b(v) = v - \frac{\int_{\underline{v}}^{v} F(x)^{n-1}dx}{F(v)^{n-1}} \tag{249}$$

As another application of this procedure we can consider an "all pay" auction where:

- the highest bid gets the object,

- every bidder pays his own bid.

In this case we use REP (since all its hypotheses are satisfied) to impose:

$$b(v) = P_i(v)E[Y_1|Y_1 < v] \tag{250}$$

where the first member is the expected payment in an "all pay" auction and the second member contains the expected payment in an ascending auction as given by the first equality of relation (248). Since we have $P_i(v) = F(v)^{n-1}$ replacing all these quantities in relation (250) we get:

$$b(v) = vF(v)^{n-1} - \int_{\underline{v}}^{v} F(x)^{n-1}dx \tag{251}$$

as the bidding strategy for a bidder with value v in an "all pay" auction with other $n-1$ bidders that have the same common distribution function $F(v)$.

Appendix: three more auction types

Introductory remarks

In this closing section we briefly introduce three more types of auctions either for the allocation of an item with a negative value for both the bidders and the auctioneer or for the sharing of a cost among a certain number of players.
For the former purpose we introduce the following models:

- negative auctions (Cioni (2009)),
- candle auctions,

whereas for the latter purpose we examine the applicability of a sort of an "all pay auction" where the players/bidders compensate a given common cost through their bids.
This Appendix simply aims at presenting the three formats with their main features and so it is relatively short and written in a rather informal style.

Negative auctions

A **negative auction** is a type of auction for the allocation of a **chore** where the bidders bid for not getting it.
With the term **chore** we denote an item that the seller/auctioneer wishes to allocate to one of the bidders but that the bidders prefer to avoid getting so they wish to be paid (as a compensation) for accepting it. We can therefore

state that a chore has a negative value for both the auctioneer and the bidders.
The aim of the proposed mechanism is therefore the identification of one of the bidders as the **losing bidder** whereas the others are the winning bidders. The losing bidder gets the auctioned chore and a compensation from the winning bidders whereas each of the winning bidders has a gain from not having the chore allocated to himself but must pay a fraction of the compensation to the losing bidder.
From these premises we derive the motivations for the mechanism:

- the allocation of the chore to the less demanding bidder that is supposed to be the one who is less damaged from the allocation of the chore,
- the participation of the other bidders to the compensation of that bidder,
- all this happens without any involvement of the auctioneer that may even know very imperfectly the bidders and so cannot choose one of them in any direct way.

The allocation therefore occurs from the auctioneer A to one of the bidders from a set B on n elements. In this Appendix we present a simplified version of this type of auctions and so:

(1) without any fee,

(2) without any support among the bidders,

(3) without any support to the bidders from other actors that we call supporters.

The **fee** represents a sum of money that each of the selected bidders may pay so to be able to avoid attending the auction. In this way the collected fees represent an extra compensation for the losing bidder. Such fees are collected in a phase that precedes the auctioning phase and represent a private information of the auctioneer that is revealed to the losing bidder only after the allocation of the auctioned item and the payment of the proper compensation.
As to the point (2) we note what follows. In this simplified version we assume that the bidders are fully independent one from the others so that the damage received from one bidder from the allocation of the chore has no effect on the other bidders.
In real world cases this is not true since the allocation of the chore to one of the bidders may have side effects also on other bidders. These potentially damaged bidders therefore may act in two ways:

- each of them may promise some funds to his most preferred bidder (as the one from whom he receives the lowest damage) so to induce him to accept the chore;

- each of them may promise some funds to his less preferred bidder (as the one from whom he receives the highest damage) so to help him to avoid the allocation of the chore.

In both cases we have an additional phase before the proper auctioning phase where the bidders can exchange "promises of payments" among each other committing themselves to honor those payments under a penalty if they fail to do so.
As to the point (3) we note what follows. The bidders that are invited to the auctions are selected by the auctioneer in arbitrary ways. Once they have been selected and made public other actors may wish to participate in the auction in the ways we have already seen before.
Such actors are called **supporters** and decide to join the auction on voluntary basis without receiving any direct compensation but the gain they can get from the fact that the chore is allocated to one of the bidders from whom they receive a reduced damage.
Also in this case we have an extra phase that precedes the proper auctioning phase. In this way, in the most complex case, we may have up to five phases if we include also the compensation phase.
In both cases we plan to extend the basic model we are going to present here so to make it capable of representing more complex and more realistic situations such those we have listed before.
In all such cases the basic idea is simple: to take into account the interactions among different actors and the damages that each of them either receives from or causes to other actors owing to the allocation of the chore.
A different approach could be the following:

- to keep the auction simple without any additional phase,

- to allocate the chore and the corresponding compensation,

- to allow for local second level compensation among the losing bidder and his neighboring bidders that can turn also into a transfer of the allocated chore.

In this case we must introduce some stopping mechanisms so to prevent the procedure from cycling forever.
Coming back to the basic model we note what follows. We underline how A has the responsibility of choosing:

(a) the chore to be auctioned,

(b) the bidders of the set B that, in this simplified version, are forced to attend the auction.

