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Auctions and barter

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Abstract

In this technical report we face the problem of the fair sharing of **goods**, **bads** and possibly **services** (also collectively termed **items**) among a set of players that cannot (or do not want to) use a common cardinal scale for their evaluation owing to the very qualitative and non economical nature of the items themselves.

To solve this problem we present two families of protocols (barter protocols and auction protocols) and use a set of classical fairness criteria (mainly for barter protocols) and performance criteria (mainly for auction protocols) for their evaluation.

The protocols are either based on auctions mechanisms or on barter mechanisms and are presented in detail, discussed and evaluated using the suitable fairness and performance criteria.

Keywords: allocations, auctions, barters, fairness criteria, performance criteria

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Introductory remarks

This technical report is a revised and enriched version of the paper of the same title accepted for presentation in the PhD session of the EAEPE (European Association for Evolutionary Political Economy) Conference to be held in Amsterdam, from Friday 6 until Sunday 8 November 2009. It also contains portions of the following papers:

- a paper written with Giorgio Gallo, entitled “Goodwill hunting: how to allocate bads or disagreeable chores” and accepted at 23rd European Conference on Operational Research that was held in Bonn, Germany, on July 5-8, 2009
- a paper entitled “Iterative barter models” that has been accepted at the Conference “S.I.N.G. 5” that was held in Amsterdam, Holland, on July 1-3 2009,
- a paper entitled “[Additive] Barter models” that has been accepted at the Conference S.I.N.G.4 that was held at Wroclaw, Poland, on June 26-28, 2008.

Last but not least, in the Appendix A we present also unpublished materials concerning an auction mechanism that we call “candle auctions” whereas in Appendix B we list some mathematical facts and findings.

1 Introduction

In this technical report we face the problem of the fair sharing of **goods**, **bads** and possibly **services** (also collectively termed **items**) among a set of players that cannot (or do not want to) use a common cardinal scale for their evaluation owing to the very qualitative and non economical nature of much of the items themselves.

To solve this problem we present two families of protocols (barter protocols and auction protocols) and use a set of classical fairness criteria (mainly for barter protocols) and performance criteria (mainly for auction protocols) for their evaluation.

As to the **fairness criteria** (Brams and Taylor (1999) Brams and Taylor (1996)) we use envy-freeness, proportionality, equitability and [Pareto] efficiency with some modifications and adjustments in order to make them suitable for the new contexts.

The **performance criteria** that we use include: guaranteed success, [Pareto] efficiency, individual rationality, stability and simplicity.

As to the families of protocols we have a **family F_1** of protocols that are based on auctions mechanisms and that can involve any number of players as an auctioneer and a set of bidders and a **family F_2** of protocols that are based on barter mechanisms and that involve a pair of players at a time but can involve an arbitrary number of such pairs.

All these protocols are presented in detail, discussed and evaluated using the suitable fairness and performance criteria.

The technical report closes with a section devoted some concluding remarks and to future research plans.

2 The family F_1

The family F_1 contains three types of auction mechanisms (Cioni (2008a), Cioni (2008b) and Cioni (2009b)):

- (a_1) a sort of Dutch auction with negative prices/bids,
- (a_2) a sort of English auction with negative prices/bids,
- (a_3) a sort of first price auction with negative prices/bids.

In mechanism (a_1) the auctioneer tries to allocate a bad to one bidder by rising his offer up to a maximum value M whereas in mechanism (a_2) the auctioneer starts with an offer L and the bidders make lower and lower offerings until one of them wins the auction and gets the bad and the money. We call such mechanisms **positive auctions** since the bidders bid to get the auctioned item.

In mechanism (a_3) the bidders bid for not getting a bad¹ that is assigned to the losing bidder (the one who bid less than the others) together with a compensation from all the other bidders. We call such mechanism a **negative auction** since the bidders bid in order of not getting the auctioned chore.

Of each mechanism we provide a description and the best strategy. Once the mechanisms have been described we also prove how the first two mechanisms are really equivalent and discuss some relations between them and the last one. We also apply the performance criteria to such mechanisms for their evaluation and prove under which conditions they are satisfied.

3 The family F_2

The family F_2 contains two subfamilies of models that we present in their basic two players A and B version.

