Iterative barter models

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Abstract

In this paper we present two types of models of barter that may be seen as an extension of the models we presented in Cioni (2008a) and Cioni (2008b) and that we term as iterative since they are based on iterative algorithms through which either one or both reveal the composition of the sets of items they are willing to barter.
Both models involve indeed a pair of actors/players A and B that aim at bartering a pair of items.
In the first model neither actor reveals to the other the set of items he is willing to barter but such a revelation occurs incrementally during the process since by exchanging proposals and counter proposals the two players reveal each other the composition of such sets. The bargaining process goes on until an agreement is reached and a bargaining occurs or both players agree that no bargaining is possible so that the process ends with a failure.
In the second model, on the other hand, we have an asymmetric situation where one of the players, be it A, shows to B his set of items, be it I', on which the bargaining process starts with a proposal from B. Also in this case the process goes on with a series of proposals and counter proposals from both players until one of the foregoing cases occurs. In this case player B reveals the composition of his set of items during the course of the process.
Keywords: bargaining, costs/benefits sharing, barter, iterative barter, fairness, fairness criteria

1 Introduction

In this paper we present two types of models of barter that may be seen as an extension of the models we presented in Cioni (2008a) and Cioni (2008b) and that we term as iterative since they are based on iterative algorithms through which either one or both reveal the composition of the sets of items they are willing to barter.

Both models involve indeed a pair of actors/players A and B that aim at bartering a pair of items.

In the former model (the so called **pure model**) neither actor reveals to the other the set of items he is willing to barter but such a revelation occurs incrementally during the process since by exchanging proposals and counter proposals the two players reveal each other the composition of such sets. The bargaining process goes on until an agreement is reached and a bargaining occurs or both players agree that no bargaining is possible so that the process ends with a failure.

In the latter or **mixed model**, on the other hand, we have an asymmetric situation where one of the players, be it A, shows to B his set of items, be it I', on which the bargaining process may start with a proposal from B. Also in this case the process goes on with a series of proposals and counter proposals from both players until one of the foregoing cases occurs. In this case player B reveals the composition of his set of items during the course of the process.

We note that, in both cases, each player can be said to know which are the items he is willing to bargain in the barter process.

This knowledge may be verified in advance by asking to each player if a given proposal would be or not in his bargaining set (or the set of the acceptable proposals).

From this perspective we can say that each player is characterized by a bargaining set, I_A for A and J_B for B, whose structures and whose preference orderings are private knowledge of each player and can be only partly revealed during the barter process. The main difference between the two models that we propose is that in the latter model the set I_A is, at least partly, a common knowledge of the two players under the form of the set I'.

The paper is structured as follows.

It opens with some comments on the concept of barter as we use it in this paper then we present in sequence the criteria we plan to use to evaluate the proposed models and make some comments on the incremental construction or revelation of sets. Then we present the two models (and some of their possible applications) and discuss both the degree at which they satisfy our criteria. At this level we disclose the metaphor and discuss the utility of System Dynamics as a tool for the devising of a proposal and the evaluation of a counter proposal. The paper closes with a discussion of the possible extensions and a section devoted to partial conclusions and plans for future research.

2 Some notes about the barter

In this paper with the term **barter** we denote a process through which two players A and B can exchange a pair of items[1] (i, j) where both items are evaluated according to each player's private evaluation system that determines either his rejection or his acceptance of the proposed items.
The exchange, if it occurs, is in kind so that the items (i, j) are the only involved items and there is no parallel or compensatory exchange of money or any other numerary good between the players (see section 3).
In any generic pair (i, j) the identifier i identifies what passes from A to B either under the form of a good or a bad or a service in exchange of the item identified by j, of the same types, from B to A.
In Table 1 we show the possible ownership of the items of each pair depending on their types.
The admissible types are:

(a) **good** (or g) or an item that has a positive value for both players;

(b) **bad** (or b) or an item that has a negative value for both players;

(c) **service** (or s) or an item that has a instrumental value for one or both players and may represent a task that a player carries out for the other.

Each ownership is provided under the form of a pair (p_1, p_2) (p_1 for i and p_2 for j) where both identifier may be A or B. In this way we identify the provider or owner of an item so that, for instance, in correspondence of g, g we have (A, B) to mean that the former good is owned by A and provided to B whereas the latter is owned by B and provided to A so that, as receivers, we have (B, A) We now examine the entries of Table 1:

- in the case g, g we have A, B since A gives to B a good in exchange for another good of B;

- in the case g, b we have B, B since B gives to A both a good and a bad;

- in the case g, s we have A, B since A gives to B a good in exchange for a service from B;

- in the case b, g we have A, A since A gives to B both a bad and a good as a compensation;

[1] In what follows we sometimes refer to the former element of each pair (i, j) as the $i-$item and to the latter as the $j-$item.