For every bidder $b_i \in B$ we define the following quantities:

- the evaluation v_i of the chore;

- the bid x_i;

- the probability p_i of losing the auction and the complementary probability $q_i = 1 - p_i$ of winning it.

The value[11] $v_i \in [0, M]$ (for a suitable common positive value M) is a private information of each bidder and represents both the damage that b_i receives from the allocation of the chore and the missed damage deriving him from the fact that the chore has been allocated to another bidder.
The bid x_i is the realization of a random variable X_i, one for each bidder. The random variables X_i are assumed to be independent and identically distributed over $[0, M]$. The bid represents what each bidder claims as a compensation in the case he is the losing bidder and may define the fraction of the compensation he has to pay if he is one of the winning bidders.
At the end of the auction each bidder may be either the losing bidder or one of the winning bidders so that his payoff is:

$$u_i(x_i, v_i) = \begin{cases} x_i - v_i \; if \; x_i < X_j \; \forall j \neq i \\ v_i - c_i \; otherwise \end{cases} \tag{252}$$

where c_i is the fraction of the compensation to the losing bidder. Possible ties are resolved with a properly designed random device but are supposed to occur with a null probability.
We note that $x_i - v_i$ represents the difference between the received compensation and the damage whereas $v_i - c_i$ is the difference between the missed damage and the fraction of the compensation to be paid to the losing bidder.
From relation (252) it is possible to derive the expected gain for bidder b_i as:

$$e_i(x_i, v_i) = p_i(x_i - v_i) + (1 - p_i)(v_i - c_i) \tag{253}$$

where p_i is the probability for bidder b_i of losing the auction and $q_i = 1 - p_i$ is the probability of winning it. We can moreover state that the bid x_i depends on the value v_i according to a continuous function β such that:

[11]We use the term V_i to denote the corresponding random variable.

- $\beta(v_i) = x_i$,

- $x_i \leq \beta(M)$.

According to a traditional approach in auction theory (see Krishna (2002)), the aim would be to find a function β that maximizes the expected gain as expressed from relation (253) and then prove it is a strategy of equilibrium. In order to do this we have to characterize the expression for p_i and the value c_i.
For what concerns c_i we have the following two possibilities. In what follows we suppose that b_1 is the losing bidder who bid x_1 having a value v_1. This can be obtained simply renumbering the bidders when the auction is over.

(1) A constant share as:
$$\frac{x_1}{n-1} \tag{254}$$
where x_1 is the compensation requested by the losing bidder.

(2) A proportional share as:
$$x_1 \frac{x_i}{X} \tag{255}$$
where x_1 is the compensation requested by the losing bidder and $X = \sum_{k \neq 1} x_k$ are the bids of all the bidders but the losing one.

For what concerns p_i we note that we want to express it as a function of the values V_i and the searched for strategy β. We evaluate it by supposing that the random variables V_i are:

- independent,

- identically distributed so to have the same distribution function and therefore the same density function.

To evaluate p_i we use symmetry and consider one of the bidders, be it b_1. We have that b_1 loses the auction if and only if he makes the lowest bid or if and only if:
$$\forall i \neq 1 \; \beta(V_i) > x_1 \tag{256}$$
or:
$$\cap_{i \neq 1} (V_i > \beta^{-1}(x_1)) \tag{257}$$
so that (by using both independence and identical distribution) we can write:
$$P(\cap_{i \neq 1} V_i > \beta^{-1}(x_1)) = \Pi_{i \neq 1} P(V_i > \beta^{-1}(x_1)) = \Pi_{i \neq 1}[1 - P(V_i \leq \beta^{-1}(x_1))] \tag{258}$$

or:

$$P(\cap_{i\neq 1} V_i > \beta^{-1}(x_1)) = \Pi_{i\neq 1}[1-F(\beta^{-1}(x_1))] = [1-F(\beta^{-1}(x_1))]^{n-1} = G(\beta^{-1}(x_1)) \quad (259)$$

in this way we have defined p_i as a function of the searched for strategy β.
At this point we can rewrite relation (253) as:

$$e(x, v) = G(\beta^{-1}(x))(x - v) + (1 - G(\beta^{-1}(x)))(v - c) \quad (260)$$

where we generically denote with $x = \beta(v)$ the bid of a bidder, with $v = \beta^{-1}(x)$ the corresponding valuation of the chore from a bidder and with c the contribution to the losing bidder.
At this point we should impose on relation (260) a first order condition as:

$$\frac{de(x, v)}{dx} = 0 \quad (261)$$

In the present paper we follow a somewhat different approach. To introduce such approach we start with some preliminary considerations in the simple case of three bidders to extend it to the general case of n bidders but only in the case of uniform distributions.

