# Auctions and barters

Lorenzo Cioni

**Department of Computer Science, University of Pisa**
largo Pontecorvo 3, 56127, Pisa, Italy
tel.: (+39) 050 2212741 fax: (+39) 050 2212726
e-mail: lcioni@di.unipi.it

**Abstract**

In this paper we face the problem of the fair sharing of **goods**, **bads** and possibly **services** (also collectively termed **items**) among a set of players that cannot (or do not want to) use a common cardinal scale for their evaluation owing to the very qualitative and non economical nature of the items themselves.
To solve this problem we present two families of protocols (barter protocols and auction protocols) and use a set of classical fairness criteria (mainly for barter protocols) and performance criteria (mainly for auction protocols) for their evaluation.
The protocols are either based on auctions mechanisms or on barter mechanisms and are presented in detail, discussed and evaluated using the suitable fairness and performance criteria.

*Keywords*: allocations, auctions, barters, fairness criteria, performance criteria

# 1 Introduction

In this paper we face the problem of the fair sharing of **goods**, **bads** and possibly **services** (also collectively termed **items**) among a set of players that cannot (or do not want to) use a common cardinal scale for their evaluation owing to the very qualitative and non economical nature of much of the items themselves.
To solve this problem we present two families of protocols (barter protocols and auction protocols) and use a set of classical fairness criteria (mainly for barter protocols) and performance criteria (mainly for auction protocols) for

their evaluation.
As to the **fairness criteria** (Brams and Taylor (1999) Brams and Taylor (1996)) we use envy-freeness, proportionality, equitability and [Pareto] efficiency with some modifications and adjustments in order to make them suitable for the new contexts.
The **performance criteria** that we use include: guaranteed success, [Pareto] efficiency, individual rationality, stability and simplicity.
As to the families of protocols we have a family $F_1$ of protocols that are based on auctions mechanisms and that can involve any number of players as an auctioneer and a set of bidders and a family $F_2$ of protocols that are based on barter mechanisms and that involve a pair of players at a time but can involve an arbitrary number of such pairs.
All these protocols are presented in detail, discussed and evaluated using the suitable fairness and performance criteria.
The paper closes with a section devoted some concluding remarks and to future research plans.

# 2 The family $F_1$

The family $F_1$ contains three types of auction mechanisms:

($a_1$) a sort of Dutch auction with negative prices/bids,

($a_2$) a sort of English auction with negative prices/bids,

($a_3$) a sort of first price auction with negative prices/bids.

In mechanism ($a_1$) the auctioneer tries to allocate a bad to one bidder by rising his offer up to a maximum value $M$ whereas in mechanism ($a_2$) the auctioneer starts with an offer $L$ and the bidders make lower and lower offerings until one of them wins the auction and gets the bad and the money. We call such mechanisms **positive auctions** since the bidders bid to get the auctioned item.
In mechanism ($a_3$) the bidders bid for not getting a bad[1] that is assigned to the losing bidder (the one who bid less than the others) together with a compensation from all the other bidders. We call such mechanism a **negative auction** since the bidders bid in order of not getting the auctioned chore.
Of each mechanism we provide a description and the best strategy. Once the mechanisms have been described we also prove how the first two mechanisms are really equivalent and define some relations between them and the last

[1] We use as a synonym also the term *chore*.

one. We also apply the performance criteria to such mechanisms for their evaluation and prove under which conditions they are satisfied.

# 3 The family $F_2$

The family $F_2$ contains two subfamilies of models that we present in their basic two players $A$ and $B$ version.
The former subfamily contains a set of **explicit barter models** whereas the latter subfamily contains an **implicit barter model** and a **mixed barter model**.
In the **explicit barter models** the players $A$ and $B$ show each other the set of items that each of them is willing to barter within a procedure that is characterized by either simultaneous or consecutive requests from one player to the other in which the barter may involve either a single item or a subset of items.
An explicit barter is an iterative procedure that may end either with a success (and so with an exchange of items) or with a failure but, at each step, may also involve a reduction of the items each player is willing to barter.
In the **implicit barter model** none of the players show his items to the other so that each player, in his turn, proposes to the other a pair of items $(i, j)$ that he is willing to barter so that the other may either accept or reply with a counter proposal. The barter ends when an agreement is reached or both agree to give up since they decide that no barter is possible. During the barter each player reveals to the other the items he is willing to barter and this can ease the reaching of an agreement.
Last but not least in the **mixed barter model** we have that one player (be it $A$) shows his items to $B$ that, on the other hand, behaves as in the implicit case. Also in this case the barter goes on as a series of proposals and counter proposals with an incremental definition of the bartering set of the player $B$.
The implicit barter model and the mixed barter model are classified, in the paper, as **iterative barter models**.

# 4 The classical criteria

In this section we recall the classical definitions of both the evaluation and the performance criteria as they are found in the literature. Such criteria will be specialized, whenever needed, for the various models we introduce in the paper.

## 4.1 The performance criteria

As **performance criteria** we use: **guaranteed success**, **individual rationality**, **simplicity** and **stability**.
With **guaranteed success** we denote the fact that a procedure is guaranteed to end with a success, with **individual rationality** we denote the fact that it is in the best interest of the players to adopt it so that they both use a procedure only if they wish to use it and can withdraw from it without any harm or a penalty greater than their potential damage.
**Simplicity** is a feature of the rules of a procedure that must be easy to understand and implement for the players without being too demanding in terms of rationality and computational capabilities.
Last but not least with **stability** we denote the availability to the players of equilibrium strategies that they can follow to attain stable outcomes in the sense that none of them has any interest in individually deviating from such strategies (Myerson (1991), Patrone (2006)).

## 4.2 The evaluation criteria

As **evaluation criteria** we use a set of classical criteria (Brams and Taylor (1999), Brams and Taylor (1996)) that allow us to verify if a barter can be termed **fair** or not.
Such criteria are:

- envy-freeness;
- proportionality;
- equitability;
- [Pareto] efficiency.

We say a barter is fair if they are all satisfied and is unfair if any of them is violated.
In the case of two players (Brams and Taylor (1999), Brams and Taylor (1996)) envy-freeness and proportionality are equivalent, as it will be shown shortly.
Generally speaking, we say that an agreement turns into an allocation of the items between the players that is **envy-free** if (Brams and Taylor (1996), Brams and Taylor (1999) and Young (1994)) none of the actors involved in that agreement would prefer somebody's else portion, how it derives to him from the agreement, to his own. If an agreement involves the sharing of benefits it is considered envy-free if none of the participants believes his share

to be lower than somebody's else share, whereas if it involves the share of burdens or chores it is considered envy-free if none of the participants believes his share to be greater that somebody's else share. In other words a procedure is envy-free if every player thinks to have received a portion that is at least tied for the biggest (goods or benefits) or for the lowest (burdens or chores).
If an allocation is envy-free then (Brams and Taylor (1999)) it is **proportional** (so that each of the $n$ players thinks to have received at least $1/n$ of the total value) but the converse is true only if $n = 2$ (as in our case). If $n = 2$ proportionality means that each player thinks he has received at least an half of the total value so he cannot envy the other. If $n > 2$ a player, even if he thinks he has received at least $1/n-$th of the value, may envy some other player if he thinks that player got a bigger share at the expense of some other player.
As to **equitability** in the case of two players (and therefore in our case) we say (according to Brams and Taylor (1999)) that an allocation is equitable if each player thinks he has received a portion that is worth the same in one's evaluation as the other's portion in the other's evaluation. It is easy to see how equitability is generally hard to ascertain (Brams and Taylor (1996) and Brams and Taylor (1999)) since it involves inter personal comparisons of utilities. In our context we tried to side step the problem by using a definition that considers both utilities with respect to the same player.
Last but not least, as to **[Pareto] efficiency**, we say (according to Brams and Taylor (1999)) that an allocation is efficient if there is no other allocation where one of the players is better off and none of them is worse off. In general terms efficiency may be incompatible with envy-freeness but in the case of two players where we have compatibility.

# 5 The positive auctions

## 5.1 Introduction

We present two types of **positive auctions** where a seller $A \notin \mathscr{B}$ offers a chore to a distinct set $\mathscr{B}$ of buyers/bidders so to sell/allocate it to one of them with a mechanism where the seller gives away the proper sum of money and the chore and the selected buyer accepts that sum of money and the chore.
As a seller $A$ wants to maximize his revenue so wishes to pay the lowest sum of money to allocate the chore to one $b_i \in \mathscr{B}$.
On the other hand each bidder/buyer $b_i \in \mathscr{B}$ wants to get the chore according

to each one's evaluation of it, evaluation that subjectively include losses and gains and that is a private information of each bidder.
Such mechanisms coincide with the usual mechanisms of buying and selling if we imagine negative payments so that the seller gives away the chore for a negative sum of money and the selected buyer accepts the chore but pays for it a negative sum of money.
For this reason we speak of a **positive auction** if the bidders bid for getting a chore in contrast with a **negative auction** mechanism where the bidders bid for not getting the chore that is allocated through the application of simple and common knowledge rules.

## 5.2 The auction mechanisms

We present two algorithms that can be used in all the cases where the auctioneer wants to "sell a chore" to the "worst offering" or to have a chore carried out by somebody else by paying him the least sum of money[2].
In the former mechanism the auctioneer offers a chore and a sum of money $m$ and raises the offer (up to an upper bound $M$) until when one of the bidders accepts it and gets both the chore and the money. The auction ends if either one of the bidders calls "stop" or if the auctioneer reaches $M$ without any of the bidders calling "stop". In the latter case we have a void auction sale. The auctioneer has a maximum value $M$ that he is willing to pay for having somebody else carry out the chore otherwise he can either give up with the chore or choose a higher value of $M$ so to repeat the auction with a different (new or wider) set of bidders.
This type of auction is a sort of **Dutch auction with negative bids** paid by the bidders to get the chore.
In the latter mechanism the auctioneer offers the chore and fixes a starting sum of money $L$. The bidders start making lower and lower bids. The bidder who bid less gets the chore and the money. Of course the auctioneer has no lower bound. Under the hypothesis that the bidders are not willing to pay for getting the chore we can suppose a lower bound $l = 0$. If this hypothesis is removed we can, at least theoretically, have $l = -\infty$. We can have a void auction sale if no bidder accepts the initial value $L$. The auctioneer can avoid

[2] Of course when an auction is over and a bidder has got the chore and the corresponding sum of money there is the risk that the chore is not carried out. The analysis and resolution of such problems is out of the scope of an auction mechanism. We can imagine the presence of binding agreements for the winning bidders that turn into either reinforcing rules or penalties. Among the reinforcing rules we can imagine a linkage between the payment and the degree of fulfillment of the chore with the full payment occurring only if and when the chore has been fully accomplished.

this by fixing a high enough value $L$. In this case the bidders are influenced by the value of $L$ that can act as a threshold since if it is too low none of them will be willing to bid. This case is as if the bidders start bidding from $-L$ and raise their bids up to $-l$ so that the one who bids the most gets the chore and pays that negative sum of money. In this case we have a sort of **English auction with negative bids**.

## 5.3 Dutch auction with negative bids

In this section we examine the mechanism[3] where the auctioneer offers the chore and a sum of money and raises the offer (up to an upper bound $M$) until when one of the bidders accepts it and gets both the chore and the money.
The auction we are describing is a sort of reversed Dutch auction where we have an increasing offer instead of a decreasing price and a chore instead of a good.
The value $M$ represents the maximum amount of money that the auctioneer is willing to pay to get the chore performed by one of the bidders. We note that the value $M$ is a private information of the auctioneer and is not known by the bidders. This fact prevents the formation of consortia and the collusion among bidders since $M$ may be not high enough to be gainful for more than one bidder (see also section 5.6).
If $x$ is the current offer of the auctioneer $A$ we can see $M - x$ as a measure of his utility.
As to the bidders $b_i$, each of them has the minimum sum he is willing to accept $m_i$ as his own private information so that $x - m_i$ may be seen as a measure of the utility of the bidder $b_i$.
We note that, if we define the set:

$$F = \{i \mid m_i \leq M\} \tag{1}$$

as the feasible set, the problem may have a solution only if $F \neq \emptyset$.
In this case the algorithm is the following:

1. $A$ starts the game with a starting offer $x_0 < M$;
2. bidders $b_i$ may either accept (by calling "stop") or refuse;
3. if one $b_i$ accepts[4] the auction is over, go to 5 ;

[3] We call it also the ascending mechanism or the ascending case.
[4] Possible ties may be resolved with a random device.

4. if none accepts and $x_i < M$ then $A$ rises the offer as $x_{i+1} = x_i + \delta$ with $0 < \delta < M - x_i$, go to 2 otherwise go to 5;

5. end.