- in the case b, b we have A, B since A gives to B a service and B gives to A another service in exchange;

- in the case b, s we have A, A since A gives to B both a bad and a service;

- in the case s, g we have A, B since A gives a service to B and receives from B a good as a compensation;

- in the case s, b we have B, B since B gives to A both a bad and a service;

- in the case s, s we have A, B since A gives to B a service and B gives to A another service in exchange.

i and j	g	b	s
g	A,B	B,B	A,B
b	A,A	A,B	A,A
s	A,B	B,B	A,B

Table 1: *Ownership of the elements of the pair (i,j)*

By using the content of Table 1 we can define another Table that, for each element of each pair (i, j), defines which player is the giver and which is the receiver. In Table 2 for each pair of items (i, j) we define:

id	case	i giver	i receiver	j giver	j receiver
1	g,g	A	B	B	A
2	g,b	B	A	B	A
3	g,s	A	B	B	A
4	b,g	A	B	A	B
5	b,b	A	B	B	A
6	b,s	A	B	A	B
7	s,g	A	B	B	A
8	s,b	B	A	B	A
9	s,s	A	B	B	A

Table 2: *Pairs (giver, receiver) for each item of each pair (i,j)*

- who is the giver of the item i and who is the corresponding receiver;

- who is the giver of the item j and who is the corresponding receiver.

For instance in correspondence of the pair $(i, j) = (g, g)$ we have A the giver of i that is received by B and vice versa for the item j.
In this way we can identify the items each player gets or loses in a barter so to assign a value to such acquisitions and losses. In section 3 we are going to use such values in the definition of the evaluation criteria that we are going to apply in section 7 to the models we are going to describe in section 5.
For player A this means that we imagine he uses two private values $v_A(j)$ and $v_A(i)$ to evaluate, on a private scale, what A gets from the barter and what A loses from it int that order.
In this way A can evaluate the ratio:

$$\rho_A = \frac{v_A(j)}{v_A(i)} \tag{1}$$

as a dimensionless quantity.
For player B, in a similar way, this means that we imagine she uses two private values $v_B(i)$ and $v_B(j)$ to evaluate, on a private scale, what B gets from the barter and what B loses from it int that order.
In this way B can evaluate the ratio:

$$\rho_B = \frac{v_B(i)}{v_B(j)} \tag{2}$$

as a dimensionless quantity.
As to the quantities that are involved in relations (1) and (2) we note that they:

- represent private information of each player;

- are measured according to private scales that may not be common knowledge between the players;

- include possibly independent discount factors for each player so to account for damages occurring to each of them from the passing of time.

Relations (1) and (2) are used respectively by player A and player B to accept or refuse a proposed barter (see section 5).
Before going on we must fix in some way four problematic cases that are contained in Table 2 and that we list again in Table 3.
We say that the case $(i, j) = (b, g)$ $(id = 2)$ is problematic[2] since it does not

[2]Similar arguments hold also for the other three cases.

id	case	i giver	i receiver	j giver	j receiver
2	g,b	B	A	B	A
4	b,g	A	B	A	B
6	b,s	A	B	A	B
8	s,b	B	A	B	A

Table 3: *Problematic cases*

present a A, B, B, A structure (a bidirectional transfer) but a structure that defines a transfer of both items from one player to the other (an unidirectional transfer). We can solve this case by imagining that the transfer of a bad from B to A is equivalent to the transfer of a good (of the proper compensating value) from A to B. In this way this case is brought back to the case with $id = 1$. In a similar way we can solve the case with $id = 4$.
For what concerns the case $(i, j) = (b, s)$ $(id = 6)$ we can solve it by imagining that the transfer of a bad from B to A is equivalent to the transfer of a service (of the proper compensating value) from B to A. In this way this case is brought back to the case with $id = 9$.
In a similar way we can solve the case with $id = 8$.

3 The performance and evaluation criteria

For the evaluation of the proposed barter procedures we use both a set of **performance criteria** and a set of **evaluation criteria**.
As to the **performance criteria** we use: **guaranteed success**, **individual rationality**, **simplicity** and **stability**.
With **guaranteed success** we denote the fact that a procedure is guaranteed to end with a success, with **individual rationality** we denote the fact that it is in the best interest of the players to adopt it so that they both use a procedure only if they wish to use it and can withdraw from it without any harm nor penalty higher than the damage they can receive from carrying it on.
Simplicity is a feature of the rules of a procedure that must be easy to understand and implement for the players without being too demanding in terms of rationality and computational capabilities.
Last but not least with **stability** we denote the availability to the players of equilibrium strategies that they can follow to attain stable outcomes in the sense that none of them has any interest in individually deviating from such strategies (Myerson (1991), Patrone (2006)).

As to the **evaluation criteria** we use a set of classical criteria (Brams and Taylor (1999), Brams and Taylor (1996)) that allow us to verify if a barter can be termed **fair** or not.
Such criteria are:

- envy-freeness;
- proportionality;
- equitability;
- [Pareto] efficiency.