Some preliminary considerations

In order to make some quantitative considerations and to define the best strategies for the players in this simple version of the negative auction mechanism we consider the simplest case of three players $i = 1, 2, 3$ each with an evaluation v_i of the chore and each making a bid x_i. If we focus on player 1 we can have the following cases:

(1) $x_1 < x_2 \leq x_3$

(2) $x_2 < x_1 \leq x_3$

(3) $x_2 < x_3 \leq x_1$

In the case (1) the bidder 1 loses and has a gain $x_1 - v_1$. If we impose it is non negative we have $x_1 \leq v_1$.
In the case (2) the bidder 1 wins so that he has to pay a sum defined as:

$$\frac{x_1}{x_1 + x_3} x_2 \leq \frac{x_2}{2} \quad (262)$$

since the losing bidder is the bidder 2. The last inequality derives easily from the fact that $x_1 \leq x_3$. In this case 1 has a gain given by:

$$v_1 - \frac{x_1}{x_1 + x_3} x_2 \geq v_1 - \frac{x_2}{2} > v_1 - \frac{x_1}{2} \quad (263)$$

since $x_1 > x_2$. From the last inequality of relation (263) we derive that 1 has a positive gain if the following inequality holds:

$$x_1 \leq 2v_1 \tag{264}$$

Also in the case (2) the bidder 1 wins so that he has to pay a sum defined as:

$$\frac{x_1}{x_1 + x_3} x_2 \leq \frac{x_1}{2x_2} x_2 = \frac{x_1}{2} \tag{265}$$

from the relations $x_2 < x_3 \leq x_1$. In this case 1 has a gain expressed as:

$$v_1 - \frac{x_1}{x_1 + x_3} x_2 > v_1 - \frac{x_1}{2} \tag{266}$$

Also in this case we derive that in order for 1 to have a positive gain the relation (264) must hold.
We have therefore derived that if 1 loses he gets $l_1 = x_1 - v_1$ (increasing with x_1) and if he wins he gets not less than:

$$w_1 = v_1 - \frac{x_1}{2} \tag{267}$$

(decreasing with x_1). At this point we can try to define the best strategy for bidder 1 in this particular case and in order to do that we can:

- evaluate for which value of x_1 we have $l_i = w_i$ (so to minimize the maximum loss);

- evaluate for which value of x_1 we have the maximum expected value of the gain for bidder 1.

According to the **minimize the maximum loss** approach we have:

$$x_1 - v_1 = v_1 - \frac{x_1}{2} \tag{268}$$

or:

$$\hat{x}_1 = \frac{4}{3} v_1 \tag{269}$$

to which it corresponds:

$$l_i = w_i = \frac{1}{3} v_1 \tag{270}$$

In this case the best strategy for 1 is to bid more than his evaluation of the chore so to get, in the worst case, the gain given by relation (270).
This result has been obtained under the hypothesis that losing and winning are events with the same probability of occurring.

At this point we can introduce probability considerations so to use to the **maximize the expected value of the gain** approach. For this aim we need to assess the value of the probability p_1.
In this simplified case we have:

$$p_1 = P[\{x_1 \leq x_2\} \cap \{x_1 \leq x_3\}] \tag{271}$$

If we consider x_2 and x_3 as random variables uniformly distributed on the interval $[0, M]$ for a suitable $M > 0$ we have:

$P[\{x_2 \leq x_1\}] = \frac{x_1}{M}$

$P[\{x_3 \leq x_1\}] = \frac{x_1}{M}$

so that, if we suppose they are also independent, we get:

$$p_1 = (1 - \frac{x_1}{M})^2 \tag{272}$$

In this way we may define the lower bound of the expected gain for bidder 1 as:

$$E(x_1) = (1 - \frac{x_1}{M})^2(x_1 - v_1) + [1 - (1 - \frac{x_1}{M})^2](v_1 - \frac{x_1}{2}) \tag{273}$$

From relation (273) we derive the following constraints on x_1:

- $x_1 \geq v_1$,
- $x_1 \leq 2v_1$,

since bidder 1 does not want negative gains either if he wins or if he loses.
Expression (273) can be rewritten as:

$$E(x_1) = p_1 \cdot (x_1 - v_1) + p_2 \cdot (v_1 - \frac{x_1}{2}) \tag{274}$$

where:

$p_1 = (1 - \frac{x_1}{M})^2$

$p_2 = 1 - (1 - \frac{x_1}{M})^2$

If we recall that $x_1 \in [v_1, 2v_1]$, in relation (274) we have the sum of two products and in each product we have, with regard to x_1:

- one decreasing term (either p_1 or $(v_1 - \frac{x_1}{2})$);
- one increasing term (either p_2 or $(x_1 - v_1)$).

We moreover have that p_1 and $p_2 = 1 - p_1$ have reciprocal behaviors as functions of x_1.
If we impose $p_1 = p_2$ we get:

$$p_1 = (1 - \frac{x_1}{M})^2 = \frac{1}{2} \tag{275}$$

or:

$$\bar{x}_1 = (1 - 2^{-1/2})M \tag{276}$$

From the expressions of p_1 and p_2 and from relation (276) we have that (given the constraints we have seen on x_1):

- if v_1 is lower than $\bar{x}_1/2$ (so that $x_1 < \bar{x}_1$) then p_1 dominates over p_2,
- if v_1 is higher $\bar{x}_1/2$ (so that $x_1 > \bar{x}_1$) then p_2 dominates over p_1.

In the former case bidder 1 must maximize the term $(x_1 - v_1)$ and so must bid $x_1 = 2v_1$ (given the constraints we have seen on x_1).
In the latter case bidder 1 must maximize the term $(v_1 - \frac{x_1}{2})$ and so must bid $x_1 = v_1$ (given the constraints we have seen on x_1).
Since these results have been obtained using M that is not known to the bidders we can argue that the result represented by relation (269) is a good suggestion for bidder 1 also in this probabilistic context.
In the next section we are going to generalize these arguments to the case of $n > 3$ bidders.