The **best strategy for** $A$ is to use a very low value of $x_0$ (or $x_0 \simeq 0$) so to be sure to stay lower that the lowest $m_i$ and, at each step, to rise it of a small fraction $\delta$ with the rate of increment of $\delta$ decreasing the more $x$ approaches $M$. Though this strategy may indirectly reveal to the bidders the possible value of $M$ it is of a little harm to $A$ since in any case no bidder is willing to accept the chore for a value lower than his own value $m_i$.
The **bidder** $b_i$**'s best strategy** is to refuse any offer that is lower than $m_i$ and to accept when $x = m_i$ since if he refuses that price he risks to lose the auction in favor of another bidder who accepts that offer.
We have moreover to consider what incentives a bidder may have to act strategically when defining his value $m_i$. Of course there is no reason for $b_i$ to define a value of $m_i$ lower than the real one (since he has no interest in accepting lower prices). He could be tempted to define a higher value $m'_i > m_i$ so losing the auction in favor of all the bidders who are willing to accept any offer within the range $[m_i, m'_i]$. This means that $b_i$ may use a value higher than $m_i$ only if he is sure that the private values of all the other bidders are still higher. Since no bidder can be sure of this, each of them has a strong incentive to behave truthfully.
In this case, if $F \neq \emptyset$ (see relation (49)), the sum $A$ would expect to pay is equal to $m_j$ where $j \in F$ is such that $m_j < m_i$ for all $i \neq j$, $i \in F$. Of course $A$ does not know such a sum in advance since we are in a game of incomplete information and that value is revealed to $A$ only at the end of the game as an ex-post condition.

## 5.4 English auction with negative bids

In this section we examine the mechanism[5] where the **auctioneer** offers a chore and a starting amount of money $L$.
On their turn **the bidders** start making a succession of lower and lower bids until when a bid is not followed by a still lower bid: the bidder who made this last bid gets both the chore and his bid as a payment for the chore.
As to the **auctioneer** we note that the only parameter he can fix is the value $L$.
The auctioneer can choose $L$ so that it is the maximum amount of money he is willing to pay (see also section 5.5) but it is neither too low (since in this

[5]We call it also the descending mechanism or the descending case.

case the auction could be void) nor too high (since in this case it could also favor the rising of collusions among the bidders (see section 5.6).
As to the **bidders** we note that if each bidder has an evaluation $m_i$ of the chore as his private information his best strategy is to start bidding at any moment when the current value of the bids is grater than $m_i$, go on until the current descending price reaches the value $m_i$ and then stop.
We note, indeed, that if $x$ is the current value of the bids, the bidder $b_i$ has a net gain equal to $x - m_i$ that is positive for $x > m_i$, null for $x = m_i$ and negative for $x < m_i$ so that the least acceptable outcome is $x = m_i$ with a null net gain.

## 5.5 The equivalence of the mechanisms

We wish to verify if the two proposed mechanisms are equivalent or not with regard to the values of some parameters and the revenue for the auctioneer.
The first thing we do is a comparison between $M$ and $L$. We saw that $M$ is the maximum amount of money the auctioneer is willing to pay to sell the chore (see section 5) and the same role is played by $L$ so we can reasonably expect that $L = M$ is true.
We can reason as follows. We suppose to have the same chore and the same set of bidders in the two auction types we examine.
It cannot be $L > M$ otherwise $A$ would risk to pay in the descending case a sum greater than his maximum willingness to pay $M$ in the ascending case.
On the other hand it cannot be $L < M$ since $A$ in the ascending case would risk to pay a sum higher than the maximum sum he his willing to pay in the descending case.
From all this we see how it must be $L = M$.
We now examine the auction's revenue from the auctioneer/seller point of view.
If we suppose that each of the $n$ bidders $b_i$ has the evaluation $m_i$ of the chore we can easily see how the chore is allocated to the bidder $b_j$ where:

$$j = argmin\{m_i \mid i = 1, \ldots, n\} \tag{2}$$

and possible ties are resolved with the use of a properly designed random device. In both cases the revenue for the auctioneer is given by:

$$M - m_j = L - m_j \tag{3}$$

From this we can say that the two mechanisms are equivalent with respect to the seller/auctioneer.

Let us consider things from the bidders point of view.
From their point of view, though they may prefer a descending mechanism to an ascending one, things are equivalent since the chore is allocated to the bidder $b_j$ where $j$ satisfies relation (2).
We underline how in both mechanisms the bidders attend on a voluntary basis so that their individual evaluations $m_i$ represent how much each of them is willing to get to accept the chore. This implies that $m_i$ hides in itself both costs and gains for each $b_i$ from the chore but the relative importance and weight of costs and gains is a private information of each bidder.

## 5.6 The possible collusions

Up to now we have supposed that the bidders act one independently from the others. Now we examine the possible collusions $(c_1)$ among all the bidders and $(c_2)$ among the auctioneer and some of the bidders.
As to $(c_1)$ we start with an analysis of the collusions in the **descending case**. In this case, indeed, the bidders could agree that one of them (be it $b_j$) bids $L$, is compensated with $\hat{m}_j = max\{\frac{L}{n}, m_j\}$ and all the others share the resulting surplus among themselves.
This strategy is potentially fragile since $b_j$ may decide to keep the chore for himself with a net gain of $L - m_j$ since any violation of the binding agreement among the bidders can be hardly punished and every bidder $b_h$ whose $m_h$ is greater than the share of the surplus may have an incentive to deviate from that strategy.
Of course any coalition not including all the bidders (a limited coalition) is fragile since the excluded bidders are free to make their bids so to incentive the others to leave the coalition.
If a limited coalition includes $b_j$ (see section 5.5) its members may not be sure to get the chore but for a price equal to $m_j$ so that no share of a surplus is possible.
In the **ascending case** the bidders do not know the value of $M$ so collusions are more risky and less profitable.
A possible strategy could be that the bidders keep from bidding until when the price offered by $A$ reaches a minimal ex-ante agreed-on value $\hat{m}$. At this point one of them, be it $b_i$ who evaluates the chore as $m_i < \hat{m}$, accepts $\hat{m}$ and the chore so that the auction is over and $b_i$ gets the chore and the sum $m_i$ whereas the other $n - 1$ bidders share equally the surplus $\hat{m} - m_i$ among themselves.
The choice of $\hat{m}$ is risky since the bidders may agree on a value that is higher than $M$ so it is never reached in the auction. This risk may be minimized by reducing $\hat{m} > m_j$ (with $j$ defined with the rule (2)) so correspondingly

reducing the surplus deriving from the auction.
This strategy is, however, fragile since $b_i$ has strong incentives to deviate unilaterally from it and keep the chore for himself (since any violation of the binding agreement among the bidders can be hardly enforced) with a net gain of $\hat{m} - m_i$ and no surplus to be shared among the other bidders.
In order to maximize the surplus $\hat{m} - m_i$ the bidder $b_i$ should be $b_j$ with $j$ defined by relation (2). On the other hand, this choice maximizes also the temptation for $b_j$ to deviate unilaterally from that strategy (since he is the bidder who gains the most from such a deviation).
As to $(c_2)$ we have that both in in the ascending case and in the descending case we can hardly imagine the possibility of collusions between the auctioneer and [part of] the bidders owing to the nature of the proposed mechanisms and to the fact that the auctioneer pays for allocating the chore to one of the bidders that, in his turn, is paid for getting the chore.

### 5.7 Some possible applications

In this section we list some possible applications of the proposed auction mechanisms under the hypothesis of the **independent bidders**. In this case we can use the proposed mechanisms to define the allocation or **localized** or **point wise** chores to one of the bidders.
This is the case of incinerators, solid wastes disposal sites, chemical plants and the like. The main point is that the carrying out of the chore requires the assent of a single bidder. This is a rather gross simplification since the damages from the allocation of a chore are hardly confined to the single bidder but exert their influence also on "adjacent" bidders. We could try to account for the presence of collateral damages among bidders through a mechanism of the side effects whose description is out of the scope of the present paper.

## 6 The negative auction

### 6.1 Introduction

We present here a mechanism for the allocation of a chore from the auctioneer $A$ to one of the **bidders** from a distinct set of actors $\mathscr{B}$. Both the chore and the members of $\mathscr{B}$ are arbitrarily selected by $A$: the bidders $b_i$ bid for not getting the chore whence the denomination of **negative auctions**.
The final aim of such auctions is the transfer of the chore from the auctioneer to one of the bidders (the losing bidder) together with a monetary

compensation from the other bidders (the so called winning bidders).

## 6.2 The auction mechanism

The auction mechanism has the following features:

- a set of $n$ independent bidders $\mathscr{B}$;
- each bidder $b_i \in \mathscr{B}$ has an evaluation $m_i$ of the chore as his private value;
- the auctioneer $A$ can either fix or not a fee $f$ that can be paid by the bidders that want to escape the auction mechanism;
- the bidders may therefore decide either to pay the fee or not so that the set $\mathscr{B}$ reduces to a set $\hat{\mathscr{B}}$ (possibly with the same cardinality if no bidder pays the fee);
- the members of the set $\hat{\mathscr{B}}$ attend both the **auction phase** and the **allocation and compensation phase**.

The auction mechanism is characterized by the following simple **auction phase** rule: the bidders of $\hat{\mathscr{B}}$ submit their bids in a sealed bid one-shot auction and the bidder who submits the lowest bid (be it $b_1$ with evaluation $m_1$) loses the auction. Possible ties are resolved with the aid of a properly devised random device.
Once the losing bidder has been identified we have the **allocation and compensation phase**. Bidder $b_1$ gets the chore (allocation) and a compensation made by:

- the evaluation $m_1$ of the chore;
- the fees payment revenue $mf$ if $m$ bidders of $n$ decided to pay the fee.

Under the hypothesis of truthful bidding (so that each bidder's submitted bid $x_i$ is equal to $m_i$) the sum $m_1$ is paid to $b_1$ from each of the other winning bidders $b_i \in \hat{\mathscr{B}}$ with $i \neq 1$ as $e_i(m_i) = m_1 \frac{m_i}{\sum_{j \neq 1} m_j}$ for $i \neq 1$.
We note that the composition of the set $\mathscr{B}$ is a private information of $A$ whereas the composition of the set $\hat{\mathscr{B}}$ is made publicly known. In this way the bidders who attend the auction are not able to know the amount of extra compensation deriving from the paid fees.

## 6.3 The fee and its meaning

The fee has been introduced in the auction mechanism so to implement the principle of **individual rationality** since the bidders are chosen by the auctioneer at his will and do not attend the auction on voluntary basis.
The auctioneer therefore fixes a fee $f$ and must properly choose it in order for not having a void auction.
From this we can argue that at least we must have $f > 0$. If, moreover, $A$ can guess the values $\tilde{m}_i$ of the $m_i$ he can choose $f > min\, \tilde{m}_i$ so to force some of the less damaged bidders to attend the auction. Last but not least the higher $f$ is fixed the more all the members of $\mathscr{B}$ have a strong incentive to attend the auction hoping in a certain number of payments and therefore in a substantial additional compensation.
We note that fixing a fee $f = 0$ is different from not having a fee so as having a free ticket is different from no ticket: a fee $f = 0$ allows the bidders to escape the auction at no cost whereas if no fee is fixed all the bidders are forced to attend the auction and submit to its rules.
We recall that during the fee payment phase the bidders do not known either each others identities or their number so that the decision of either paying or not the fee is up to each bidder.
Let us consider what could happen if such hypothesis should prove false so that the bidders of $\mathscr{B}$ may guess how many bidders are willing to pay the fee and even may agree on some common strategies (see further on).
At the offset $A$ contacts the $n$ members of $\mathscr{B}$ and it may happen that $m$ decide to pay the fee whereas $k = n - m$ decide to attend the auction.
In this way the auctioneer collects a sum $mf$ to be given to the losing bidder as an extra compensation.
A critical case may occur if $m = n$ so that all the all the bidders of $\mathscr{B}$ pay the fee, $\hat{\mathscr{B}} = \emptyset$ and the auction is void. In this eventuality the fees are given back to the bidders, since no auction occurs, so that each of them has a null utility (since he pays $f$ but gets back $f$). It is easy to show that this all pay strategy where the bidders have 0 utility is fragile if there is at least on bidder $b_i$ such that $(n-1)f \geq m_i$ since the deviating bidder gets a payoff equal to $(n-1)f$ whereas all the other get a payoff equal to $-f$.

## 6.4 The compensation rule and the strategies

In section 6.2 we supposed that each bidder would truthfully bid $x_i = m_i$. In this way, assuming for simplicity that no bidder paid the fee, we have:

(1) the losing bidder $b_1$ gets a compensation equal to $m_1$ and coincident with the damage $m_1$ that derives him from the allocation of the chore;

(2) each winning bidder $b_i$ with $i \neq 1$ gets a gain deriving from the fact that he succeeded in not getting the chore (and so equal to $m_i$) but has to undergo a compensation rule and so has to pay a sum equal to:

$$e_i(m_i) = m_1 \frac{m_i}{\sum_{j \neq 1,i} m_j + m_i} \tag{4}$$

with $i \neq 1$.

If we denote with $p$ the probability for $b_i$ of winning the auction (and therefore $(1-p)$ is the probability of losing the auction) we have:

$$E_T = (1-p)(m_i - m_i) + p[m_i - e_i(m_i)] = p[m_i - e_i(m_i)] \tag{5}$$

where $E_T$ denotes the expected gain of the generic bidder $b_i$ in the case of truthful bidding $x_i = m_i$.
In a similar way we can denote as:

(1) $E_G$ the expected gain for $b_i$ from bidding $x_i > m_i$ with the corresponding probabilities of winning $(p' > p)$ and of losing $(1 - p' < 1 - p)$;

(2) $E_L$ the expected gain for $b_i$ from bidding $x_i < m_i$ with the corresponding probabilities of winning $(p' < p)$ and of losing $(1 - p' > 1 - p)$.