We say a barter is fair if they are all satisfied and is unfair if any of them is violated.
In the case of two players (Brams and Taylor (1999), Brams and Taylor (1996)) envy-freeness and proportionality are equivalent, as it will be shown shortly.
Generally speaking, we say that an agreement turns into an allocation of the items between the players that is **envy-free** if (Brams and Taylor (1996), Brams and Taylor (1999) and Young (1994)) none of the actors involved in that agreement would prefer somebody's else portion, how it derives to him from the agreement, to his own. If an agreement involves the sharing of benefits it is considered envy-free if none of the participants believes his share to be lower than somebody's else share, whereas if it involves the share of burdens or chores it is considered envy-free if none of the participants believes his share to be greater that somebody's else share. In other words a procedure is envy-free if every player thinks to have received a portion that is at least tied for the biggest (goods or benefits) or for the lowest (burdens or chores).
If an allocation is envy-free then (Brams and Taylor (1999)) it is **proportional** (so that each of the n players thinks to have received at least $1/n$ of the total value) but the converse is true only if $n = 2$ (as in our case). If $n = 2$ proportionality means that each player thinks he has received at least an half of the total value so he cannot envy the other. If $n > 2$ a player, even if he thinks he has received at least $1/n-$th of the value, may envy some other player if he thinks that player got a bigger share at the expense of some other player.
As to **equitability** in the case of two players (and therefore in our case) we say (according to Brams and Taylor (1999)) that an allocation is equitable if each player thinks he has received a portion that is worth the same in one's evaluation as the other's portion in the other's evaluation. It is easy to see

how equitability is generally hard to ascertain (Brams and Taylor (1996) and Brams and Taylor (1999)) since it involves inter personal comparisons of utilities. In our context we tried to side step the problem by using a definition that considers both utilities with respect to the same player.
Last but not least, as to **[Pareto] efficiency**, we say (according to Brams and Taylor (1999)) that an allocation is efficient if there is no other allocation where one of the players is better off and none of them is worse off. In general terms efficiency may be incompatible with envy-freeness but in the case of two players where we have compatibility.
Such criteria, in order to be used in our context of two players without either any common scale or any numerary good, must be adapted or must be redefined someway so to be in agreement either with the essence of their classical definitions or with intuition or with both.
We start with **envy-freeness**.
If we denote with $v_A(j)$ and $v_A(i)$ the values in A's opinion and evaluation, respectively, of what A obtains and loses from the barter (and with $v_B(i)$ and $v_B(j)$ the same quantities for player B) we say that the allocation deriving from a barter (or a barter tout court) is **envy-free** if we have for A:

$$\rho_A = \frac{v_A(j)}{v_A(i)} \geq 1 \tag{3}$$

and for B:

$$\rho_B = \frac{v_B(i)}{v_B(j)} \geq 1 \tag{4}$$

Relation (3) means that the value that A assigns to what he gets from the barter is at least equal to the value that A assigns to what he loses from the barter. We assign a similar meaning to relation (4) with regard to B.
Since, in the case of two players, we want to maintain the equivalence between proportionality and envy-freeness we must give a definition that mirrors the classical definition of proportionality and reflects this equivalence.
For player A we may define a barter as proportional if it satisfies the following condition:

$$\frac{v_A(j)}{v_A(j) + v_A(i)} \geq \frac{v_A(i)}{v_A(j) + v_A(i)} \tag{5}$$

so that the fractional value of what A gets from the barter is at least equal to that of what he loses from it. We remark that $v_A(j) + v_A(i)$ represents the value that A assigns to the bartered items.
A similar condition holds also for B:

$$\frac{v_B(i)}{v_B(i) + v_B(j)} \geq \frac{v_B(j)}{v_B(i) + v_B(j)} \tag{6}$$

We say that a barter is proportional if both (5) and (6) hold.
It is easy to see how from equation (5) it is possible to derive equation (3) and vice versa. The same holds also for equations (6) and (4) so that the equivalence of the two definitions has been maintained in the case of two players.
We now pass to the criterion of **equitability**.
We must adapt its definition to our framework in the following way. We need firstly some definitions. With respect to the occurrence of the barter of the items (i, j) we define for player A:

- V_A a measure for A himself of his current welfare before the barter occurs;

- V'_A a measure for A himself of his current welfare after the barter has occurred;

and for player B:

- V_B a measure for B himself of his current welfare before the barter occurs;

- V'_B a measure for A himself of his current welfare after the barter has occurred.

With the term **welfare** we denote a personal and private evaluation from each player of his global situation (under the hypothesis of either additivity or superadditivity[3]) through a single value that is used to rank the items that are entering the barter process.
We therefore define a barter of the items (i, j) as **equitable** for A if the fractional value of what he gets is at least equal to the fractional value he gives to what he loses from the barter or:

$$\frac{v_A(j)}{V'_A} \geq \frac{v_A(i)}{V_A} \tag{7}$$

On the other hand the barter is equitable for B if, using the corresponding quantities we used in equation (7) but referred to player B, we have:

$$\frac{v_B(i)}{V'_B} \geq \frac{v_B(j)}{V_B} \tag{8}$$

[3]With reference to sets with the term **additivity** we denote the fact that the value of a set is given by the sum of the values of its components whereas if this value is at least equal to that sum we speak of **superadditivity** and of **strict superadditivity** if it is strictly greater.

If both relations hold we say that the barter is **equitable**.
We note that if $V'_A \geq V_A$ then relation (7) implies envy-freeness for A whereas if $V'_B \geq V_B$ then relation (8) implies envy-freeness for B.
To make inequalities (7) and (8) of more practical use we may rewrite them, for instance for player A, as follows (for every turn after the first one):

$$\frac{v_A(j)}{\downarrow_A} \geq \frac{v_A(i)}{\uparrow_A} \tag{9}$$

where:

> $\downarrow_A$ is the minimum value of all the $j-$items that A has bargained before the current turn of bargaining;
>
> $\uparrow_A$ is the maximum value of all the $i-$items that A has bargained before the current turn of bargaining.