The possible strategies in the general case

In this section we examine the general case of n players but keeping the basic hypothesis we have made in section 12.11 for the distributions of the random variables (uniformity and independence). It is easy to verify how he relations we derive in this section are in accordance with what we derived for $n = 3$ in section 12.11.
In this case we focus on a generic bidder i that bids x_i and may either lose or win.
If i loses he gets a gain $x_i - v_i$ whereas if he wins he has to pay:

$$\frac{x_1}{x_1 + \cdots + x_i + \ldots x_n} x_i \leq \frac{x_1}{(n-1)x_1} x_i = \frac{x_i}{n-1} \tag{277}$$

since 1 is the losing bidder that bid $x_1 < x_j$ for any $j \neq 1$.
In this case i has a gain given by:

$$v_i - \frac{x_1}{x_1 + \cdots + x_i + \ldots x_n} x_i \geq v_i - \frac{x_i}{(n-1)} \tag{278}$$

From such expressions of the gain in the two cases we obtain the following constraints on x_i:

$x_i \geq v_i$

$x_i \leq (n-1)v_i$

since if they are both satisfied bidder i is sure to have a positive gain in every possible case.
If we follow the same approach that we used in section 12.11 we can impose:

$$x_i - v_i = v_i - \frac{x_i}{(n-1)} \tag{279}$$

In this way we get that i should bid:

$$x_i = \frac{2(n-1)}{n} v_i \tag{280}$$

with a benefit:

$$x_i - v_i = \frac{n-2}{n} v_i \tag{281}$$

It is easy to see how the higher is the number of the bidders n the more the bid tends to $2v_i$ whereas the benefit tends to v_i.
If we adopt the probabilistic approach we may define the lower bound of the expected gain for bidder i as:

$$E(x_i) = (1 - \frac{x_i}{M})^{(n-1)}(x_i - v_i) + [1 - (1 - \frac{x_i}{M})^{(n-1)}](v_i - \frac{x_i}{n-1}) \tag{282}$$

In this case we have:

$p_1 = (1 - \frac{x_i}{M})^{(n-1)}$

$p_2 = 1 - (1 - \frac{x_i}{M})^{(n-1)}$

If we impose $p_1 = p_2$ we get:

$$(1 - \frac{x_1}{M}) = 2^{-(n-1)} \tag{283}$$

or:

$$\bar{x}_i = (1 - 2^{-(n-1)})M \tag{284}$$

that tends to 0 as the number of the bidders n grows, independently from M.

In this case p_2 dominates over p_1 so that the best strategy for i is to maximize the term:

$$(v_i - \frac{x_i}{n-1}) \tag{285}$$

under the constraints $x_i \in [v_i, (n-1)v_i]$ so that the best strategy for i is to bid $x_i = v_i$ with a benefit given by:

$$v_i - \frac{v_i}{n-1} = \frac{n-2}{n-1}v_i \tag{286}$$

higher than the benefit expressed by relation (281).
In this case the strategy suggested by relation (280) may be seen as a conservative strategy to be adopted in cases where the number of the bidders n is not too high.

Candle auctions

Candle auctions have been used in the past as a variant of the English auction with a random termination time associated either to the going out of a candle or to the falling of a needle inserted in a random position in a burning candle.
We are planning to use such auctions for the allocation of a chore at one bidder from a set of bidders that have been selected by the auctioneer through a set of private criteria that do not depend on the willingness to attend of the single bidders.
The main motivation of this type of auctions is the following.
We have an actor that wants to allocate a chore to another actor to be chosen among a set of actors and the information he has about these actors are imprecise so that he cannot profitably choose one of them being sure to have chosen the best one. On the other hand he wants the selected actor to be compensated from the other actors for having been selected. For these reasons he can use the proposed auction based mechanism.
In this section we present the simplest form of a candle auction for these purposes whose basic ingredients are:

- an auctioneer A and a set B of n bidders b_i, $i = 1, \ldots, n$;
- a random integer $L > 0$ and a counter t that starts at 0 and stops at L;
- a fee f and a common pot P (initialized at 0) that is the compensation for the winning bidder;

- a random number generator that generates (according to an uniform distribution) an integer in the interval $[1, n]$ at each tick of the counter;
- a private value v_i that represents the damage that each bidder receives from the allocation of the chore.

At each tick of the counter every bidder is selected with a probability equal to $p_i = 1/n$ whereas the complementary probability is $q_i = (n-1)/n$. We note that successive selections are independent events and that if we consider every selection as a success we are dealing with a binomial distribution on each bidder.
We note how both the value of L and the entity of the fee f play an important role in the mechanism. If L is too small the probability that all the bidders refuse for the whole duration of the auction is high and the same is true if f is too small. On the other hand it is meaningless to have L too high so that at each step from one value of the counter on all the bidders accept. In this case the pot is no more incremented and the auction is a mere waste of time. The same may be true also if the fee f is fixed too high since in this case the bidders tend to accept too often and the content of the pot remains low.
The rules of the auctions are the following:

- at each tick of the counter a random integer i is generated and a bidder b_i is selected;
- the bidder b_i can either accept or refuse;
- if he refuses he adds a fee f to the common pot so that $P = P + f$
- if he accepts he qualifies as the current chore holder;
- when the counter expires the current chore holder wins the auction and gets both the chore and the content of the common pot P.