If we imagine the variables $x_j$ (with $j = 1, \ldots, k$) as independent random variables identically (possibly uniformly) distributed over the interval $[0, M]$ for a suitable value $M > 0$ we may state that:

(1) $p' \to 0$ if $x_i \to 0$;

(2) $p' = p$ if $x_i = m_i$;

(3) $p' \to 1$ if $x_i \to M$.

We have two cases of strategic bidding or:

(1) $x_i < m_i$;

(2) $x_i > m_i$.

For both cases we assume that only player $b_i$ deviates from the strategy of truthful bidding whereas all the others go on with that strategy.
In case **(1)** we have that, by bidding $x_i < m_i$, $b_i$ can either become or remain a loser with a **loss** equal to $m_i - x_i$ or can remain a winner with a gain that can be proved to be lower than his losses. In this way we have that, by bidding $x_i < m_i$, $b_i$ at the best has a gain that is lower than his losses so

that strategy is dominated by the one of truthful bidding and is surely not used by bidder $b_i$.
In case **(2)** we have that, by bidding $x_i > m_i$, $b_i$ can either remain a loser (with a **gain** equal to $x_i - m_i$) or became a winner (so to pay instead of being compensated) or remain a winner(so to pay a higher compensation to $b_1$). In this case **(2)** we note that:

- the probability of the first event is lower than the corresponding probability of $b_i$ being a loser by bidding $m_i$;
- the probability of the last event is higher than the corresponding probability of $b_i$ being a winner by bidding $m_i$;
- under the hypotheses we made on the variables $x_i$ we have that:

$$(1 - p')(x_i - m_i) \to 0 \qquad (6)$$

as $x_i \to M$.

In this way we may argue that the higher $b_i$ bids the more he risks having a loss. This is the main reason why bidding $x_i > m_i$ is a bad strategy for $b_i$.
Another reason may be found by leaving the individual deviation approach and using a Pareto-like reasoning.
We note first of all that compensations and payments are made on the basis of the declared bids $x_j$ and not on the basis of the private evaluations $m_j$.
This means that the ordering of the bidders $b_j$ according to their effective bids $x_j$ may differ from the ordering made according to their private evaluations $m_j$. Since, if one player $b_j$ has the incentive to deviate and so to bid $x_j > m_j$, this is true for all the players we have that we may have $m_i > m_1$ for some bidders $b_i$. In this case bidders $b_i$ would possibly pay a sum higher than their due share. In order to avoid this every bidder has one more reason for avoiding a bid $x_i > m_i$.

## 6.5 The properties and the applications

In order for the description of the basic mechanism to be completed we have to verify whether the proposed basic mechanism satisfies or not a minimal set of basic **performance criteria** and to describe the **possible applications** of the proposed basic mechanism.
The performance criteria that we use include: guaranteed success, [Pareto] efficiency, individual rationality, stability and simplicity.
The property of **guaranteed success**, from the auctioneer point of view, is satisfied whenever the auction is not void. This happens for sure if $A$ does

not fix any fee (and therefore the auction cannot be void and a losing bidder exists for sure) so conflicting with the requirement of **individual rationality** (see further on) but can occur also if he fixes a fee on condition that the value $f$ is fixed at the right level .
From the bidders point of view the property of **guaranteed success** is satisfied since the auctioned item is assigned to the bidder who makes the lowest bid and the others compensate him for this according to a simple rule of cost sharing.
As to **[Pareto] efficiency** we may state that the bidder with the lowest evaluation gets the chore and an equivalent compensation and all the other bidders pay a sum that is lower than the missed damage from not having the chore allocated to each of them so have a positive gain.
If the chore was allocated to another bidder with a higher evaluation all the winning bidders would pay a higher fraction of the compensation so they would be worse off. This is enough to say that the allocation to the lowest evaluation bidder is [Pareto] efficient.
The satisfaction of the property of **individual rationality** is guaranteed by the presence of the fee $f$ that allows some bidders to escape the auction by paying it.
As to **stability** we have argued in section 6.4 that for each bidder the truthful bidding is the best strategy. Though this arguing must be justified on more solid grounds we think that the intuition we have given should be enough to assure the satisfaction of this property.
Last but not least **simplicity** is assured by the fact that the rules of the auction are simple enough so to be implemented even by bidders with a bounded rationality.
As to the **applications** of the basic mechanism we mention here all those cases where a chore must be allocated from the auctioneer to one of the bidders from an equivalent (with respect to the chore) set of bidders.

# 7 Some remarks about the explicit barter models

## 7.1 The basic motivation

The basic motivation of the models we propose in section 8 is the need to describe how an exchange of goods can happen without the intervention of any transferable utility such that represented by money or by any other numerary good. In this way the involved actors do not need to share anything, such as preferences or utilities as shared information, but the will to propose

pool of goods that they present each other so to perform a barter.
All the barters are in kind and are essentially based on the following very simple basic scheme (see section 8.2): we have two actors that show each other the goods, each of them chooses one of the goods of the other and, if they both assent, they have a barter otherwise some rearrangement is needed and the process is repeated until either a barter occurs or both agree to give up.

## 7.2 Goods and chores as services

The key point of the proposed models is that each of the two players owns a set of items that enters it the barter process, $I$ for $A$ and $J$ for $B$.
In the paper we suppose both $I$ and $J$ contain goods or elements that have a positive value for both players. From this point of view a good may also be a service that one player is willing to perform on behalf of the other.
In this case, for instance, player $A$ asks to player $B$ for one of the available $B$'s services in exchange for one of the available $A$'s services that player $B$ asks to player $A$. Of course this occurs in the one-to-one barter case.
Another perspective sees the two sets $I$ and $J$ as containing chores or items that have a negative value for both players.
In this latter case the two players try to allocate each other their chores so that a chore allocated from $A$ to $B$ can be seen as a service performed by $B$ on behalf of $A$ to solve a problem of $A$. In this way we can unify the two perspectives and consider the goods case as a general case.

## 7.3 Some definitions

With the term **barter** we mean, see section 8, an exchange of goods for other goods without any involvement of money or any other numerary good.
We can have either a **one shot barter** or a repeated or **multi shot barter**.
In the **one shot case** the two actors execute the barter only once by using a potentially multi stage process that aims at a single exchange of goods and can involve a reduction of the sets of goods to be bartered.
In the **multi shot case** they repeatedly execute the barter process, every time either with a new set of goods or with a possibly partially renewed set of goods but usually excluding previously bartered goods.
In section 8 we are going to examine only the one shot barter between the two actors so that there is no possibility of retaliation owing to repetitions of the barter.
In order to avoid interpersonal comparisons and the use of a common scale we let the two players show each other their goods and ask separately to each of

them if he thinks the goods of the other are worth bartering. If both answer affirmatively we are sure that such interval exists otherwise we cannot be sure of its existence. Anyway the bartering process can go on, though with a lower possibility of a successful termination.
In this way we describe the absence of a common market (as a place where goods have a common and exogenously fixed evaluation in monetary terms) between the two players as well as the absence of any outer evaluator that can impose or even only suggest common evaluations according to a common numerary quantity to both players.

# 8 The explicit barter models

## 8.1 Introduction

We suppose the actor $A$ with his pool $I = \{i_1, \ldots, i_n\}$ of $n$ heterogeneous goods and the actor $B$ with her pool $J = \{j_1, \ldots, j_m\}$ of $m$ heterogeneous goods.
The sets $I$ and $J$ represent all the goods that both players are willing to barter on that occasion so that there is no "hidden good" that can be added at later stages. This is a design choice that qualifies the proposed models as models of **explicit barter**. If we imagine that the players have "hidden goods" that can be revealed and added to the sets at later stages we deal with what we may define an **implicit barter**. In the present section we deal only with barters of the former type.
In this case $A$ assigns a **private** (i.e. known only by him) vector $v_A$ of $n$ values to his goods of the set $I$, one value $v(i)$ for each good $i \in I$.
Also $B$ assigns a private vector $v_B$ of $m$ values to her goods of the set $J$. These vectors are fixed before the barter begins and cannot be modified during the barter. From these hypotheses, for any subset $K \subseteq I$, player $A$ once for all can evaluate, by using a property of additivity, the quantity:

$$v_A(K) = \sum_{i_k \in K} v_X(i_k) \tag{7}$$

A similar quantity may be independently evaluated by player $B$.
In a similar way we can define a private vector $s_A$ of $m$ values of the appraisals of the goods of $B$ from $A$ and a vector $s_B$ of $n$ values of the appraisals of the goods of $A$ from $B$. In this case $A$ can evaluate:

$$s_A(H) = \sum_{j_h \in H} s_X(j_h) \tag{8}$$

for any subset $H \subseteq J$. A similar quantity may be independently evaluated by player $B$.
These assignments reflect the basic hypotheses that $A$ can see the goods of $B$ but does not know $v_B$ (the values that $B$ assigns to her goods) and the same holds for $B$ with respect to $A$.
In this way we can define four types of barter:

1. **one-to-one** or one good for one good;

2. **one-to-many** or one good for a basket of goods;

3. **many-to-one** or a basket of goods for one good;

4. **many-to-many** or a basket of goods for a basket of goods.

The second and the third case are really two symmetric cases so they will be examined together in a single section.

## 8.2 One-to-one barter

Even in this simple type of barter there must be a pre-play agreement between the two actors that freely and independently agree that each other's goods are suitable for a one-to-one barter. The barter can occur either with **simultaneous** (or "blind") requests or with **sequential** requests.
In the case of **simultaneous requests**, at the moment of having a barter we can imagine that the two actors privately write the identifier of the desired good on a piece of paper and reveal such information at a fixed time after both choices have been made. In this case we have that $A$ requires $j \in J$ and $B$ requires $i \in I$ so that $A$ has a gain $s_A(j)$ but suffers a loss $v_A(i)$ and$B$ has a gain $s_B(i)$ but suffers a loss $v_B(j)$.
The two actors can, therefore, evaluate privately the two changes of value of their goods (that we may slightly improperly call **utilities**):

$$u_A(i, j) = s_A(j) - v_A(i) \tag{9}$$

$$u_B(i, j) = s_B(i) - v_B(j) \tag{10}$$

since all the necessary information is available to both actors after the two requests have been devised and revealed.
Equations such (9) and (10) are privately evaluated by each player that only declares **acceptance** or **refusal** of the barter, declaration that can be verified to be true by an independent third party upon request. We note that a possible strategy for both players is to maximize the value they get

from the barter (and so $s_A(j)$ and $s_B(i)$). Owing to the simultaneity of the requests this is not a guarantee for each player of maximizing his own utility since in equations (9) and (10) we have a loss due to what the other player asks for himself (and so $v_A(i)$ and $v_B(j)$) (see section 8.4).
The basic rule for $A$ is the following[6]:

$$\textbf{if}(u_A \geq 0)\ \textbf{then}\ accept_A\ \textbf{else}\ refuse_A \tag{11}$$

and a similar rule holds also for $B$.
We have therefore the following four cases:

1. both players accept, $accept_A$ and $accept_B$,
2. player $A$ refuses and $B$ accepts, $refuse_A$ and $accept_B$,
3. player $A$ accepts and $B$ refuses, $accept_A$ and $refuse_B$,
4. both players refuse, $refuse_A$ and $refuse_B$.

that we are going to describe in detail in section 8.3.
In the case of **sequential requests** we can imagine that there is a chance move (such as the toss of a fair coin) to choose who moves first and makes a public request. In this way both $A$ and $B$ have a probability of 0.5 to move first.
If $A$ moves first (the other case is symmetric) and requires $j \in J$, $B$ (since she knows her possible request $i \in I$) may evaluate her utility in advance using equation (10) whereas the same does not hold for $A$ that, when he makes the request, does not know the choice $i \in I$ of $B$ and so cannot evaluate $v_A(i)$.
At this level $B$ can either explicitly refuse (if $u_B < 0$) or implicitly accept (if $u_B \geq 0$).
In the refuse case $B$ can only take the good $j$ off her set so that the process restarts with a new deliberation of the possibility of the barter and a new chance move.
In the accept case the implicit acceptance is revealed by the fact that also $B$ makes a request. In this case $B$ may be tempted to chose $i \in I$ so to evaluate:

$$max\ u_B(i,j) = max\ (s_B(i) - v_B(j)) = max\ s_B(i) \tag{12}$$

where the quantity $v_B(j)$ is fixed (since it depends on the already expressed choice of $A$) and cannot be modified by $B$.
Acting in this way, $B$ may harm $A$ by causing $u_A < 0$ and this would prevent

[6] In the general case we have $u_A \geq \varepsilon$ with $\varepsilon > 0$ if there is a guaranteed minimum gain or with $\varepsilon < 0$ if there is an acceptable minimum loss.

the barter from occurring at this pass. Roughly speaking we can say that since $B$ choses after $A$ she can act accommodatingly or in an exploiting way: in the first case the probability that the barter occurs are higher than in the second case. Anyway $B$ makes a request of $i \in I$ so that also $A$ can evaluate his utility through equation (9).
Now, using rules such as (11), we may have the cases we have already seen but except for the case of double refusal since the case where who choses as the second refuses is handled at a different stage of the algorithm (see section 8.3).
All this goes on until both accepts so the barter occurs or one of them empties his set of goods or both decide to give up since no barter is possible, how it will be clear from the description that we are going to make in section 8.3.