We say that (9) is easier to use than relation (7) because keeping track of both a maximum and a minimum value in a multi step process is easier than evaluating at each step the new value of the welfare under the hypothesis that the proposed barter occurs.
A similar relation holds also for player B:

$$\frac{v_B(i)}{\downarrow_B} \geq \frac{v_A(j)}{\uparrow_B} \tag{10}$$

Last but not least we examine the criterion of **[Pareto] efficiency**.
A barter of the items (i, j) is **[Pareto] efficient** if there is not another pair of items (i', j') that gives at least to one player a better result without hurting the other, under the hypothesis that at least one of the following inequalities hold:

1. $i' \neq i$
2. $j' \neq j$

For players A and B this means that there is no barter (i', j') that satisfies the following inequalities:

$$\frac{v_A(j)}{v_A(i)} \leq \frac{v_A(j')}{v_A(i')} \tag{11}$$

$$\frac{v_B(i)}{v_B(j)} \leq \frac{v_B(i')}{v_B(j')} \tag{12}$$

with at least one of them satisfied with the $<$ relation.
In such relations the pairs $v_A(i)$, $v_A(j)$ and and $v_A(i')$, $v_A(j')$ are related to

A and are associated respectively to (i, j) and to (i', j'). Similar quantities are defined also for player B.
We note that if the barter is such that both players attain:

$$\frac{v_{A_{max_j}}}{v_{A_{min_i}}} \tag{13}$$

and

$$\frac{v_{B_{max_i}}}{v_{B_{min_j}}} \tag{14}$$

we are sure to have an **efficient barter** whereas if both attain:

$$\frac{v_{A_{min_j}}}{v_{A_{max_i}}} \tag{15}$$

and

$$\frac{v_{B_{min_i}}}{v_{B_{max_j}}} \tag{16}$$

we are sure that the barter is surely inefficient. In (13), (14), (15) and (16) we have:

$v_{A_{max_j}}$ is the best $j-$item that A can get from the barter;

$v_{A_{min_j}} < v_{A_{max_j}}$ is the worst $j-$item that A can get from the barter;

$v_{B_{max_i}}$ is the best $i-$item that B can get from the barter;

$v_{B_{min_i}} < v_{B_{max_i}}$ is the worst $i-$item that B can get from the barter.

We remark how conditions (13), (14), (15) and (16) are sufficient conditions of efficiency but may also be hints for either good or bad strategies for both players.
Last but not least, we note, from the equations (11) and (12), how **efficiency** of a barter cannot be always guaranteed and must be verified case by case.

4 Incremental construction/revelation

One of the key point of the proposed models is the fact that either one or both players reveal incrementally the set of pairs of items (i, j) each of them is willing to barter with the other. Each pair is seen as a single element of the set and from two pairs (i, j) and (h, k) we can obtain one of the following pairs:

- (i, h)

- (i, k)
- (h, j)
- (k, j)

and all their possible convex combinations, depending on the nature of the involved items.
It is therefore necessary to understand the ways through which a set is incrementally enlarged from the initial empty set to a maximal set, the so called bargaining set, that include all the possible elements that a player is willing to barter.
The first way we can use is the following (in what follows we consider the case of A, the case of B is analogous and will not be explicitly considered). A may start with $I_0 = \emptyset$ and add one element at a time according to some insertion criteria until a criterion of stop is met so that the process is interrupted and the final set I_∞ is constructed. In this way A builds up the following succession of sets:

$$I_0 \subset I_1 \subset \cdots \subset I_\infty \tag{17}$$

where the set I_∞ may, at least in theory, contain infinitely many elements.
In this way A proceeds **bottom up** since he starts from the empty set and eventually ends with the whole set of the items I_∞ that may coincide or not with I_A (since it contains also elements from the [counter] proposals of B).
This process gives to A the greatest flexibility since it allows him to build up new elements by mixing and or merging the existing ones so that the construction process can adapt better to the course of the barter process.
The main problem with this approach is that the barter may prove a very time consuming process since the number of the possible combinations increase exponentially with the increase of the number of the available elements.
Another way that A can use is the following that we may call **top down**. A starts with his fixed and predefined set I_A of n elements. Each item is initially set as invisible (so that again A starts with a publicly known set $I_0 = \emptyset$) and during the process one element at a time is set visible. In this way A builds up the following succession of sets:

$$I_0 \subset I_1 \subset \cdots \subset I_n \tag{18}$$

with:

$$I_n = I_A \tag{19}$$

This incremental disclosure may be obtained by using a set of n flags initially set at *invisible* and by setting at each step one flag at a time at *visible* so to

reveal the associated element.
The process ends when a barter occurs or when all the elements of the set I_n are revealed without any barter occurring.
In this case we get the lowest flexibility since the elements are fixed from the start but we are sure the process has a fixed bound that must occur when all the elements have been revealed without any barter occurring.
A this point we have to define:

- in the bottom up approach, which criterion can be used by player A (and similarly by player B) to add a new element to the current set I_i;

- in the top down approach, which criterion can be used by player A (and similarly by player B) to set as visible a new element to the current set I_i.