The former subfamily contains a set of **explicit barter models** whereas the latter subfamily contains an **implicit barter model** and a **mixed barter model**.

In the **explicit barter models** the players A and B show each other the set of items that each of them is willing to barter within a procedure that is characterized by either simultaneous or sequential requests from one player to the other in which the barter may involve either a single item or a subset of items.

An explicit barter is an iterative procedure that may end either with a success

¹We use as a synonym also the term *chore*.

(and so with an exchange of items) or with a failure but, at each step, may also involve a reduction of the items each player is willing to barter.

In the **implicit barter model** none of the players shows his items to the other so that each player, in his turn, proposes to the other a pair of items (i, j) that he is willing to barter so that the other may either accept or reply with a counter proposal. The barter ends when an agreement is reached or both agree to give up since they decide that no barter is possible. During the barter each player reveals to the other the items he is willing to barter and this can ease the reaching of an agreement.

Last but not least in the **mixed barter model** we have that one player (be it A) shows his items to B that, on the other hand, behaves as in the implicit case. Also in this case the barter goes on as a series of proposals and counter proposals with an incremental definition of the bartering set of the player B . The implicit barter model and the mixed barter model are classified, in this technical report, as **iterative barter models**.

4 The classical criteria

In this section we recall the classical definitions of both the evaluation and the performance criteria as they are found in the literature. Such criteria will be specialized, whenever needed, for the various models we introduce in this technical report.

4.1 The performance criteria

As **performance criteria** we use: **guaranteed success**, **individual rationality**, **simplicity**, **stability** and **[Pareto] efficiency** (that we define in section 4.2).

With **guaranteed success** we denote the fact that a procedure is guaranteed to end with a success, with **individual rationality** we denote the fact that it is in the best interest of the players to adopt it so that they both use a procedure only if they wish to use it and can withdraw from it without any harm or a penalty greater than their potential damage.

Simplicity is a feature of the rules of a procedure that must be easy to understand and implement for the players without being too demanding in terms of rationality and computational capabilities.

Last but not least with **stability** we denote the availability to the players of equilibrium strategies that they can follow to attain stable outcomes in the sense that none of them has any interest in individually deviating from such strategies (Myerson (1991), Patrone (2006)).

4.2 The evaluation criteria

As **evaluation criteria** we use a set of classical criteria (Brams and Taylor (1999), Brams and Taylor (1996)) that allow us to verify if a barter can be

termed **fair** or not.

Such criteria are:

- envy-freeness;
- proportionality;
- equitability;
- [Pareto] efficiency.

We say a barter is fair if they are all satisfied and is unfair if any of them is violated.

In the case of two players (Brams and Taylor (1999), Brams and Taylor (1996)) envy-freeness and proportionality are equivalent, as it will be shown shortly.

Generally speaking, we say that an agreement turns into an allocation of the items between the players that is **envy-free** if (Brams and Taylor (1996), Brams and Taylor (1999) and Young (1994)) none of the actors involved in that agreement would prefer somebody's else portion, how it derives to him from the agreement, to his own. If an agreement involves the sharing of benefits it is considered envy-free if none of the participants believes his share to be lower than somebody's else share, whereas if it involves the share of burdens or chores it is considered envy-free if none of the participants believes his share to be greater than somebody's else share. In other words a procedure is envy-free if every player thinks to have received a portion that is at least tied for the biggest (goods or benefits) or for the lowest (burdens or chores).

If an allocation is envy-free then (Brams and Taylor (1999)) it is **proportional** (so that each of the n players thinks to have received at least $1/n$ of the total value) but the converse is true only if $n = 2$ (as in our case). If $n = 2$ proportionality means that each player thinks he has received at least an half of the total value so he cannot envy the other. If $n > 2$ a player, even if he thinks he has received at least $1/n$ -th of the value, may envy some other player if he thinks that player got a bigger share at the expense of some other player.

As to **equitability** in the case of two players (and therefore in our case) we say (according to Brams and Taylor (1999)) that an allocation is equitable if each player thinks he has received a portion that is worth the same in one's evaluation as the other's portion in the other's evaluation. It is easy to see how equitability is generally hard to ascertain (Brams and Taylor (1996) and Brams and Taylor (1999)) since it involves inter personal comparisons of utilities. In our context we tried to side step the problem by using a definition that considers both utilities with respect to the same player.