## 8.3 Formalization of the models

In this section we present a concise but fairly detailed listing of the two models of the one-to-one barter, starting from the case of **simultaneous or "blind" requests**.
In this case the algorithm is based on the following steps:

(1) both $A$ and $B$ show each other their goods;

(2) both players decide if the barter is [still] possible or not;

  (a) if it is not possible then go to step (6);
  (b) if it is possible then continue;

(3) both simultaneously perform their choice (so $A$ chooses $j \in J$ and $B$ chooses $i \in I$);

(4) when the choices have been made and revealed both $A$ and $B$ can make an evaluation (using equations (9) and (10)) and say if each accepts or refuses (using rules such as (11));

(5) we can have one of the following cases:

  (a) if ($accept_A$ and $accept_B$) then go to step (6);
  (b) if ($refuse_A$ and $accept_B$) then \\at $A$'s full discretion
    i. either $A$ executes $I = I \setminus \{i\}$ and if ($I \neq \emptyset$) then go to step (2) else go to step (6);
    ii. or $A$ only executes a new choice and then go to step (4);
  (c) if ($accept_A$ and $refuse_B$) then \\at $B$'s full discretion

i. either $B$ executes $J = J \setminus \{j\}$ and if $(J \neq \emptyset)$ then go to step (2) else go to step (6);

ii. or $B$ only executes a new choice and then go to step (4);

(d) if $(refuse_A$ and $refuse_B)$ then

i. $A$ executes either $I = I \setminus \{i\}$ or a new choice; \\at $A$'s full discretion

ii. $B$ executes either $J = J \setminus \{j\}$ or a new choice; \\at $B$'s full discretion

iii. if (both $A$ and $B$ make a new choice) then go to (4);

iv. if (only one of $A$ and $B$ makes a new choice and the reduced set of the other is not empty) then

- if (the barter is still possible) then go to (4);
- if (the barter is not possible) then go to (6);

v. if (only one of $A$ and $B$ makes a new choice and the reduced set of the other is empty) then go to step (6);

vi. if (both reduce each one's set and $I \neq \emptyset$ and $J \neq \emptyset$) then go to step (2) else go to step (6);

(6) end of the barter.

The solution we have adopted at point (5)($d$) is the most flexible since it mixes the two cases (5)($b$) and (5)($c$) and gives the two players the full spectrum of possibilities at the same time remaining simple enough to be understood and implemented by the players.
We remark how at the very beginning of the process we suppose that the barter is possible though this is not necessarily true at successive interactions.
We now give the description of the model with **sequential requests**. We denote the player who moves first as 1 (it can be either $A$ or $B$) and the player who moves second as 2 (it can be either $B$ or $A$) and for both we use male forms. With a similar convention we denote as $I_1$ the set of goods and $i_1$ a single good of player 1 whereas for player 2 we have respectively $I_2$ and $i_2$:

(1) both players show each other their goods;

(2) both players decide if the barter is [still] possible or not;

(a) if it is not possible then go to step (10);

(b) if it is possible then continue;

(3) there is a chance move to decide who moves first and makes a choice;

(4) 1 reveals his choice $i_2 \in I_2$;

(5) 2 can now perform an evaluation of all his possibilities;

(6) if 2 refuses he takes $i_2$ off his barter set then go to (2);

(7) if 2 accepts he can reveal his choice $i_1 \in I_1$;

(8) both 1 and 2 can make an evaluation (using equations such as (9) and (10)) and say if each accepts or refuses (using rules such as (11));

(9) we can have one of the following cases:

   (a) if ($accept_1$ and $accept_2$) then go to step (10);

   (b) if ($refuse_1$ and $accept_2$) then \\at 1's full discretion

      i. either 1 performs $I_1 = I_1 \setminus \{i_1\}$ and if ($I_1 \neq \emptyset$) then go to step (2) else go to step (10);

      ii. or 1 only performs and reveals a new choice and then go to step (8);

   (c) if ($accept_1$ and $refuse_2$) then \\at 2's full discretion

      i. either 2 performs $J_1 = J_1 \setminus \{j_1\}$ and if ($J_1 \neq \emptyset$) then go to step (2) else go to step (10);

      ii. or 2 only performs and reveals a new choice and then go to step (8);

(10) end of the barter.

We note that the case (9.$c$) ($accept_1$ and $refuse_2$) can occur as a consequence of the case (9.$b$).

## 8.4 Possible strategies in the one-to-one barters

We now make some comments on the possible strategies that the players can adopt in the case of the algorithms we have shown in section 8.3.
In the case of **simultaneous requests** both players perform their choice without knowing the choice of the other. If they evaluate their utilities according to equations such as (9) and (10) their best strategy would seem to choose the good of the other that each value at the most.
In this case we have that:

$A$ requires $\hat{j} = argmax_{j \in J} s_A(j)$ and causes $B$ a loss that $A$ may only roughly estimate;

$B$ requires $\hat{i} = argmax_{i \in I} s_B(i)$ and causes $A$ a loss that $B$ may only roughly estimate.

Acting in this way each of them may have the other player to refuse the barter. As we have seen a refusal may turn into the withdrawal of a good from one of the sets $I$ or $J$. This fact is surely unfavorable for each player. Both players therefore have strong incentives to devise better strategies.
In what follows we introduce one possible strategy under the hypothesis the both players use a more slack rule than rule (11) so that acceptance or refusal are rather discretionary than linked to a condition satisfaction criterion.
We devise a strategy for player $A$ whereas for player $B$ we have two possibilities:

(1) $B$ follows a generic non systematic strategy,

(2) $B$ follows a similar strategy.

The strategy for $A$ is the following.
$A$ orders the set $J$ of $B$ in increasing order (from the lowest to the highest) according to the values he gives to its elements.
In the case **(1)** $B$ uses a generic strategy of selection whereas in the case **(2)** she uses an analogous strategy over the set $I$ of $A$.
The process of choice and request involves a certain number of pass until an agreement is reached either in a positive or in a negative sense. At the generic $l$-th pass (with $l = 1, \dots$) $A$ requires the current item of higher value $j_l \in J$ whereas $B$ chooses $i \in I$.
After the $l-$th choice from both $A$ and $B$ at pass $l$ we may have:

($a_1$) $A$ accepts so that everything depends on the decision taken by $B$,

($a_2$) $A$ refuses so that both goes at pass $l+1$-th.

In this way $A$ (but a similar argument holds also for $B$) scans the vector $J$ from higher to lower values goods looking for the right opportunity to perform a barter and having as the last choice the remaining good of lowest value. We recall indeed that at any pass both players may decide to prune their own sets of goods.
In the case of **sequential requests** we have that the two players make the choice one after the other according to an order that, at each step, depends on a random device. In this case, therefore, the players can adopt strategies similar to those we have seen for the simultaneous requests case but can try to exploit the advantage of being second mover.
Let us suppose we are at a generic step where $A$ moves as first and $B$ as

second. We consider $B$'s point of view but similar considerations hold also for $A$'s point of view. $B$ has ordered the goods of $I$ in increasing order of value. In this case we have:

- $A$ chooses $j \in J$ so that $B$ is able to evaluate $v_B(j)$
- $B$ can choose $i \in I$ so to get a high value of his utility $u_B(i, j) = s_B(i) - v_B(j)$ but
- without hurting $A$ since in that case $A$ could refuse the barter.

We recall that a refusal may turn into the pruning of a set and so in a unfavorable outcome for the requesting player that had requested the pruned good. From these considerations we derive that the step-by-step strategy that we have seen in the simultaneous requests case can be profitably used also in this case.
Similar strategies can be conceived, with the proper modifications and adaptations, for the other three models of barter that we are going to describe in the next two sections.

## 8.5 One-to-many and many-to-one barters

In these two symmetric cases one of the two actors has the possibility to require one good whereas the other has the possibility to require a basket of goods (that can even contain a single good) and so any subset of the goods offered by the former. This kind of barter must be agreed on by both actors and can occur only if one of the two actor agrees to be offering a pool of "light" goods whereas the other agrees to be offering a pool of "heavy" goods.
The meaning of the terms "light" and "heavy" may depend on the application and must be agreed on during a pre-barter phase by the actors themselves. We remark how the adopted perspective (lack of any quantitative common scale) turns into qualitative evaluations of the goods so that they are termed **light** if they are assigned **qualitatively low values** whereas they are termed **heavy** if they are assigned **qualitatively high values**.
The aim of this preliminary phase is to give one of the two actors the possibility of asking for any set of goods whereas this same possibility is denied to the other. If there is no agreement during this phase, three possibilities are left: they may decide either to give up (so the bather process neither starts) or to switch to a one-to-one barter (see section 8.2) or to a many-to-many barter (see section 8.6).
If there is a pre-barter agreement we may have two symmetrical cases. In

this section we are going to examine only the "one-to-many" case. In this case we have that $A$ owns "light" goods and may require only a single good $j \in J$ but $B$ owns "heavy" goods and may require (at her free choice) a subset $\hat{I}_0 \subseteq I$ of goods with $|\hat{I}_0| \leq n$ and the two requests may be either simultaneous or sequential.
In the case of **simultaneous requests** both actors can evaluate their respective utilities, soon after the requests have been revealed, by using equivalent relations to (9) and (10):

1. $u_A(\hat{I}_0, j) = s_A(j) - v_A(\hat{I}_0)$
2. $u_B(\hat{I}_0, j) = s_B(\hat{I}_0) - v_B(j)$

where both players use equations like (7) and (8) and the additivity hypothesis.
Also in this case we can have the four cases we have seen in section 8.2. We note, however, how in this case if, for instance, $A$ refuses, using a rule such as (11), he can either repeat his request (with $B$ keeping fixed her request ) or can act as we are going to show in section 8.6. In the latter case indeed $A$ can partitions his goods in subsets that he is willing to barter, possibly updating these subsets at every refusal. Except for this fact the barter goes on as in the $one - to - one$ case with simultaneous requests.
In the case of **sequential requests** the procedure does not use a chance move to assign one of the two actors the right to move first but gives this right to the actor that owns the pool of "light" goods. After this first move the barter goes on as in the $one - to - many$ case with sequential requests but without any chance move and with the modification we have introduced for the case of the refusal (see section 8.6).

## 8.6 Many-to-many barter

In this case $A$ may choose and require any subset $\hat{J}_0 \subseteq J$ with $1 \leq |\hat{J}_0| \leq m$ of the goods of $B$ whereas $B$ may chose and require any subset $\hat{I}_0 \subseteq I$ with $1 \leq |\hat{I}_0| \leq n$ of the goods of $A$ and the two requests may be either **simultaneous** or **sequential**.
Also this kind of barter must be agreed on by both actors in a pre-barter phase during which they both agree that in the course of the barter each of them can ask for a subset of the goods of the other player.
Since also in this case we can have either simultaneous or sequential requests the algorithms are basically the same that in cases of one-to-one barter. The main differences are about the use of the subsets and the way in which every case of refusal is managed

As to the point **(1)** we note that in the algorithms we must replace single elements with subsets of the pool of goods so that the evaluations must be performed on such subsets by using equations (7) and (8) and so the additivity hypothesis.
As to the point **(2)** in the algorithms for the one-to-one barter the solution we adopted was the possible pruning of the set of the goods from the refusing actor (see the points 5 or 9 ($b$), ($c$) and ($d$) of the algorithms of section 8.3). This solution cannot be applied in the present case since this policy could empty one of the two initial pools or both in a few steps. To get a solution in this case we can devise an independent partitioning strategy of the two sets of goods from both actors $A$ and $B$.
In this case at the very start of the barter the two players show each other their sets of goods so to hide their preferences that are partially revealed only after each refusal. After every (possibly double) refusal the player who refuses (be it $A$) uses the procedure $partitioning_A(I)$ to split $I$ in labeled disjoint subsets so to make clear to $B$ which are the subsets of goods that he is inclined to barter at that stage. The case of $B$ is fully symmetric. We note that under the additivity hypothesis the sets $I$ and $J$ can be partitioned at will by their respective owner.
This solution is implemented by replacing all the occurrences of the assignment instructions $I = I \setminus \{i\}$ and $J = J \setminus \{j\}$ respectively with the following assignments:

$$I = partitioning_A(I) = \{I_i \mid \cup_i I_i = I \; I_i \cap I_j = \emptyset \, \forall \, i \neq j\} \tag{13}$$

$$J = partitioning_B(J) = \{J_i \mid \cup_i J_i = J \; J_i \cap J_j = \emptyset \, \forall \, i \neq j\} \tag{14}$$

so to replace a flat set with a set of disjoint labeled subsets.
In this case, referring to $A$, we have that if $A$ refuses the barter proposed by $B$ he can either repeat his request with $B$ keeping fixed her request or he can partition his set in subsets as collective goods that he is willing to barter with subsets of the goods of $B$. The case of $B$ is fully symmetrical.
We recall how the barter in this case may evolve as follows. At the very start the two players propose each other their sets of goods. Then we can have the following cases. (1) Both players make a request and both accept. In this easy case the barter is successful and ends. (2) Both players make a request but one accepts whereas the other refuses. The refusing player has the possibility to rearrange his set of goods. This rearrangement is a partitioning of the player's set of goods according to the rules (13) or (14) so that the other player, at the next step, knows which are the subsets that can enter successfully into a barter.
(3) Both players make a request and both refuses. The rearrangement is

performed by both players at the same time.
For further details we refer to section 8.3.