In both cases the simplest criterion is the following: a player either adds or sets visible an element that is expected to give him an advantage greater than the one deriving him from the current proposal.
We recall that such an element represents the new [counter] proposal and so an advantageous elements for the proposing player (otherwise he would not propose it).
The other point is to clarify how player A (but the same is true also for player B) can derive at step i the element (h, k) such that:

$$I_i = I_{i-1} \cup (h, k) \tag{20}$$

In the bottom up approach such an element can be either an element of I_A or an element composed by using elements from J_{i-1} or a mixture of both cases.
In the top down approach such an element is simply one of the elements of I_A.

5 The proposed barter models

In this paper we propose two iterative barter models. In the former model, that we call **pure model**, neither A nor B shows each other the sets of the items they wish to barter.
In the latter model, that we call **mixed model**, we suppose that one player, be it A, shows the items he is willing to barter from the offset of the process whereas the other, in this case B, keeps her items hidden but reveal them during the process by making either proposals or counter proposals aiming at the reaching of an agreement and therefore a barter.

5.1 General remarks

Both models are described by using simple algorithms of which we present the general structure and the various options that the each of two players has at each step. In order to keep the structure of each algorithm simple and readable we may use strings to describe sub procedures that we verbally describe separately.
Since both models are based on a succession of proposals and counter proposals we firstly need to define what do we mean with the terms proposal and counter proposal. We also list which are the moves that each player can use during the process.
A **proposal** is a pair of item identifiers (i, j) that a player proposes to the other as the object of the barter and whose ownership is defined as we have seen in Table 1.
On the other hand, given a proposal (i, j) a **counter proposal** is a pair (i', j') such that either $i' \neq i$ or $j' \neq j$ is true since:

- if $i' = i$ and $j' = j$ we have an implicit acceptance so that the process ends with a success;

- if $i' \neq i$ and $j' \neq j$ we define it as a new proposal.

It is obvious that a proposal in reply to a counter proposal is termed a counter proposal and not a counter counter proposal so to avoid the chaining of counter prefixes.
Within our perspective we have that:

- a counter proposal may follow only a proposal;

- a new proposal may follow either a pass move or a reject move (see further on).

Both a proposal and a counter proposal can be followed by one of the following moves from the listening player:

- **pass**,

- **give up**,

- **accept**,

- **reject**.

A **pass** move is a way through which a player may signal to to the other that it is necessary that he shows some more goodwill in order for the process to go on.
A **give up** move is a way for one player to signal to the other that he thinks the process is not worth being carried on any more.
We note that two successive give up moves (one from each player) cause the process termination with a failure.
An **accept** move closes the barter with a success since it signals that a player accepts the last [counter] proposal made by the other player.
A **reject** move means that the received [counter] proposal made by the other player cannot be accepted.
Both an **accept move** and a **reject move** can follow any [counter] proposal but if the answer is a reject move the turn remains to the rejecting player that can make his counter proposal.
A **pass move** gives the turn to the other player and may be answered by either a new proposal, by a pass move or by a give up move. We note that there cannot be more than two consecutive pass move so that a natural succession for a closure with a failure may be: $pass_A$, $pass_B$, $give\,up_A$, $give\,up_B$.
Last but not least a give up move may be followed either by a new proposal or by another give up move from the other player: in the former case the process goes on whereas in the latter it is interrupted with a failure and in a way that does not necessarily involve the use of pass moves.

5.2 The pure model

In the case of the **pure model** the situation we are interested in can be described in the following terms.
We have one player that wants to exchange an item with another player but none of them has a knowledge of the items the other is willing to barter.
The only way to proceed is through an iterative process. At each step of the process a pair of items (i, j) is proposed and such a pair may be either accepted or refused in some way.
In the former case the process ends with a success so that the barter occurs.
In the latter case we may have:

- a pass move so the next move is up to the other player,
- a reject move so the next move is up to the same player;
- a counter proposal.

At the beginning of the barter we have:

the set of pairs-of-items-to-be-bartered of A is I_0;

the set of pairs-of-items-to-be-bartered of B is J_0.

We therefore have an *initialization phase* where we put:

- $I_0 = \emptyset$, $J_0 = \emptyset$, $i = 1$ and $j = 1$;
- we select at random who moves first, be it A. The other case being fully symmetrical will not be examined here.

In the description of the algorithm we use the notation $propose_A$ to summarize the execution of the following steps:

1. A presents a [counter] proposal $p_A = (i, j)$;
2. $I_i = I_{i-1} \cup p_A$;
3. $i = i + 1$;

and symmetrically we use the notation $propose_B$ to summarize the execution of the following steps:

1. B presents a [counter] proposal $p_B = (i', j')$;
2. $J_j = J_{j-1} \cup p_B$;
3. $j = j + 1$;

The main structure of the algorithm in this case is the following:

(0) *initialization phase*;

(1) $propose_A$;

(2) B may:

- (2_a) *accept*; *go to (4)*;
- (2_b) *reject*; $propose_B$; *go to (3)*;
- (2_c) $propose_B$; *go to (3)*;
- (2_d) *give up*; go to (5);
- (2_e) *pass*; go to (6);

(3) A may:

- (3_a) *accept*; *go to (4)*;

(3_b) *reject*; *propose*$_A$; *go to (2)*;