The counter is incremented of one unit at each acceptance or refusal and runs for $L + 1$ ticks (from 0 to L) and at $t = L$ it stops with no selection so that we have L useful ticks.
If all the bidders refuse at every tick at the end of the auction we have $P = Lf$ and the auctioneer can use this sum to allocate the chore to a further player not included in the set B.
From the rules of the auction we can expect an initial succession of refusals (when the content of the pot is low) followed by a succession of acceptances and refusals (as the content of the pot increases) to end with a succession where the acceptances are more than the refusals (when the content of the pot is big enough so to be higher than the value v_i for most of the bidders). From these premises we derive that:

(1) at the generic step $t = h \in [0, L]$ we can have $k \in [0, h]$ refusals so that $P = kf$;

(2) for each bidder b_i we can define with k_i the number of his refusals and with k_{-i} the number of the other bidders' refusals so that we have $k = k_{-i} + k_i$;

(3) at the generic step $t = h$ we have $t = h = k_{-i} + k_i + k_a$ where k_a is the total number of acceptances from all the bidders;

(4) at $t = L$ (when the counter stops) b_i may be the last accepting bidder so he has a gain only if $k_{-i}f > v_i$;

(5) at $t = L$ (when the counter stops) b_i is not the last accepting bidder so he has a gain only if $k_i f < v_i$;

(6) we can define the value $\hat{k} = int(\frac{v_i}{f})$ (where *int* is the integer part of the ratio) as the maximum number of refusals that b_i can make before having a loss in the case (5) since $(\hat{k}_i + 1)f > v_i$;

(7) at the generic step $t = h$, bidder b_i may be selected and can (according to strategies to be specified shortly) either accept or refuse (and in this case he pays the fee f).

We note that a bidder loses if he does not get the chore (and the pot) at the end of the auction otherwise he is said to win (and so receives both the chore and the content of the pot).
Before listing the possible strategies of each bidder b_i we note that from the relation $t = h = k_{-i}+k_i+k_a$ we cannot impose any constraint on the relation between k_{-i} and k_i since they are independent one from the other.
We underline the fact that if b_i wins he gets $P = kf = k_{-i}f + k_i f$ so he gets back what he paid during the auction so his real payoff depends only on k_{-i}.
On the other hand if b_i loses his real payoff depends on what he paid and so on k_i. In order to summarize the possible cases at the end of the auction (and so at $t = L$) we define the following Table 2.
From Table 2 (where $k_i f$ is what b_i has paid in the pot and $k_{-i}f$ is what has been paid from the other bidders) we have that at $t = L$:

- in case (1) we have $k_i f < v_i$ and $k_{-i}f > v_i$ so b_i is better off both if he loses and if he wins;

- in case (2) we have $k_i f > v_i$ and $k_{-i}f > v_i$ so b_i is better off only if he wins;

	$k_i f, v_i$	$k_{-i} f, v_i$
(1)	$<$	$>$
(2)	$>$	$>$
(3)	$<$	$<$
(4)	$>$	$<$

Table 2: *Possible relations between payments and evaluation*

- in case (3) we have $k_i f < v_i$ and $k_{-i} f < v_i$ so b_i is better off only if he loses;

- in case (4) we have $k_i f > v_i$ and $k_{-i} f < v_i$ so b_i is always worse off.

We can now present the possible strategies that a bidder b_i can use at each step $t = h$. We recall that the counter is incremented of one unit after each acceptance or refusal from the selected bidder but at $t = L$ when the auction ends.
We have the following strategies:

(1) **strong,**

(2) **weak**,

(3) **very weak**.

Each strategy is applied at the generic step $t \in [0, L-1]$. We remind that the following strategies must be considered within the framework where there is a random selection mechanism of the bidders and only the selected bidder can select one of the two actions **accept** r **refuse**.
In the **strong** strategy b_i accepts only if $k_{-i} f > v_i$ (hoping to be the last accepting bidder) and $(k_i + 1)f > v_i$ otherwise he refuses. We note that the sum $k_{-i} f$ may only increase so that the condition $k_{-i} f > v_i$ may only become more favorable to the bidder b_i.
With $(k_i + 1)f$ we denote the sum that b_i would end up paying by refusing once more. In this case we have that b_i accepts only if he has a gain either if he wins or if he loses.
In the **weak** strategy b_i accepts only if $k_{-i} f > v_i$ otherwise he refuses. In this case he hopes to be the last accepting bidder since this condition does not guarantee that also $k_i f < v_i$ is satisfied and there is no guarantee that b_i will be the last accepting bidder.
In the **very weak** strategy b_i accepts only if $(k_i + 1)f > v_i$ otherwise he refuses. In this case he hopes not to be the last accepting bidder since this

condition does not guarantee that also $k_{-i}f > v_i$ is satisfied and there is no guarantee that b_i will not be the last accepting bidder.
A common feature with these three strategies is the following: if bidder b_i (upon being selected) accepts at $t = h$ then he will be accepting at every $t > h$ (again upon being selected). This derives easily from the fact the the **accepting conditions** once they have been verified cannot be falsified by successive action of either the same bidder or of the other bidders.
The foregoing strategies can be used as follows:

- to define three types of bidders to be put into a competition among themselves in the same auction;

- to define one type of bidders where each bidder can follow one of such strategies depending on the current situation of the auction.