## 8.7 The basic criteria

In this section we refer to the criteria we introduced in section 4.
Such criteria, in order to be used in our context of two players without either any common scale or any numerary good, must be adapted or must be redefined someway so to be in agreement either with the essence of their classical definitions or with intuition or with both. In what follows we are going to make use of a general notation that must be specialized in the single models we have already presented in the proper past sections.
We start with **envy-freeness**. If we denote with[7,8] $a_A(\cdot)$ and $l_A(\cdot)$ the values in $A$'s opinion and evaluation, respectively, of what $A$ obtains and loses from the barter (and with $a_B(\cdot)$ and $l_B(\cdot)$ the same quantities for player $B$) we say that the allocation deriving from a barter (or a barter tout court) is **envy-free** if we have for $A$:

$$\frac{a_A(\cdot)}{l_A(\cdot)} \geq 1 \tag{15}$$

and for $B$:

$$\frac{a_B(\cdot)}{l_B(\cdot)} \geq 1 \tag{16}$$

As we have already seen from section 8.1 on, if a barter actually occurs it is guaranteed to be envy-free. Relation (15) means that the value that $A$ assigns to what he gets from the barter is at least equal to the value that $A$ assigns to what he loses from the barter. We assign a similar meaning to relation (16) with regard to $B$.
Since, in the case of two players, we want to maintain the equivalence between proportionality and envy-freeness we must give a definition that mirrors the classical definition of proportionality and reflects this equivalence.

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[7]With $\cdot$ we denote a generic set of bartered goods. This set may contain also a single element.

[8]In the one-to-one barter model, for instance, we have that:

1. $a_A(\cdot) = s_A(j)$
2. $l_A(\cdot) = v_A(i)$
3. $a_A(\cdot) = s_B(i)$
4. $l_B(\cdot) = v_B(j)$

whereas in the other cases the single elements must be replaced by the properly defined subsets.

For player $A$ we may define a barter as proportional if it satisfies the following condition:

$$\frac{a_A(\cdot)}{a_A(\cdot)+l_A(\cdot)} \geq \frac{l_A(\cdot)}{a_A(\cdot)+l_A(\cdot)} \tag{17}$$

so that the fractional value of what $A$ gets from the barter is at least equal to that of what he loses from it. We remark that $a_A(\cdot)+l_A(\cdot)$ represents the value that $A$ assigns to the bartered goods.
A similar condition holds also for $B$:

$$\frac{a_B(\cdot)}{a_B(\cdot)+l_B(\cdot)} \geq \frac{l_B(\cdot)}{a_B(\cdot)+l_B(\cdot)} \tag{18}$$

We say that a barter is proportional if both (17) and (18) hold.
It is easy to see how from equation (17) it is possible to derive equation (15) and vice versa. The same holds also for equations (18) and (16).
As to **equitability** we must adapt its definition to our framework in the following way. We need firstly some definitions. We define (with respect to the occurrence of the barter itself) $I$ and $I'$, respectively, as the ex-ante and ex-post sets of goods of $A$ and $J$ and $J'$, respectively, as the ex-ante and ex-post sets of goods of $B$. If $(i, j)$ denotes the bartered goods ($i$ from $A$ to $B$ and $j$ from $B$ to $A$) in a one-to-one barter, we have:

$$I' = I \setminus \{i\} \cup \{j\} \tag{19}$$

$$J' = J \setminus \{j\} \cup \{i\} \tag{20}$$

In the case of other kind of barters involving also subsets of goods we must appropriately replace single goods with subsets.
On the sets $I'$ and $J'$ we define, for the player $A$, the quantities that represent the values for $A$ himself, after the bater, of his goods and $B$'s goods, respectively, as $a_A(I')$ and $l_A(J')$. We therefore define a barter as **equitable** for $A$ if the fractional value of what he gets is at least equal to the fractional value he gives to what he loses from the barter or:

$$\frac{a_A(j)}{a_A(I')} \geq \frac{l_A(i)}{a_A(I)} \tag{21}$$

On the other hand the barter is equitable for $B$ if, using the corresponding quantities we used in equation (21) but referred to player $B$, we have:

$$\frac{a_B(i)}{a_B(J')} \geq \frac{l_B(j)}{a_B(J)} \tag{22}$$

If both relations hold we say that the barter is **equitable**. We remark that we are under an additivity hypothesis where the value of a set is given by

the sum of the values of its elements so that the value that a player assigns to a set, such as $I'$ or $J'$, is the sum of the values that the player assigns to the elements of that set.
As to **efficiency** we say that a barter of the two goods $(i, j)$ (or of the one-to-one type) is efficient if there is not another pair of goods $(i', j')$ that gives at least to one player a better result without hurting the other.
For players $A$ and $B$ this means that there is no barter $(i', j')$ that satisfies the following inequalities:

$$\frac{a_A(j)}{l_A(i)} \leq \frac{a_A(j')}{l_A(i')} \tag{23}$$

$$\frac{a_B(i)}{l_B(j)} \leq \frac{a_B(i')}{l_B(j')} \tag{24}$$

with at least one of them satisfied with the $<$ relation.
In such relations the pairs $l_A(i)$, $a_A(j)$ and and $l_A(i')$, $a_A(j')$ are related to $A$ and are associated respectively to $(i, j)$ and to $(i', j')$. Similar quantities are defined also for player $B$.
We remark how we are under the hypothesis that at least one of the following inequalities hold:

1. $i' \neq i$
2. $j' \neq j$

Also in this case if the barter involves subsets of goods such relations must be modified by replacing single goods with properly defined subsets of goods.
We note that if the barter is such that both players attain:

$$\frac{a_{A_{max}}}{l_{A_{min}}} \tag{25}$$

and

$$\frac{a_{B_{max}}}{l_{B_{min}}} \tag{26}$$

we are sure to have an **efficient barter** whereas if both attain:

$$\frac{a_{A_{min}}}{l_{A_{max}}} \tag{27}$$

and

$$\frac{a_{B_{min}}}{l_{B_{max}}} \tag{28}$$

we are sure that the barter is surely inefficient. In (25) and (27) with respectively $a_{A_{max}}$ and $a_{A_{min}} \leq a_{A_{max}}$ we denote the maximum and minimum values

that $A$ assigns to the goods he can get from the barter and with $l_{A_{max}} \geq l_{A_{min}}$ and $l_{A_{min}}$ we denote the maximum and minimum values that $A$ assigns to the goods he may lose from the barter. In (26) and (28) we have the same quantities assigned to the corresponding goods by player $B$.
We remark how conditions (25), (26), (27) and (28) are sufficient conditions of efficiency and do not represent effective strategies for each player since condition (25), for instance, has a quantity that depends on the choice of $A$ at the numerator but a quantity that depends on the choice of $B$ as denominator.
Last but not least, we note, from the equations (23) and (24), how **efficiency** of a barter cannot be always guaranteed and must be verified case by case.

## 8.8 Fairness of the proposed solutions

In this section we aim at verifying if the solutions we have proposed in the previous sections satisfy the criteria we stated in section 8.7 so that we can say whether they produce fair barters or not.
We start with **envy-freeness** in the one-to-one barter. In this case a barter occurs if and only if both $A$ and $B$ get a non negative utility form it or if both players think each of them gets no less than one looses. This turns, in the simplest case, in the following conditions (involving strictly positive quantities):

$(b_1)$ $s_A(j) - v_A(i) \geq 0$ or $\frac{s_A(j)}{v_A(i)} \geq 1$

$(b_2)$ $s_B(i) - v_B(j) \geq 0$ or $\frac{s_B(i)}{v_B(j)} \geq 1$

so that $(b_1)$ coincides with relation (15) and $(b_2)$ coincides with relation (16). In this way we can derive that if a barter occurs then it is guaranteed to be **envy-free** (and therefore **proportional**, since we have maintained the equivalence between the two concepts in the current case of two players).
In more complex settings things can be more tricky to prove but, following similar guidelines, it is possible to show that whenever a barter occurs it is guaranteed to be envy-free.
We recall that in every case where a set of goods is involved we can evaluate its worth by using the additivity hypothesis.
As to **equitability** (see relations (21) and (22)) we refer only to player $A$ since the case of $B$ is completely analogous. In this case we remark that:

$(eq_1)$ $a_A(j) < a_A(I')$

$(eq_2)$ $l_A(i) < a_A(I)$

From $(eq_1)$ and $(eq_2)$ we can easily derive $a_A(j)l_A(i) < a_A(I')a_A(I)$ or:

$$\frac{a_A(I')}{a_A(j)} > \frac{l_A(i)}{a_A(I)} \tag{29}$$

On the other hand from $(eq_1)$ it is possible to derive:

$$\frac{a_A(I')}{a_A(j)} > \frac{a_A(j)}{a_A(I')} \tag{30}$$

If we compare relations (29) and (30) with relation (21) we can easily see that there may be possibilities to have an equitable barter for $A$ and, in a similar way, an equitable barter for $B$ so to get an **equitable barter**.
For $A$ this occurs if we get:

$$\frac{a_A(I')}{a_A(j)} > \frac{a_A(j)}{a_A(I')} > \frac{l_A(i)}{a_A(I)} \tag{31}$$

since the rightmost inequality is equivalent to relation (21).
In order for this to happen we must have:

$$a_A(j)a_A(I) > a_A(I')l_A(i) \tag{32}$$

or:

$$a_A(j)a_A(i) = a_A(j)l_A(i)\alpha > a_A(I')l_A(i) \tag{33}$$

so that we need to find the minimum value $\alpha > 1$ such that:

$$\alpha a_A(j) > a_A(I') \tag{34}$$

holds. Instead than using $(eq_1)$ we could have used $(eq_2)$ so to derive the corresponding necessary value for $\beta$.
In this way, since we do not use at all the condition of envy-freeness, we establish an independence between the two concepts but for the fact that if a barter is not envy-free it does not occur so that it is not possible to evaluate its degree of equitability.
Last but not least we deal with the verification of the **efficiency** of a barter $(i, j)$ in the case of a one-to-one barter. In this case we must verify that there is not another barter $(i', j')$ such that the relations (23) and (24) hold.
Even if $A$ choses $\hat{j}$ (see section 8.4) $B$ could have chosen $i'$ such that $l_A(i') < l_A(i)$ so that relation (23) (with $j = \hat{j}$) would be verified implying that the current barter $(i, \hat{j})$ is not efficient.
Similar considerations hold also for $B$. From these considerations we derive that **efficiency** for both players can be verified only a posteriori. If it is

violated we derive **inefficiency** from which both actors may derive a **regret** that could be (at least partially) compensated through repeated barters (see section 8.9).
Summing up, we can say that, in the case of one-to-one barter:

- envy-freeness is guaranteed every time a barter occurs,
- equitability may be guaranteed at every barter,
- efficiency must be verified a posteriori at every barter,

so that the **fairness** of a barter is a by-product of the barter process itself and is not a-priori guaranteed by its structure.
Similar considerations hold also for the other three models.

## 8.9 Extensions

The planned extensions include the possibility of (1) **repeated barter** involving (2) even **more than two** players and (3) the **relaxing of additivity**.
If we allow the execution of **repeated barters** we must introduce and manage the possibility of the retaliations between the players from one barter session to the following sessions and how the pool of goods are defined and/or modified between consecutive barter sessions. In the proposed algorithms (currently stateless) we can deal with the presence of the **retaliation** through **state variables** that account for past attitudes of the players (Axelrod (1985) and Axelrod (1997)).
If we allow the presence of **more than two actors** we must introduce the mechanisms for the execution of parallel and concurrent negotiations.
If, for instance, we have three actors $A$, $B$ and $C$ we can have (in the case of one-to-one barter with simultaneous requests) the following possibilities: circular one-to-one requests where, for instance, $A$ makes a request to $B$, $B$ to $C$ and $C$ to $A$ or one-to-many requests so that $A$ makes a request to $B$ and $C$, $B$ makes a request to $A$ and $C$ and $C$ makes a request to $B$ and $A$.
In the former case there can be no conflict/concurrence whereas in the latter it can occur that two actors ask the same item to the third causing a conflict that must be resolved some way.
In both cases we have: the barter occurs if and only if every actor accepts what is proposed by the others; if all actors refuse the others' proposals a rearrangement (that depends on the nature of the barter) of the respective pools occurs followed by a repetition of the barter; in all the other cases the procedure must allow the refusing actors (two at the most) to repeat their