(3_c) *propose*$_A$; *go to (2)*;

(3_d) *give up*; go to (5);

(3_e) *pass*; go to (6);

(4) end;

If we denote with the player i either A or B and with j either B or A we can define the moves that can follow either a *pass* or a *give up* move in the following ways:

(5) a *give up*$_i$ move may be followed by:

(5_a) *propose*$_j$; go to (3) if $j = B$ else go to (2);

(5_b) *give up*$_i$; go to (4);

(6) a *pass*$_i$ move may be followed by:

(6_a) *propose*$_j$; go to (3) if $j = B$ else go to (2);

(6_b) *pass*$_j$; *give up*$_i$; *give up*$_j$; go to (4);

(6_c) *give up*$_j$; go to (5);

The execution of either an *accept* or a *reject* move on a proposal (i, j) from player A (the case of player B is fully symmetrical and will not be analyzed) is based on the consideration of the values $v_A(j)$ and $v_A(i)$ through a function:

$$eval_A(i, j) = f_A(v_A(j), v_A(i)) \tag{21}$$

where the function f_A synthesizes the procedure of comparison from A between the values $v_A(j)$ and $v_A(i)$. In its simplest form we can express it as:

$$f_A(v_A(j), v_A(i)) = v_A(j) - v_A(i) \tag{22}$$

Such function can be used in rules such as the following[4]:

$$\textbf{if}(eval_A(i, j) \geq 0)\ \textbf{then}\ accept_A\ \textbf{else}\ refuse_A \tag{23}$$

so to establish a strict preference ordering $\succ$ on the proposals. We can indeed say[5]:

$$(i, j) \succ_A (i', j') \Leftrightarrow eval_A(i, j) > eval_A(i', j') \tag{24}$$

and the same holds also for B.

[4]We could have $eval_A(i, j) \geq \varepsilon$ with $\varepsilon > 0$ if a minimum gain is required or $\varepsilon < 0$ if a maximum loss is acceptable.

[5]It is obvious that with $\succ_A$ we denote the strict preference relation of player A.

5.3 The mixed model

In the **mixed model** we have an asymmetric situation where one of the players, be it A, shows to the other, B, his set of items I' whereas the first move is up to the other player, B in this case.
The algorithm in this case has the following structure that is very similar to the one we have seen in section 5.2 but for the *initialization phase* and the starting move.

(0) *initialization phase*;

(1) $propose_B$;

(2) A may:

- (2_a) *accept*; *go to (4)*;
- (2_b) *reject*; $propose_A$; *go to (3)*;
- (2_c) $propose_A$; *go to (3)*;
- (2_d) *give up*; go to (5);
- (2_e) *pass*; go to (6);

(3) B may:

- (3_a) *accept*; *go to (4)*;
- (3_b) *reject*; $propose_B$; *go to (2)*;
- (3_c) $propose_B$; *go to (2)*;
- (3_d) *give up*; go to (5);
- (3_e) *pass*; go to (6);

(4) end;

For the points (5) and (6) and for the description of the $propose_A$ and $propose_B$ moves we refer to the same points and the same description we gave in section 5.2.
The *initialization phase* is unchanged for B (so that again we have $J_0 = \emptyset$ and $j = 1$) but for A we must account for the presence of the set I' whose content is defined by A freely and completely at his will for what concerns both the type and the number of the contained items.
From this we have:

$$I_0 \subset I' \otimes I' \qquad (25)$$

where:

$$I' \otimes I' = \{(i,j) \mid i,j \in I' \; but \; i \neq j\} \tag{26}$$

so that I_0 is rather fuzzily defined for B so reducing his seemingly initial advantage over A.

6 Possible strategies

We now present and analyze the possible strategies of the two players in either the pure model or the mixed model.
In the **pure model** both players act without any knowledge of the other's player set of elements but with each player only knowing his bargaining set, I_A if A and J_B if B.
In this case the player who moves first, be it A, can do no better than choosing an element (i,j) from his bargaining set I_A so to maximize the value:

$$eval_A(i,j) = f_A(v_A(j), v_A(i)) \tag{27}$$

This is possible for A if he chooses one of the two cases $id = 4$ or $id = 6$ of Table 2 where he has full control of both the items of the proposed element $I_1 = \{(i,j)\}$.
In his turn B may use both J_B and I_1 to devise his [counter] proposal to A so defining his set J_1 after his first step.
At the generic step $i > 1$ for A and $j > 1$ for B we have that each player:

- knows the current proposal (i,j) and is able to evaluate it using the function[6] $eval_h$;
- can possibly devise a [counter] proposal (i',j') and propose it if it is preferred by the player to the current one according to the preference ordering we introduced with relation (24) or if:

$$(i',j') \succ_h (i,j) \Leftrightarrow eval_h(i',j') > eval_h(i,j) \tag{28}$$

- can frame the new [counter] proposal on the knowledge of both I_i and J_j so to favor at the best also the other player (therefore increasing the probability of an acceptance) by using the common knowledge between them;

[6] With the subscript h we denote either the player A or the player B depending on the case.

- if the [counter] proposal of the foregoing steps cannot be devised the player can either accept the current proposal, press the other for a change of route (with a *pass* move) or signal the will to stop the process (with a *give up* move).