As to the second use, we recall indeed that each bidder b_i has a private value v_i so that each bidder has his own values for which the condition of acceptance is satisfied in each of the foregoing strategies.

Auctions for cost sharing

Given a set N of n players and a cost $C(N)$ there are many classical ways through which the cost can be shared among the members of N.
Among such methods we mention both the Shapley value and the nucleolus together with a bunch of methods that fall under the collective name of methods of the separable costs.
Such methods are based on the definition of the **separable cost** or individual cost of each player as the marginal contribution of each player to the cost of the grand coalition (the set N). The separable cost for player i is defined as:

$$m_i = C(N) - C(N \setminus \{i\}) \tag{287}$$

If we have $\sum_{i=1}^{n} m_i \leq C(N)$ we can define the **individual separable cost** as $\sum_{i=1}^{n} m_i$ and the **non separable cost** as the part of the cost $C(N)$ that cannot be compensated directly by the players or:

$$g(N) = C(N) - \sum_{i=1}^{n} m_i \tag{288}$$

It is possible to prove that if the core of the cost game associated to this problem of cost sharing is non empty then we have $g(N) \geq 0$.
In this case from the definition we have:

$$m_i = C(N) - C(N \setminus \{i\}) = \sum_{j \in N} x_j - C(N \setminus \{i\}) \tag{289}$$

where we have:

- $C(N) = \sum_{i \in N} x_i$,
- x_j is the fraction of the cost for the j-th player,
- from the definition of the core (see further on) we get $x(N \setminus \{i\}) \leq C(N \setminus \{i\})$ where $x(N \setminus \{i\}) = \sum_{j \in N \setminus \{i\}} x_j$.

In this way we get:

$$m_i = C(N) - C(N \setminus \{i\}) \leq \sum_{j \in N} x_j - \sum_{j \in N \setminus \{i\})} x_j = x_i \tag{290}$$

or:

$$m_i \leq x_i \tag{291}$$

From this we have that in the associated cost game with a non empty core (see the definition further on) the condition of individual rationality ($x_i \leq C(i)$) imposes that we have:

$$\sum_{i \in N} m_i \leq \sum_{i \in N} x(i) = C(N) \tag{292}$$

so that we have $g(N) \geq 0$.
The various methods that can be used to share the cost $C(N)$ depend on how the non separable cost $g(N)$ is shared among the n players since the portions m_i are allocated depending on the marginal contribution of each player as expressed by relation (287).
The classical methods include (Fragnelli (2008)):

- equal cost allocation or ECA,
- alternative cost allocation or ACA,
- cost gap allocation or CGA.

According to the ECA method the non separable cost $g(N)$ is equally shared among the players so each player has to pay a total cost defined as:

$$ECA_i = m_i + \frac{g(N)}{n} \tag{293}$$

A possible variation is represented by a proportional share where each player pays a fraction of the non separable cost determined by his individual separable cost. In this case every player's total payment is defined as:

$$PCA_i = m_i + \frac{m_i}{\sum_{j=1}^{n} m_j} g(N) \tag{294}$$

where P stands for **proportional**. If m_j is the same for all the players (or $m_j = m$ for every j) we obviously get relation (293).
In the ACA method the non separable cost is shared among the players proportionally to the saving of each player that is defined as:

$$r_i = C(i) - m_i \tag{295}$$

or as the difference between the individual cost $C(i)$ (that the player i suffers acting as a singleton) and the marginal contribution to the cost for the player i within the grand coalition N.
In this case from the definitions we again have:

$$m_i \leq x_i \tag{296}$$

From this we have that in the associated cost game with a non empty core (see the definition further on) the condition of individual rationality ($x_i \leq C(i)$) imposes that we have:

$$m_i \leq x_i \leq C(i) \tag{297}$$

so that we have $r_i \geq 0$.
With this definition we have that the total payment of each player is defined as:

$$ACA_i = m_i + \frac{r_i}{\sum_{j=1}^{n} r_j} g(N) \tag{298}$$

according to a proportional allocation of the non separable cost defined as a proportion of the individual savings.
Last but not least we introduce the method CGA. Also in this case we have to define a way to share the quantity $g(N)$ among the players. To do such a sharing we introduce the non separable cost of a coalition $S \subset N$ as:

$$g(S) = C(S) - \sum_{i \in S} m_i \tag{299}$$

Also in this case from the condition of non emptiness of the core for the associated cost game and from the condition $m_i \leq x_i$ we get:

$$\sum_{i \in S} m_i \leq \sum_{i \in S} x_i \leq C(S) \tag{300}$$

or:

$$g(S) = C(S) - \sum_{i \in S} m_i \geq 0 \tag{301}$$

At this point we consider the non separable cost of all the coalitions to which a player i can belong and, for each player, determine the minimum of such non separable costs as:

$$g_i = min\{g(S) \mid i \in S\} \quad (302)$$

Such values can be used to define the total payment of each player as:

$$CGA_i = m_i + \frac{g_i}{\sum_{j \in N} g_j} g(N) \quad (303)$$

again according to a scheme of proportional allocation.
All these methods, to be applied, require the availability of the values of the associated cost game and so essentially:

- $C(i)$ for every $i \in N$,
- $C(S)$ for every $S \subset N$,
- $C(N)$.