request.
Obviously in all the other cases the interactions tend to be more and more complex. Analysis of such extensions can be carried out using the tools suggested in Myerson (1991), section 9.5 where *graphical cooperation structures* are introduced and used.
As a last extension we mention the **relaxing of additivity**. Additivity is undoubtedly a simplifying assumption and is based on the hypothesis of the relative independence of the goods that the actors want to barter among themselves. This hypothesis in many cases is not justified since functional links, for instance, make the goods acquire a value when and only when they are properly combined. In such cases the goods must be bartered as dynamically chosen subsets and cannot enter properly in a one-to-one barter. The issue is very complex (so complex that Brams and Taylor (1996) and Brams and Taylor (1999) deal with it only marginally) and here we only make some basic comments and considerations and present a toy example.
We recall that player $A$ choses among the goods of $B$ and vice versa. What $A$ loses, owing to the choice performed by $B$, belongs to the set $I$ and is evaluated according to the values of $v_A$ and what he gets belongs to $J$ and is evaluated according to the values of $s_A$. Similar considerations hold also for player $B$.
Up to now we have supposed that $A$ evaluates subsets of the goods involved in the barter with additive rules and that the same holds also for $B$. From this point on we are going to consider both subadditivity and superadditivity for player $A$ but similar considerations hold also for player $B$.
We note that as to $s_A$ subadditivity (or the case where the value of the set is lower than the sum of the values of its composing elements) is meaningless since in this case $A$ would be better off by simply asking for a single good from $B$. On the other hand subadditivity on $v_A$ is highly implausible since there is no reason to believe that $A$ would bring to the barter goods that taken as sets are worth less than the single goods.
From these considerations we derive that (1) $A$ sees $J$ in a superadditive way by hypothesis and (2) $I$ in a superadditive way as his worst case and similar considerations hold also for the player $B$.
As to (1) this means that $\forall K \subseteq J$:

$$s_A(K) \geq \sum_{j_k \in K} s_A(j_k) \tag{35}$$

$A$ is of course more interested in subsets $K \subseteq J$ such that:

$$s_A(K) > \sum_{j_k \in K} s_A(j_k) \tag{36}$$

We call the subsets for which relation (35) holds the **superadditive subsets** of $J$ and those for which relation (36) holds the **strictly superadditive subsets** of $J$.
As to (2) we recall that $I$ contains the goods that $A$ loses in the barter so that the condition:

$$v_A(H) \geq \sum_{i_h \in H} v_A(i_h) \tag{37}$$

(for $H \subseteq I$) represents a worst condition for $A$ with regard to the additive case in the evaluation of his utility in the one-to-many and many-to-many barter cases. At this point we have the cases of Table 1 where we show the

| *A vs. B* | additive | superadditive |
|---|---|---|
| additive | one-to-one | one-to-many |
| superadditive | many-to-one | many-to-many |

Table 1: *Possible types for the ways in which each player evaluates their requested goods*

possible typologies of the players with regard to the values $s_A$ for $A$ and $s_B$ for $B$.
From this perspective, the fact that $A$ is superadditive means that at least relation (35) holds and the same is true for $B$ if she is superadditive.
From that Table we se that if both players are superadditive they are more willing to agree on a many-to-many barter, if they are both additive they may prefer a one-to-one barter whereas if one is superadditive and the other is additive they may agree on either a many-to-one or a one-to-many barter depending on which is the superadditive player.
In the closing part of this section we are going to deal only with the **many-to-many** barter case with **simultaneous requests** where $A$ asks for the goods of the set $J_0 \subseteq J$ and loses the goods of the set $I_0 \subseteq I$ whereas $B$ asks for the goods of the set $I_0 \subseteq I$ and loses the goods of the set $J_0 \subseteq J$.
Also in this case the core of the algorithms (see sections 8.3 and 8.6) is composed by the four cases that may occur at each pass:

(a) both $A$ and $B$ accepts the proposed barter so that the process ends with a success;

(b) $A$ accepts but $B$ refuses;

(c) $A$ refuses whereas $B$ accepts;

(d) both $A$ and $B$ refuse.

In the symmetric cases ($b$) and ($c$) the accepting player keeps his request fixed while the refusing player has two possible mutually exclusive strategies:

- can repeat his choice;
- can partition (on the first refusal) or rearrange a partitioning (on successive refusals) his set of goods so that another round may occur.

In the case ($d$) each player has both the repeater and the modifier strategies at his disposal.
The fact that a player rearranges in some way his goods through the definition of variable partitions interfere with the superadditive evaluations of the other player and this may cause both players agree that there is no possibility for the process to go on (see the step (2)($a$) of the simultaneous requests algorithm of section 8.3).

# 9 Some remarks about the iterative barter models

## 9.1 Introduction

In this section we present two types of models of barter that may be seen as an extension of the models we presented in Cioni (2008a) and Cioni (2008b) and that we term as iterative since they are based on iterative algorithms through which either one or both reveal the composition of the sets of items they are willing to barter.
Both models involve indeed a pair of actors/players $A$ and $B$ that aim at bartering a pair of items.
In the former model (the so called implicit or pure model) neither actor reveals to the other the set of items he is willing to barter but such a revelation occurs incrementally during the process since by exchanging proposals and counter proposals the two players reveal each other the composition of such sets. The bargaining process goes on until an agreement is reached and a bargaining occurs or both players agree that no bargaining is possible so that the process ends with a failure.
In the latter or mixed model, on the other hand, we have an asymmetric situation where one of the players, be it $A$, shows to $B$ his set of items, be it $I'$, on which the bargaining process starts with a proposal from $B$. Also in this case the process goes on with a series of proposals and counter proposals from both players until one of the foregoing cases occurs. In this case player $B$ reveals the composition of his set of items during the course of the process.

We note that in both cases each player can be said to know which are the items he is willing to bargain in the barter process.
This knowledge may be verified in advance by asking to each player if a given proposal would be or not in his bargaining set (or the set of the acceptable proposals).
From this perspective we can say that each player is characterized by a bargaining set, $I_A$ for $A$ and $J_B$ for $B$, whose structures and whose preference orderings are private knowledge of each player and can be only partly revealed during the barter process. The main difference between the two models that we propose is that in the latter model the set $I_A$ is, at least partly, a common knowledge of the two players under the form of the set $I'$.

## 9.2 Some notes about the barter

In this section with the term **barter** we denote a process through which two players $A$ and $B$ can exchange a pair of items[9] $(i, j)$ where both items are evaluated according to each player's private evaluation system that determines either his rejection or his acceptance of the proposed items.
The exchange, if it occurs, is in kind so that the items $(i, j)$ are the only involved items and there is no parallel or compensatory exchange of money or any other numerary good between the players.
In any generic pair $(i, j)$ the identifier $i$ identifies what passes from $A$ to $B$ either under the form of a good or a bad or a service in exchange of the item identified by $j$, of the same types, from $B$ to $A$. Both items $i$ and $j$ are characterized by their ownership (in the sense of who is the provider and who is the receiver) and by their type as $(a)$ a **good** or an item that has a positive value for both players; $(b)$ a **bad** or an item that has a negative value for both players; $(c)$ a **service** or an item that has a instrumental value for one or both players and represents a task that a player carries out for the other.
Each ownership is provided under the form of a pair $(p_1, p_2)$ where both identifier may be $A$ or $B$ and identify the direction of transfer from a provider to a receiver. The basic idea is to have a transfer from $A$ to $B$ of $i$ and from $B$ to $A$ of $j$. In order to obtain this it is necessary in some cases to manipulate the transfer so to replace a bad in one direction with a good or with a service in the opposite direction.
When things have been arranged so to have two unidirectional transfers (from $A$ to $B$ and vice versa) we may quantify the transfers so to evaluate them.

[9] In what follows we sometimes refer to the former element of each pair $(i, j)$ as the $i-$item and to the latter as the $j-$item.

For player $A$ this means that we imagine he uses two private values $v_A(j)$ and $v_A(i)$ to evaluate, on a private scale, what $A$ gets from the barter and what $A$ loses from it int that order.
In this way $A$ can evaluate the ratio:

$$\rho_A = \frac{v_A(j)}{v_A(i)} \tag{38}$$

as a dimensionless quantity.
For player $B$, in a similar way, this means that we imagine she uses two private values $v_B(i)$ and $v_B(j)$ to evaluate, on a private scale, what $B$ gets from the barter and what $B$ loses from it int that order.
In this way $B$ can evaluate the ratio:

$$\rho_B = \frac{v_B(i)}{v_B(j)} \tag{39}$$

as a dimensionless quantity.
As to the quantities that are involved in relations (38) and (39) we note that they represent private information of each player, are measured according to private scales that may not be common knowledge between the players and include possibly independent discount factors for each player so to account for damages occurring to each of them from the passing of time.
Relations (38) and (39) are used respectively by player $A$ and player $B$ to accept or refuse a proposed barter (see section 10).

## 9.3 The performance and evaluation criteria

For the evaluation of the proposed barter procedures we use the **performance criteria** and the **evaluation criteria** we introduced in section 4.
Also in this case such criteria must be adapted or must be redefined someway so to be in agreement either with the essence of their classical definitions or with intuition or with both.
We start with **envy-freeness**.
If we denote with $v_A(j)$ and $v_A(i)$ the values in $A$'s opinion and evaluation, respectively, of what $A$ obtains and loses from the barter (and with $v_B(i)$ and $v_B(j)$ the same quantities for player $B$) we say that the allocation deriving from a barter (or a barter tout court) is **envy-free** if we have for $A$:

$$\rho_A = \frac{v_A(j)}{v_A(i)} \geq 1 \tag{40}$$

and for $B$:

$$\rho_B = \frac{v_B(i)}{v_B(j)} \geq 1 \tag{41}$$

Relation (40) means that the value that $A$ assigns to what he gets from the barter is at least equal to the value that $A$ assigns to what he loses from the barter. We assign a similar meaning to relation (41) with regard to $B$.
Since, in our case of two players, we want to maintain the equivalence between proportionality and envy-freeness we must give a definition that mirrors the classical definition of proportionality and reflects this equivalence.
For player $A$ we may define a barter as proportional if it satisfies the following condition:

$$\frac{v_A(j)}{v_A(j)+v_A(i)} \geq \frac{v_A(i)}{v_A(j)+v_A(i)} \tag{42}$$

so that the fractional value of what $A$ gets from the barter is at least equal to that of what he loses from it. We remark that $v_A(j)+v_A(i)$ represents the value that $A$ assigns to the bartered items.
A similar condition holds also for $B$:

$$\frac{v_B(i)}{v_B(i)+v_B(j)} \geq \frac{v_B(j)}{v_B(i)+v_B(j)} \tag{43}$$

We say that a barter is proportional if both (42) and (43) hold.
It is easy to see how from equation (42) it is possible to derive equation (40) and vice versa. The same holds also for equations (43) and (41) so that the equivalence of the two definitions has been maintained in the case of two players.
We now pass to the criterion of **equitability**.
We must adapt its definition to our framework in the following way. We need firstly some definitions. With respect to the occurrence of the barter of the items $(i, j)$ we define for player $A$:

- $V_A$ a measure for $A$ himself of his current welfare before the barter occurs;
- $V'_A$ a measure for $A$ himself of his current welfare after the barter has occurred;

and for player $B$:

- $V_B$ a measure for $B$ himself of his current welfare before the barter occurs;
- $V'_B$ a measure for $A$ himself of his current welfare after the barter has occurred.

With the term **welfare** we denote a personal and private evaluation from each player of his global situation (either under the hypothesis of additivity or subadditivity[10]) through a single value that is used to rank the items that are entering the barter process.
We therefore define a barter of the items $(i, j)$ as **equitable** for $A$ if the fractional value of what he gets is at least equal to the fractional value he gives to what he loses from the barter or:

$$\frac{v_A(j)}{V'_A} \geq \frac{v_A(i)}{V_A} \tag{44}$$

On the other hand the barter is equitable for $B$ if, using the corresponding quantities we used in equation (44) but referred to player $B$, we have:

$$\frac{v_B(i)}{V'_B} \geq \frac{v_B(j)}{V_B} \tag{45}$$

If both relations hold we say that the barter is **equitable**.
We note that if $V'_A \geq V_A$ then relation (44) implies envy-freeness for $A$ whereas if $V'_B \geq V_B$ then relation (45) implies envy-freeness for $B$.
To make inequalities (44) and (45) of more practical use we may rewrite them, for instance for player $A$, as follows:

$$\frac{v_A(j)}{\downarrow_A} \geq \frac{v_A(i)}{\uparrow_A} \tag{46}$$

where, for each turn following the first one:

> $\downarrow_A$ is the minimum value of all the $j-$items that $A$ has bargained before the current turn of bargaining;
>
> $\uparrow_A$ is the maximum value of all the $i-$items that $A$ has bargained before the current turn of bargaining.

We say that (46) is easier to use than relation (44) because keeping track of both a maximum and a minimum value in a multi step process is easier than evaluating at each step the new value of the welfare under the hypothesis that the proposed barter occurs.
A similar relation holds also for player $B$:

$$\frac{v_B(i)}{\downarrow_B} \geq \frac{v_A(j)}{\uparrow_B} \tag{47}$$

[10]With reference to sets with the term **additivity** we denote the fact that the value of a set is given by the sum of the values of its components whereas if this value is at least equal to that sum we speak of **superadditivity** and of **strict superadditivity** if it is strictly greater.