As the last step we note how each player can use also his bargaining set for framing a new [counter] proposal but with a lower probability of if being accepted by the other player.
In the **mixed model** the presence of an asymmetry does not modify very much what we have seen before since the initial knowledge of B of the items of A is rather fuzzy and, moreover, she is the player who must move first. We can, therefore, state that the strategies we have seen for the **pure model** case can be profitably used also in the n the **mixed model** case.

7 Satisfaction of the criteria and applications

We now examine if the proposed models satisfy or not the performance and evaluation criteria we have introduced in section 3.
For what concerns the **performance criteria** we have that:

- **guaranteed success** is not always satisfied since a process of barter may end without any effective barter and without any penalty for the player who withdraws from the process;

- **individual rationality** is guaranteed to each player that feels he engaged himself in an unfavorable process from the availability of *pass* and *give up* moves;

- **simplicity** is assured by the fact that the frame of both algorithms is a sequence of proposals and counter proposals until an agreement is reached either to have a barter or to give up since no barter is effectively possible;

- **stability** is satisfied since both players have easy to follow and implement strategies (see section 6).

We recall that the acceptance or not of a proposed barter is governed by the relations (21), (22) and (23) of section 5.2.
For what concerns the **evaluation criteria** we have that if a barter occurs, according to the previously recalled relations, then it is **envy-free** (see relations (3) and (4)) for both players and therefore it is also **proportional**, from the equivalence of the two concepts in the case of two players.

It is easy to verify, by the same relations, that if a barter is fair then it occurs.
The check of the **equitability** requires the check of the relations (7) (or of the corresponding relation (9)) and (8) (or of the corresponding relation (10)).
Such a check may be performed by both players at every step of the barter so that, in the best case:

- one player may propose a barter that for him is envy-free and equitable;
- the other may verify such a proposal and accept if he too thinks that for him it is envy-free and equitable.

As to the [**Pareto**] **efficiency** we need to verify the the existence or not of a pair (i', j') of items that satisfy the relations (11) and (12)). As we have already seen both players have sufficient conditions to attain an efficient barter even if such conditions require a co-operative attitude between the players and such an attitude is not self-enforcing (Myerson (1991)) so that both players may have (more or less plausible reasons) for individually deviating from it. Such conditions, moreover, may be conflicting so that they cannot drive effectively a strategy. Moreover the check of the relations (11) and (12)) may be impossible since not all the possible elements (i, j) are known during the barter so that the inefficiency of a barter may be discovered only when both players have accepted it.
Summing up, we can say that:

- envy-freeness is guaranteed every time a barter occurs,
- proportionality is guaranteed every time a barter occurs,
- equitability may be guaranteed at every barter,
- efficiency may be attained, since the players have sufficient conditions for attaining it, but may also be easily missed.

From these considerations we have that the **fairness** of a barter is a by-product of the barter process itself and is not a-priori guaranteed by its structure.
From all this we derive that each of the two players may judge an occurred barter as either inequitable or inefficient or both and therefore unfair, since not all the fairness criteria are verified.
Last but not least we comment a little on the possible **applications** of the proposed models.
The **pure model** can be applied in all the cases where two players meet

to perform a barter but none of them has a knowledge of the items that the other may be willing to barter so that this knowledge must be acquired during the barter process itself through a try-and-error process.
In the current version of the pure model we imagined the two players as peers in the barter, this fact being represented by the random move for the choice of the first time mover. It is easy to remove this feature by giving, for some reason, the right to make the first move to one of the two players. For the range of the possible applications we refer to the tables we provided in this paper, mainly table 2, from where it is possible to understand the nature of the items that the two players may wish to exchange.
On the other hand, the **mixed model** can be used in all the cases where one of the two players cannot conceal or is forced to show the initial set of his items so that the other has some information about the possible advantageous (for him) proposals that he can make without even revealing any of the items of his private set.
Also in this case we refer to the tables we provided in this paper, mainly table 2, from where it is possible to understand the nature of the items that the two players may wish to exchange.

8 Possible extensions

Up to now we have seen the basic models involving a pair of players in a one shot barter for the exchange of a pair of items.
In this section we briefly present the following planned extensions:

- (e_1) the possibility of having more than two players i.e. a plurality of players;
- (e_2) the possibility of performing repeated barters;
- (e_3) the possibility of performing multi-pairs barter.

In the following subsections we examine the various extensions singularly and one independently from the others though it is obvious that they could be combined together in various ways. The treatment is at an introductory level since such extensions represent the core of a future research stream.

8.1 A plurality of players

Instead of a pair of players A and B we may define a set $\mathscr{P}$ of $n \geq 4$ players (with n even[7]) that can form $n/2$ pairs (so to have $n/2$ contemporaneous

[7] In abstract term we might have two cases:

barters) in the following ways:

- by a random selection,
- by raising up of hands,
- by mutual selection.