Such values are used for the evaluation of the values m_i and of the non separable cost $g(N)$ and so enter in all those methods.
Given a cost game we can define its core through to the following relations:

- $x_i \leq C(i)$,
- $\sum_{i \in S} x_i \leq C(S)$,
- $\sum_{i \in N} x_i = C(N)$.

where x_i is the cost suffered by each player from joining the grand coalition N with a cost $C(N)$. It is easily seen that if such relations are satisfied the core is not empty and defines a set to which all the mentioned solutions belong.
If the chore is not empty we have seen how we have:

- $g(S)\ geq0$,
- $g(N)\ geq0$
- $r_i \geq 0$

whereas there is no guarantee that this is true if the chore is empty.
On the other hand we can state that if the core is empty then one at least of such properties is violated and if one of such properties is violated then the core is empty.
Another feature of foregoing solutions is that they are Pareto optimal in the sense that switching from one solution to another one we have that some players are better off but at least one is worse off. This feature depends on the fact that the non separable cost amounts to $g(N)$ so that if we have that some player is better off we necessarily have that some other player must be worse off since the sum of what they pay is fixed.
This means that we cannot say that one of such solution is better than (or dominates) the others.
All the aforesaid methods are based on the idea of the sharing of the non separable cost according to a fixed rule defined in some way.
A possible alternative is to resort to an auction based mechanism defined as follows.
In this case the players know $C(N)$ (as the sum of a separable cost and a non separable cost), each of them has his marginal contribution $m_i = C(N) - C(N \setminus \{i\})$ so that what is left out to be allocated is the non separable cost $g(N) = C(N) - \sum_i m_i$.
We consider firstly how to allocate the non separable cost $g(N)$ among the various players/bidders.
The players know the entity of $g(N)$ and each of them submits a bid b_i in a sealed bid auction so that the total sum:

$$B = \sum_{i \in N} b_i \tag{304}$$

is collected.
In the case $b_i > 0$ for every i (so that $B > 0$) we can share the non separable cost among the players as:

$$g_i = \frac{b_i}{B} g(N) \tag{305}$$

so that the total payment of each player amounts to $m_i + g_i$.
If $B = 0$ (so $b_i = 0$ for every i) we can adopt the proportional solution we proposed before (see relation (294)). In this way, by using the marginal contributions, we get:

$$g_i = \frac{m_i}{\sum_{j=1}^{n} m_j} g(N) \tag{306}$$

so that also in this case each player pays $m_i + g_i$.
If $B > 0$ but $b_i = 0$ for some i we can adopt a mixed solution based on the definition of the following quantities:

- $B = \sum_{i\,:\,b_i>0} b_i$
- $M = \sum_{i\,:\,b_i=0} m_i$

In this case we have that:

- the players with $b_i = 0$ pay:

$$g_i = \frac{m_i}{M+B} g(N) \tag{307}$$

- the players with $b_i > 0$ pay:

$$g_i = \frac{b_i}{M+B} g(N) \tag{308}$$

In this way if $B = 0$ from relation (307) we derive relation (306). Also in this case each player pays $m_i + g_i$.
We now wish to verify if the strategy of biding $b_i = 0$ for every i is a Nash Equilibrium (NE) or not. To verify this we suppose that all the bidders follow that strategy but bidder 1 that bids $b_1 > 0$. If bidder 1 is worse off by deviating then the initial strategy is a NE otherwise he has an incentive for deviating and that strategy is not a NE.
If bidder 1 deviates and bids $b_1 > 0$ he pays (with $B = b_1$):

$$\frac{b_1}{M'+b_1} g(N) \tag{309}$$

instead of:

$$\frac{m_1}{M} g(N) \tag{310}$$

where we have:

- $M' = \sum_{i \neq 1} m_i$
- $M = \sum_i m_i = m_1 + M'$

In order to prove that the deviation is unworthy we have to prove that we have:

$$\frac{b_1}{M'+b_1} g(N) > \frac{m_1}{M} g(N) \tag{311}$$

or:

$$\frac{b_1}{M'+b_1} > \frac{m_1}{M'+m_1} \tag{312}$$

Relation (312) can be rewritten as:

$$b_1 M' + b_1 m_1 > m_1 M' + m_1 b_1 \tag{313}$$

This relation is true if $b_1 > m_1$ so that in this case bidder 1 is worse off by deviating. On the other hand if $b_1 < m_1$ such relation is false and so the bidder 1 is better off by deviating and bidding a small amount of money. In this way we proved that the strategy where all the bidders bid $b_i = 0$ is not a NE.
In similar ways we can verify if for each player bidding $b_i > 0$ is a NE or not. We can use the same approach by supposing that all the bidders but bidder 1 follow that strategy whereas bidder 1 either obeys (and bids $b_1 > 0$) or deviates unilaterally (and bids $b_1 = 0$).
In the former case he gets:

$$\frac{b_1}{B} g(N) \tag{314}$$

whereas in the latter he gets:

$$\frac{m_1}{B' + m_1} g(N) \tag{315}$$

with $B = B' + b_1$. We therefore have:

$$\frac{b_1}{B' + b_1} g(N) < \frac{m_1}{B' + m_1} g(N) \tag{316}$$

so that if $b_1 < m_1$ deviating is not fruitful and therefore the strategy of bidding $b_i > 0$ is a NE.
The next step is to adopt the same approach for the allocation of the whole cost $C(N)$ instead of the non separable cost $g(N)$.
In this case the players know the entity of $C(N)$ and each of them submits a bid b_i in a sealed bid auction so that the total sum:

$$B = \sum_{i \in N} b_i \tag{317}$$

is collected.
We can have three cases:

(1) $B > C(N)$,

(2) $B = C(N)$,

(3) $B < C(N)$.

In the case **(1)** we have a surplus given by $B - C(N)$that can be proportionally shared among the bidders as:

$$(B - C(N)) \frac{b_i}{B} \tag{318}$$

so that every bidders pays:

$$b_i - (B - C(N))\frac{b_i}{B} = \frac{C(N)}{B}b_i \tag{319}$$

In the case **(2)** we have neither a surplus nor a deficit to be shared among the bidders whereas in the case **(3)** we have a deficit given by $C(N) - B$ to be collected in some way among the bidders.
To understand in which ways this can be achieved we can try to define the NE in this cost game.
As a starting tentative hypothesis we can assume that the deficit is equally shared among the bidders as:

$$\frac{C(N) - B}{n} \tag{320}$$

We want to verify if:

(a) a generic strategy where the bidders bid $b_i > 0$ is a NE;

(b) a strategy where the bidders bid $b_i = 0$ is a NE.

In the case **(a)** we have that if the bidders follow that strategy each of them pays:

$$b_i + \frac{C(N) - B}{n} \tag{321}$$

We suppose that bidder 1 unilaterally deviates so to pay αb_1 with $0 \leq \alpha < 1$. In this case we have:

$$B' = \sum_{j \neq 1} b_j + \alpha b_1 = B + (\alpha - 1)b_1 \tag{322}$$

so that 1 pays:

$$\alpha b_1 + \frac{C(N) - B'}{n} \tag{323}$$

We want to verify if the individual deviation makes the deviating bidder worse off or if we have:

$$\alpha b_1 + \frac{C(N) - B'}{n} > b_1 + \frac{C(N) - B}{n} \tag{324}$$

or:

$$\alpha b_1 - \frac{\alpha - 1}{n}b_1 > b_1 \tag{325}$$

since from relation (321) we have:

$$C(N) - B' = C - B - (\alpha - 1)b_1 \tag{326}$$

From inequality (325) we derive $\alpha > 1$ in contradiction with the fact that we imposed $\alpha < 1$. This means that by deviating bidder 1 is better off and so the imposed strategy is not a NE.
As an example of this case we can verify that the strategy where each bidder bids:

$$b_i = \frac{C(N)}{n} \tag{327}$$

is not a NE. If we suppose that all the bidders follow that strategy but bidder 1 who bids:

$$\alpha\frac{C(N)}{n} \tag{328}$$

(with $\alpha < 1$) we have a deficit to be shared among that bidders that is equal to:

$$C(N) - B' = (1-\alpha)\frac{C(N)}{n} \tag{329}$$

so that b_1 pays:

$$\alpha\frac{C(N)}{n} - (1-\alpha)\frac{C(N)}{n^2} \tag{330}$$

If by deviating b_1 is worse off then the foregoing strategy (327) is a NE so we want to prove that the following inequality holds:

$$\alpha\frac{C(N)}{n} - (1-\alpha)\frac{C(N)}{n^2} > \frac{C(N)}{n} \tag{331}$$

from where we derive $\alpha > 1$ in contradiction with the hypothesis that $\alpha < 1$ so that by deviating b_1 is better off and the proposed strategy is not a NE.
If we examine the case **(b)** we have that by following that strategy (since $B = 0$) every bidder pays:

$$\frac{C(N)}{n} \tag{332}$$

We suppose that bidder 1 deviates and bids $b_1 > 0$ so that $B = b_1$. In this case bidder 1 pays:

$$b_1 + \frac{C(N) - b_1}{n-1} \tag{333}$$

Since we have:

$$b_1 + \frac{C(N) - b_1}{n-1} > \frac{C(N)}{n} \tag{334}$$

thanks to the fact that we have:

$$b_1 - \frac{b_1}{n-1} > 0 \tag{335}$$

and that we can easily prove that:

$$\frac{C(N)}{n-1} > \frac{C(N)}{n} \tag{336}$$

we have that by deviating bidder 1 is worse off so that the strategy $b_i = 0$ for all i is a NE.
From this we have that each bidder has a strong incentive to bid $b_i = 0$ in the auction so that each of them gets an equal share of the cost but this equal sharing makes this method of questionable interest unless we do not define a different way of sharing the cost $C(N)$ among the bidders that is different from the one defined by relation (320) (that turns into relation (332) if the bidders follow the equilibrium strategy).

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