Last but not least we examine the criterion of [**Pareto**] **efficiency**.
A barter of the items $(i, j)$ is [**Pareto**] **efficient** if there is not another pair of items $(i', j')$ that gives at least to one player a better result without hurting the other, under the hypothesis that at least one of the following inequalities hold:

1. $i' \neq i$
2. $j' \neq j$

For players $A$ and $B$ this means that there is no barter $(i', j')$ that satisfies the following inequalities:

$$\frac{v_A(j)}{v_A(i)} \leq \frac{v_A(j')}{v_A(i')} \tag{48}$$

$$\frac{v_B(i)}{v_B(j)} \leq \frac{v_B(i')}{v_B(j')} \tag{49}$$

with at least one of them satisfied with the $<$ relation.
In such relations the pairs $v_A(i)$, $v_A(j)$ and and $v_A(i')$, $v_A(j')$ are related to $A$ and are associated respectively to $(i, j)$ and to $(i', j')$. Similar quantities are defined also for player $B$.
We note that if the barter is such that both players attain:

$$\frac{v_{A_{max_j}}}{v_{A_{min_i}}} \tag{50}$$

and

$$\frac{v_{B_{max_i}}}{v_{B_{min_j}}} \tag{51}$$

we are sure to have an **efficient barter** whereas if both attain:

$$\frac{v_{A_{min_j}}}{v_{A_{max_i}}} \tag{52}$$

and

$$\frac{v_{B_{min_i}}}{v_{B_{max_j}}} \tag{53}$$

we are sure that the barter is surely inefficient. In (50), (51), (52) and (53) we have:

$v_{A_{max_j}}$ is the best $j-$item that $A$ can get from the barter;

$v_{A_{min_j}} < v_{A_{max_j}}$ is the worst $j-$item that $A$ can get from the barter;

$v_{B_{max_i}}$ is the best $i-$item that $B$ can get from the barter;

$v_{B_{min_i}} < v_{B_{max_i}}$ is the worst $i-$item that $B$ can get from the barter.

We remark how conditions (50), (51), (52) and (53) are sufficient conditions of efficiency but may also be hints for either good or bad strategies for both players.
Last but not least, we note, from the equations (48) and (49), how **efficiency** of a barter cannot be always guaranteed and must be verified case by case.

## 9.4 Incremental construction/revelation

One of the key point of the proposed models is the fact that either one or both players reveal incrementally the set of pairs of items $(i, j)$ each of them is willing to barter with the other. Each pair is seen as a single element of the set and from two pairs $(i, j)$ and $(h, k)$ we can obtain one of the following pairs: $(i, h)$, $(i, k)$, $(h, j)$ and $(k, j)$ depending on the nature of the involved items.
It is therefore necessary to understand the ways through which a set is incrementally enlarged from the initial empty set to a maximal set, the so called bargaining set, that include all the possible elements that a player is willing to barter.
The first way we can use is the following (in what follows we consider the case of $A$, the case of $B$ is analogous and will not be explicitly considered). $A$ may start with $I_0 = \emptyset$ and add one element at a time according to some insertion criteria until a criterion of stop is met so that the process is interrupted and the final set $I_\infty$ is constructed. In this way $A$ builds up the following succession of sets $I_0 \subset I_1 \subset \cdots \subset I_\infty$ where the set $I_\infty$ may, at least in theory, contain infinitely many elements.
In this way $A$ proceeds **bottom up** since he starts from the empty set and eventually ends with the whole set of the items $I_\infty$ that may coincide or not with $I_A$.
This process gives to $A$ the greatest flexibility since it allows him to build up new elements by mixing and or merging the existing ones so that the construction process can adapt better to the course of the barter process.
The main problem with this approach is that the barter may prove a very time consuming process since the number of the possible combinations increase with the increase of the number of the available elements.
Another way that $A$ can use is the following that we may call **top down**. $A$ starts with his fixed and predefined set $I_A$ of $n$ elements. Each item is initially set as invisible (so that again $A$ starts with a publicly known set $I_0 = \emptyset$) and during the process one element at a time is set visible. In this

way $A$ builds up the following succession of sets $I_0 \subset I_1 \subset \cdots \subset I_n$ with $I_n = I_A$.
This incremental disclosure may be obtained by using a set of $n$ flags initially set at *invisible* and by setting at each step one flag at a time at *visible* so to reveal the associated element.
The process ends when a barter occurs or when all the elements of the set $I_n$ are revealed without any barter occurring.
In this case we get the lowest flexibility since the elements are fixed from the start but we are sure the process has a fixed bound that must occur when all the elements have been revealed without any barter occurring.
A this point we have to define, in the bottom up approach, which criterion can be used by player $A$ (and similarly by player $B$) to add a new element to the current set $I_i$ and, in the top down approach, which criterion can be used by player $A$ (and similarly by player $B$) to set as visible a new element to the current set $I_i$.
In both cases the simplest criterion is the following: a player either adds or sets visible an element that is expected to give him an advantage greater than the one deriving him from the current proposal.
We recall that such an element represents the new [counter] proposal and so an advantageous elements for the proposing player.
The other point is to clarify how player $A$ (but the same is true also for player $B$) can derive at step $i$ the element $(h, k)$ such that $I_i = I_{i-1} \cup (h, k)$.
In the bottom up approach such an element can be either an element of $I_A$ or an element composed by using elements from $J_{i-1}$ or a mixture of both cases.
In the top down approach such an element is simply one of the elements of $I_A$.

# 10 The implicit and the mixed barter models

## 10.1 Introduction

In this section we propose two iterative barter models. In the former model, that we call **implicit model** or **pure model**, neither $A$ nor $B$ shows each other the sets of the items they wish to barter.
In the latter model, that we call **mixed model**, we suppose that one player, be it $A$, shows the items he is willing to barter from the offset of the process whereas the other, in this case $B$, keeps her items hidden but reveal them during the process by making either proposals or counter proposals aiming at the reaching of an agreement and therefore a barter.

## 10.2 General remarks

Both models are described by using simple algorithms of which we present the general structure and the various options that the each of two players has at each step. In order to keep the structure of each algorithm simple and readable we may use strings to describe sub procedures that we verbally describe separately.
Since both models are based on a succession of proposals and counter proposals we firstly need to define what do we mean with the terms proposal and counter proposal. We also list which are the moves that each player can use during the process.
A **proposal** is a pair of item identifiers $(i, j)$ that a player proposes to the other as the object of the barter where each item is characterized by an ownership.
On the other hand, given a proposal $(i, j)$ a **counter proposal** is a pair $(i', j')$ such that either $i' \neq i$ or $j' \neq j$ is true since:

- if $i' = i$ and $j' = j$ we have an implicit acceptance so that the process ends with a success;

- if $i' \neq i$ and $j' \neq j$ we define it as a new proposal.

It is obvious that a proposal in reply to a counter proposal is termed a counter proposal and not a counter counter proposal so to avoid the chaining of counter prefixes.
Within our perspective we have that a counter proposal may follow only a proposal and a new proposal may follow either a pass move or a reject move (see further on).
Both a proposal and a counter proposal can be followed by one of the following moves from the listening player: **pass**, **give up**, **accept** and **reject**.
A **pass** move is a way through which a player may signal to to the other that it is necessary that he shows some more goodwill in order for the process to go on.
A **give up** move is a way for one player to signal to the other that he thinks the process is not worth being carried on any more.
We note that two successive give up moves (one from each player) cause the process termination with a failure.
An **accept** move closes the barter with a success since it signals that a player accepts the last [counter] proposal made by the other player.
A **reject** move means that the received [counter] proposal made by the other player cannot be accepted.
Both an **accept move** and a **reject move** can follow any [counter] proposal

but if the answer is a reject move the turn remains to the rejecting player that can make his counter proposal.
A **pass move** gives the turn to the other player and may be answered by either a new proposal, by a pass move or by a give up move. We note that there cannot be more than two consecutive pass move so that a natural succession for a closure with a failure may be: $pass_A$, $pass_B$, $give\,up_A$, $give\,up_B$.
Last but not least a give up move may be followed either by a new proposal or by another give up move from the other player: in the former case the process goes on whereas in the latter it is interrupted with a failure and in a way that does not necessarily involve the use of pass moves.

## 10.3 The implicit model

In the case of the **implicit/pure model** the situation we are interested in can be described in the following terms.
We have one player that wants to exchange an item with another player but none of them has a knowledge of the items the other is willing to barter.
The only way to proceed is through an iterative process. At each step of the process a pair of items $(i, j)$ is proposed and such a pair may be either accepted or refused in some way.
In the former case the process ends with a success so that the barter occurs.
In the latter case we may have a pass move so the next move is up to the other player, a reject move so the next move is up to the same player or a counter proposal.
At the beginning of the barter we have the set of pairs-of-items-to-be-bartered of $A$ is $I_0$ and the set of pairs-of-items-to-be-bartered of $B$ is $J_0$.
We therefore have an *initialization phase* where we pu $I_0 = \emptyset$, $J_0 = \emptyset$, $i = 1$ and $j = 1$ and where we select at random who moves first, be it $A$. The other case being fully symmetrical will not be examined here.
In the description of the algorithm we use the notation $propose_A$ to summarize the execution of the following steps:

1. $A$ presents a [counter] proposal $p_A = (i, j)$;
2. $I_i = I_{i-1} \cup p_A$;
3. $i = i + 1$;

and symmetrically we use the notation $propose_B$ to summarize the execution of the following steps:

1. $B$ presents a [counter] proposal $p_B = (i', j')$;

2. $J_j = J_{j-1} \cup p_B$;

3. $j = j + 1$;

The main structure of the algorithm in this case is the following:

(0) *initialization phase*;

(1) $propose_A$;

(2) $B$ may:

- ($2_a$) *accept*; *go to (4)*;
- ($2_b$) *reject*; $propose_B$; *go to (3)*;
- ($2_c$) $propose_B$; *go to (3)*;
- ($2_d$) *give up*; go to (5);
- ($2_e$) *pass*; go to (6);

(3) $A$ may:

- ($3_a$) *accept*; *go to (4)*;
- ($3_b$) *reject*; $propose_A$; *go to (2)*;
- ($3_c$) $propose_A$; *go to (2)*;
- ($3_d$) *give up*; go to (5);
- ($3_e$) *pass*; go to (6);

(4) end;

If we denote with the player $i$ either $A$ or $B$ and with $j$ either $B$ or $A$ we can define the moves that can follow either a *pass* or a *give up* move in the following ways:

(5) a *give* $up_i$ move may be followed by:

- ($5_a$) $propose_j$; go to (3) if $j = B$ else go to (2);
- ($5_b$) *give* $up_i$; go to (4);

(6) a $pass_i$ move may be followed by:

- ($6_a$) $propose_j$; go to (3) if $j = B$ else go to (2);
- ($6_b$) $pass_j$; *give* $up_i$; *give* $up_j$; go to (4);
- ($6_c$) *give* $up_j$; go to (5);

The execution of either an *accept* or a *reject* move on a proposal $(i, j)$ from player $A$ (the case of player $B$ is fully symmetrical and will not be analyzed) is based on the consideration of the values $v_A(j)$ and $v_A(i)$ through a function:

$$eval_A(i, j) = f_A(v_A(j), v_A(i)) \tag{54}$$

where the function $f_A$ synthesizes the working of comparison from $A$ between the values $v_A(j)$ and $v_A(i)$. In its simplest form we can express it as:

$$f_A(v_A(j), v_A(i)) = v_A(j) - v_A(i) \tag{55}$$

Such function can be used in rules such as the following[11]:

$$\textbf{if}(eval_A(i, j) \geq 0) \textbf{ then } accept_A \textbf{ else } refuse_A \tag{56}$$

so to establish a strict preference ordering $\succ$ on the proposals. We can indeed say[12]:

$$(i, j) \succ_A (i', j') \Leftrightarrow eval_A(i, j) > eval_A(i', j') \tag{57}$$

and the same holds also for $B$.

## 10.4 The mixed model

In the **mixed model** we have an asymmetric situation where one of the players, be it $A$, shows to the other, $B$, his set of items $I'$ whereas the first move is up to the other player, $B$ in this case.
The algorithm in this case has the following structure that is very similar to the one we have seen in section 10.3 but for the *initialization phase* and the starting move.

(0) *initialization phase*;

(1) $propose_B$;

(2) $A$ may:

- ($2_a$) *accept*; *go to (4)*;
- ($2_b$) *reject*; $propose_A$; *go to (3)*;
- ($2_c$) $propose_A$; *go to (3)*;
- ($2_d$) *give up*; go to (5);

[11] We could have $eval_A(i, j) \geq \varepsilon$ with $\varepsilon > 0$ if a minimum gain is required or $\varepsilon < 0$ if a maximum loss is acceptable.

[12] It is obvious that with $\succ_A$ we denote the strict preference relation of player $A$.

($2_e$) *pass*; go to (6);

(3) $B$ may:

($3_a$) *accept*; *go to (4)*;

($3_b$) *reject*; *propose*$_B$; *go to (2)*;

($3_c$) *propose*$_B$; *go to (2)*;

($3_d$) *give up*; go to (5);

($3_e$) *pass*; go to (6);

(4) end;

For the points (5) and (6) and for the description of the *propose*$_A$ and *propose*$_B$ moves we refer to the same points and the same description we gave in section 10.3.
The *initialization phase* is unchanged for $B$ (so that again we have $J_0 = \emptyset$ and $j = 1$) but for $A$ we must account for the presence of the set $I'$ whose content is defined by $A$ freely and completely at his will for what concerns both the type and the number of the contained items.
From this we have $I_0 \subset I' \otimes I'$ where $I' \otimes I' = \{(i, j) \mid i, j \in I' \text{ but } i \neq j\}$ so that $I_0$ is rather fuzzily defined for $B$ so reducing his seemingly initial advantage over $A$.