In the case of **random selection** we may imagine a procedure that:

(0) starts with $i = n$;

(1) at the $i-$th step it chooses at random one of the $i(i-1)/2$ values so to match a pair of players;

(2) if $i = 2$ go to (3) else $i = i - 2$ go to (1);

(3) end;

At step (1) we may imagine to have a dice with i faces so that the outcome of $j \in [1, i]$ corresponds to one of the possible $i(i-1)/2$ pairings. We have to see how to assign the possible pairings to the faces of the dice. To do so we can set up a $n \times n$ matrix V with all the elements at 0 but those above the main diagonal that assume (row by row form left to right) the increasing integer values from 1 (for the element $v(1, 2)$) to $n(n-1)/2$ (for the element $v(n-1, n)$) so that $v(i, j) \neq 0$ for $j > i$.
After each step we remove the two matched players, renumber the remaining players, reduce accordingly the matrix V, reassign the values $v(i, j)$ with the same rule and repeat the procedure.
In the case of **raising hands** we can imagine that one player raises up one of his hands. If more that one player raises then each of them lowers his hand, waits for some random amount of time and then raises again his hand. This goes on until there is only one hand up at a time. At this point one or more of the others may join him to form a pair for a barter. If more than one player express the wish to join there is a random selection of one of them. The process goes on until $n/2$ pairs have been formed, at each step the number of waiting players being decremented by 2.

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(1) n is even so $n/2$ pairs of players form and no player is left out;

(2) n is odd so $n/2$ pairs form but one player is left out.

In this section we assume that n is even since the case n odd has no sense but in cases of repeated barters (see section 8.2) where the player who is left out at one stage gains some precedence at the next stage according to a form of balanced alternation (Brams and Taylor (1999)).

In the case of **mutual selection** the players are divided, through the use of any suitable random device[8], in two subsets of $n/2$ elements each: the former P_1 contains the choosers and the latter P_2 the choices.
Each player of P_1 chooses at his will a player from P_2: if the latter accepts the pair forms otherwise the two players switch from one set to the other. If there are multiple selections the choice is up to the player of P_2 that is forced to select one of his choosers. The matched players are removed from the respective sets.
When, according one of the foregoing ways, the $n/2$ pairs are formed each of them may use one of the algorithms we have seen in section 5 to perform a barter.
The basic idea is that the n players are willing to engage each other in a barter so the $n/2$ pairs form even if there is no guarantee that this turns out in $n/2$ effective barters.

8.2 Repeated barters

In this case we may use some concepts from Game Theory (Myerson (1991)) and consider each barter as a single stage of the potentially endless process of repeated barters between two players A and B.
To get this chaining effective we need to make some changes to the models of section 5. Such changes include:

- the addition of moves to implement the chaining;
- the introduction of state variables through which each player records the attitude of the confronting player during the previous stages (Axelrod (1985), Axelrod (1997));
- the definition of a barter as either with memory or memoryless.

As to the **moves** we need:

- to modify the **accept** move in a **accept and stop** move so to signal a lack of will to go on with the barter;
- to add the **accept and repeat** move so to signal to the other player the willingness to the execution of one more stage;
- to add the **repeat?** move so to allow a player to request to the other the execution of one more stage;

[8]We can imagine a urn containing n balls, $n/2$ red and $n/2$ blue, and n selections without replacement for the assignment of each extracting player to one of the two subsets according the color of the extracted ball.

- to add a **repeat** move so to answer affirmatively to an explicit or implicit request for one more stage;

- to add a **refuse** move so to answer negatively to an explicit or implicit request for one more stage.

so to allow the players to link two stages together in a repeated barter.
The **state variables** represent private information of each player through which he may record, from one stage to another, if the attitude of the other player at the previous stages has been more cooperative or more exploitative. In this way each player may implement long run strategies of retaliation so to punish spiteful attitudes.
Last but not least we define a barter as **with memory** if at the end of each stage the composition of the revealed set is a valid common knowledge (Myerson (1991)) between the players whereas it is defined as **memoryless** if it is lost and has no common knowledge value since each player is free to modify at his will the composition of his set without any notice to the other. This feature of a repeated barter must be agreed upon by both players at the very offset of the process, may be changed only after a new mutual agreement after each stage is over and forms a common knowledge between the two players.

8.3 Multi pairs barter

In this case every player at each step may propose more than one pair of items to be bartered. For instance, A may propose:

$$(i_j, j_l)(i_h, j_k) \tag{29}$$

and B can reply with a counter proposal with even more or less elements or can even accept. The acceptance might be:

- **partial** if the agreement can involve only a subset of a [counter] proposal;

- **global** if the agreement involves a [counter] proposal as a whole.

Partial agreements are fully meaningful only within a repeated barter setting whereas in a one shot barter we usually speak of global multi pairs agreement whenever the two players agree on bartering all the items contained in one proposal.

9 Concluding remarks

In the present paper we have introduced two algorithms that can be used in the case of one shot barter between a pair of players.
In the former algorithm we have a symmetric situation where the two players try to conclude a barter through an incremental revelation of the sets of items each of them is willing to barter.
In the latter algorithm, on the other hand, the situation is asymmetrical since one of the player either does not want to or cannot hide or conceal the set of his items to the other player that therefore has an initial advantage and the right to make the starting offer.
The paper presented both a description of the algorithms and their evaluation according to well defined classical performance and evaluation criteria.
In the closing part of the paper we also presented some possible extensions whose analysis and formalization are still to be completed. Similarly we need to complete the analysis of the possible applications of the proposed models to real world cases where exchanges of items occur between players that aim at attaining their objectives without sharing any common scales of qualitative or quantitative values.

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