## 10.5 Possible strategies

We now present and analyze the possible strategies of the two players in either the pure model or the mixed model.
In the **pure model** both players act without any knowledge of the other's player set of elements but with each player only knowing his bargaining set, $I_A$ if $A$ and $J_B$ if $B$.
In this case the player who moves first, be it $A$, can do no better than choosing an element $(i, j)$ from his bargaining set $I_A$ so to maximize the value:

$$eval_A(i, j) = f_A(v_A(j), v_A(i)) \qquad (58)$$

This is possible for $A$ if he chooses one of the cases where he has full control of both the items of the proposed element $I_1 = \{(i, j)\}$.
In his turn $B$ may use both $J_B$ and $I_1$ to devise his [counter] proposal to $A$ so defining his set $J_1$ after his first step.
At the generic step $i > 1$ for $A$ and $j > 1$ for $B$ we have that each player:

- knows the current proposal $(i, j)$ and is able to evaluate it using the function[13] $eval_h$;
- can possibly devise a [counter] proposal $(i', j')$ and propose it if it is preferred by the player to the current one according to the preference ordering we introduced with relation (57) or if:

$$(i', j') \succ_h (i, j) \Leftrightarrow eval_h(i', j') > eval_h(i, j) \tag{59}$$

- can frame the new [counter] proposal on the knowledge of both $I_i$ and $J_j$ so to favor at the best also the other player (therefore increasing the probability of an acceptance) by using the common knowledge between them;
- if the [counter] proposal of the foregoing steps cannot be devised the player can either accept the current proposal, press the other for a change of route (with a *pass* move) or signal the will to stop the process (with a *give up* move).

As the last step we note how each player can use also his bargaining set for framing a new [counter] proposal but with a lower probability of if being accepted by the other player.
In the **mixed model** the presence of an asymmetry does not modify very much what we have seen before since the initial knowledge of $B$ of the items of $A$ is rather fuzzy and, moreover, she is the player who must move first. We can, therefore, state that the strategies we have seen for the **pure model** case can be profitably used also in the n the **mixed model** case.

## 10.6 Satisfaction of the criteria and applications

We now examine if the proposed models satisfy or not the performance and evaluation criteria we have introduced in section 9.3.
For what concerns the **performance criteria** we have that **guaranteed success** is not always satisfied since a process of barter may end without any effective barter and without any penalty for the player who withdraws from the process. **Individual rationality** is guaranteed to each player that feels he engaged himself in an unfavorable process from the availability of *pass* and *give up* moves whereas **simplicity** is assured by the fact that the frame of both algorithms is a sequence of proposals and counter proposals until an agreement is reached either to have a barter or to give up since no

[13] With the subscript $h$ we denote either the player $A$ or the player $B$ depending on the case.

barter is effectively possible. Last but not least **stability** is satisfied since both players have easy to follow and implement strategies (see section 10.5). We recall that the acceptance or not of a proposed barter is governed by the relations (54), (55) and (56) of section 10.3.
For what concerns the **evaluation criteria** we have that if a barter occurs, according to the previously recalled relations, then it is **envy-free** (see relations (9) and (41)) for both players and therefore it is also **proportional**, from the equivalence of the two concepts in the case of two players.
It is easy to verify, by the same relations, that if a barter is fair then it occurs. The check of the **equitability** requires the check of the relations (44) (or of the corresponding relation (46)) and (45) (or of the corresponding relation (47)).
Such a check may be performed by both players at every step of the barter so that, in the best case, one player may propose a barter that for him is envy-free and equitable and the other may verify such a proposal and accept if he too thinks that for him it is envy-free and equitable.
Up to this point we have left out [Pareto] efficiency.
As to the **[Pareto] efficiency** we need to verify that the relations (48) and (49) cannot be satisfied and both players have sufficient conditions to attain such condition. Such conditions, however, may be conflicting so that they cannot drive effectively a strategy. Moreover the check of the relations (48) and (49)) may be impossible since not all the possible elements $(i, j)$ are known during the barter so that the inefficiency of a barter may be discovered only when both players have accepted it.
Summing up, we can say that envy-freeness is guaranteed every time a barter occurs, proportionality is guaranteed every time a barter occurs, equitability may be guaranteed at every barter and efficiency may be guaranteed since the players have sufficient conditions for attaining it but may also be easily missed.
From these considerations we have that the **fairness** of a barter is a by-product of the barter process itself and is not a-priori guaranteed by its structure.
From all this we derive that each of the two players may judge an occurred barter as either inequitable or inefficient or both and therefore unfair, since not all the fairness criteria are verified.
Last but not least we comment a little on the possible **applications** of the proposed models.
The **implicit model** can be applied in all the cases where two players meet to perform a barter but none of them has a knowledge of the items that the other may be willing to barter so that this knowledge must be acquired during the barter process itself through a try-and-error process.

In the current version of the pure model we imagined the two players as peers in the barter, this fact being represented by the random move for the choice of the first time mover. It is easy to remove this feature by giving, for some reason, the right to make the first move to one of the two players. For the range of the possible applications we mention: the exchange of two goods; the exchange of a pair (good, bad) or (bad, good); the exchange of a pair (good, service) or (service, good); the exchange of a pair (service, bad) or (bad, service) and the exchange of two services.
On the other hand the **mixed model** can be used in all the cases where one of the two players cannot conceal or is forced to show the initial set of his items so that the other has some information about the possible advantageous (for him) proposals that he can make without even revealing any of the items of his private set.
Also in this case we refer to the foregoing list, from where it is possible to understand the nature of the items that the two players may wish to exchange.

## 10.7 Possible extensions

Up to now we have seen the basic models involving a pair of players in a one shot barter for the exchange of a pair of items. In this section we briefly present the planned extensions and examine them singularly and one independently from the others though it is obvious that they could be combined together in various ways. This treatment is an introductory level since such extensions represent the core of a future research stream.

### 10.7.1 A plurality of players

Instead of a pair of players $A$ and $B$ we may define a set $\mathscr{P}$ of $n \geq 4$ players (with $n$ even[14]) that can form $n/2$ pairs (so to have $n/2$ contemporaneous barters) in the following ways:

- by a random selection,

- by raising up of hands,

[14] In abstract term we might have two cases:

(1) $n$ is even so $n/2$ pairs of players form and no player is left out;

(2) $n$ is odd so $n/2$ pairs form but one player is left out.

In this section we assume that $n$ is even since the case $n$ odd has no sense but in cases of repeated barters (see section 10.7.2) where the player who is left out at one stage gains some precedence at the next stage according to a form of balanced alternation (Brams and Taylor (1999)).

- by mutual selection.

In the case of **random selection** we may imagine a procedure that:

(0) starts with $i = n$;

(1) at the $i-$th step it chooses at random one of the $i(i-1)/2$ values so to match a pair of players;

(2) if $i = 2$ go to (3) else $i = i - 2$ go to (1);

(3) end;

At step (1) we may imagine to have a dice with $i$ faces so that the outcome of $j \in [1, i]$ corresponds to one of the possible $i(i-1)/2$ pairings. We have to see how to assign the possible pairings to the faces of the dice. To do so we can set up a $n \times n$ matrix $V$ with all the elements at 0 but those above the main diagonal that assume (row by row form left to right) the increasing integer values from 1 (for the element $v(1,2)$) to $n(n-1)/2$ (for the element $v(n-1,n)$) so that $v(i,j) \neq 0$ for $j > i$.
After each step we remove the two matched players, renumber the remaining players, reduce accordingly the matrix $V$, reassign the values $v(i,j)$ with the same rule and repeat the procedure.
In the case of **raising hands** we can imagine that one player raises up one of his hands. If more that one player raises then each of them lowers his hand, waits for some random amount of time and then raises again his hand. This goes on until there is only one hand up at a time. At this point one or more of the others may join him to form a pair for a barter. If more than one player express the wish to join there is a random selection of one of them. The process goes on until $n/2$ pairs have been formed, at each step the number of waiting players being decremented by 2.
In the case of **mutual selection** the players are divided, through the use of any suitable random device[15], in two subsets of $n/2$ elements each: the former $P_1$ contains the choosers and the latter $P_2$ the choices.
Each player of $P_1$ chooses at his will a player from $P_2$: if the latter accepts the pair forms otherwise the two players switch from one set to the other. If there are multiple selections the choice is up to the player of $P_2$ that is forced to select one of his choosers. The matched players are removed from the respective sets.
When, according one of the foregoing ways, the $n/2$ pairs are formed each of

[15]We can imagine a urn containing $n$ balls, $n/2$ red and $n/2$ blue, and $n$ selections without replacement for the assignment of each extracting player to one of the two subsets according the color of the extracted ball.

them may use one of the algorithms we have seen in section 10 to perform a barter.
The basic idea is that the $n$ players are willing to engage each other in a barter so the $n/2$ pairs form even if there is no guarantee that this turns out in $n/2$ effective barters.

### 10.7.2 Repeated barters

In this case we may use some concepts from Game Theory (Myerson (1991)) and consider each barter as a single stage of the potentially endless process of repeated barters between two players $A$ and $B$.
To get this chaining effective we need to make some changes to the models of sections 10.3 and 10.4. Such changes include:

- the addition of moves to implement the chaining;
- the introduction of state variables through which each player records the attitude of the confronting player during the previous stages (Axelrod (1985), Axelrod (1997));
- the definition of a barter as either with memory or memoryless.

As to the **moves** we need to modify the **accept** move in a **accept and stop** move so to signal a lack of will to go on with the barter; to add the **accept and repeat** move so to signal to the other player the availability to the execution of one more stage; to add the **repeat?** move so to allow a player to request to the other the execution of one more stage; to add a **repeat** move so to answer affirmatively to an explicit or implicit request for one more stage and to add a **refuse** move so to answer negatively to an explicit or implicit request for one more stage so to allow the players to link two stages together in a repeated barter.
The **state variables** represent private information of each player through which he may record from one stage to another if the attitude of the other player at the previous stages has been more cooperative or more exploitative. In this way each player may implement long run strategies of retaliation so to punish spiteful attitudes.
Last but not least we define a barter as **with memory** if at the end of each stage the composition of the revealed set is a valid common knowledge (Myerson (1991)) between the players whereas it is defined as **memoryless** if it is lost and has no common knowledge value since each player is free to modify at his will the composition of his set without any notice to the other. This feature of a repeated barter must be agreed upon by both players at

the very offset of the process, may be changed only after a new mutual agreement after each stage is over and forms a common knowledge between the two players.

### 10.7.3 Multi pairs barter

In this case every player at each step may propose more than one pair of items to be bartered. For instance, $A$ may propose:

$$(i_j, j_l)(i_h, j_k) \tag{60}$$

and $B$ can reply with a counter proposal with even more or less elements or can even accept. The acceptance might be either **partial** if the agreement can involve only a subset of a proposal or **global** if the agreement involves a [counter] proposal as a whole.
Partial agreements are fully meaningful only within a repeated barter setting whereas in a one shot barter we usually speak of global multi pairs agreement whenever the two players agree on bartering all the items contained in one proposal.

# 11 Concluding remarks and future plans

This section presents for the members of both the family $F_1$ and the family $F_2$ some concluding remarks and hints of future research plans.
In sections 5 and 6 we presented some auctions mechanisms of two distinct types. Those of the former type are termed **positive auctions** and resembles classical mechanisms but for the fact that they aim at the allocation of chores rather than goods.
The one of the latter type faces the problem of the allocation through an auction from a negative perspective since the chore is allocated to the bidder who proves less capable of avoiding it but that is compensated for this incapability.
Both types of mechanisms need a deeper and more formal analysis of their structure, their properties, the possibility of collusions and, most important, their practical applications in the area of environmental problem solving.
In section 8 we have introduced a family of barter models between two actors that execute a one shot barter through which they exchange, according to one among various mechanisms, the goods of two separate and privately owned pools. The various models have been introduced under the hypothesis of additivity according to which the value of a set is given by the sum of the values of its composing elements.

In that section we presented the basic algorithms for the one-to-one barter, we showed the possible uses of the proposed models, we verified if some criteria of fairness are satisfied by the proposed models or not and we also introduced some extensions.
The main extension we presented is the relaxing of the additivity hypothesis with the adoption of superadditive sets where the value of a set is at least equal to the sum of the values of its elements. In this way we model functional relations among the goods that increase their joint values.
The presentation we made in section 8 is at an introductory level and a lot of formalization is still to be done for what concerns both the presented models, their extensions and the possible uses in concrete cases .
We need indeed to examine more formally the basic models of one shot barter; to improve the proposed algorithms; to examine the properties of such algorithms and their plausibility and, last but not least, to analyze and formalize the extensions we essentially only listed in section 8.9.
In section 10 we have introduced two algorithms that can be used in the case of one shot barter between a pair of players.
In the former algorithm we have asymmetric situation where the two players try to conclude a barter through an incremental revelation of the sets of items each of them is willing to barter.
In the latter algorithm, on the other hand, the situation is asymmetrical since one of the player either does not want to or cannot hide the set of his items to the other player that therefore has an initial advantage and the right to make the starting offer.
The section presented both a description of the algorithms and their evaluation according to well defined classical performance criteria.
In the closing part of that section we also presented some possible extensions whose analysis and formalization are still to be completed. Similarly we need to complete the analysis of the possible applications of the proposed models to real world cases where exchanges of items occur between players that aim at attaining their objectives without sharing any common scales of qualitative or quantitative values.

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