Last but not least, as to **[Pareto] efficiency**, we say (according to Brams and Taylor (1999)) that an allocation is efficient if there is no other allocation where one of the players is better off and none of the others is worse off. In general terms efficiency may be incompatible with envy-freeness but in the case of two players where we have compatibility.

5 The positive auctions

5.1 Introduction

We present two types of **positive auctions** where a seller $A \notin \mathcal{B}$ offers a chore to a distinct set \mathcal{B} of buyers/bidders so to sell/allocate it to one of them with a mechanism where the seller gives away the proper sum of money and the chore and the selected buyer accepts that sum of money and the chore.

As a seller A wants to maximize his revenue so wishes to pay the lowest sum of money to allocate the chore to one $b_i \in \mathcal{B}$.

On the other hand each bidder/buyer $b_i \in \mathcal{B}$ wants to get the chore according to each one's evaluation of it, evaluation that subjectively include losses and gains and that is a private information of each bidder.

Such mechanisms coincide with the usual mechanisms of buying and selling if we imagine negative payments so that the seller gives away the chore for a negative sum of money and the selected buyer accepts the chore but pays for it a negative sum of money.

For this reason we speak of a **positive auction** if the bidders bid for getting a chore in contrast with a **negative auction** mechanism (see section 6) where the bidders bid for not getting the chore that, in all the cases, is allocated through the application of simple and common knowledge rules.

5.2 The auction mechanisms

We present two algorithms that can be used in all the cases where the auctioneer wants to “sell a chore” to the “worst offering” or to have a chore carried out by somebody else by paying him the least sum of money².

In the former mechanism the auctioneer offers a chore and a sum of money m and raises the offer (up to an upper bound M) until when one of the bidders accepts it and gets both the chore and the money. The auction ends if either one of the bidders calls “stop” or if the auctioneer reaches M without any of the bidders calling “stop”. In the latter case we have a void auction sale. The auctioneer has a maximum value M that he is willing to pay for having somebody else carry out the chore otherwise he can either give up with the chore or choose a higher value of M so to repeat the auction with a possibly different (new or wider) set of bidders.

This type of auction is a sort of **Dutch auction with negative bids** paid by the bidders to get the chore.

In the latter mechanism the auctioneer offers the chore and fixes a starting sum of money L . The bidders start making lower and lower bids. The bidder who

²Of course when an auction is over and a bidder has got the chore and the corresponding sum of money there is the risk that the chore is not carried out. The analysis and resolution of such problems is out of the scope of an auction mechanism. We can imagine the presence of binding agreements for the bidder who gets the chore that turn into either reinforcing rules or penalties. Among the reinforcing rules we can imagine a linkage between the payment and the degree of fulfillment of the chore with the full payment occurring only if and when the chore has been fully accomplished.

bid less gets the chore and the money. Of course the auctioneer has no lower bound. Under the hypothesis that the bidders are not willing to pay for getting the chore we can suppose a lower bound $l = 0$. If this hypothesis is removed we can, at least theoretically, have $l = -\infty$. We can have a void auction sale if no bidder accepts the initial value L . The auctioneer can avoid this by fixing a high enough value L . In this case the bidders are influenced by the value of L that can act as a threshold since if it is too low none of them will be willing to bid. This case is as if the bidders start bidding from $-L$ and raise their bids up to $-l$ so that the one who bids the most gets the chore and pays that negative sum of money. In this case we have a sort of **English auction with negative bids**.

5.3 Dutch auction with negative bids

In this section we examine the mechanism³ where the auctioneer offers the chore and a sum of money and raises the offer (up to an upper bound M) until when one of the bidders accepts it and gets both the chore and the money.

The auction we are describing is a sort of reverse Dutch auction where we have an increasing offer instead of a decreasing price and a chore instead of a good. The value M represents the maximum amount of money that the auctioneer is willing to pay to get the chore performed by one of the bidders. We note that the value M is a private information of the auctioneer and is not known by the bidders. This fact prevents the formation of consortia and the collusion among bidders since M may be not high enough to be gainful for more than one bidder (see also section 5.6).

If x is the current offer of the auctioneer A we can see $M - x$ as a measure of his utility.

As to the bidders b_i , each of them has the minimum sum he is willing to accept m_i as his own private information so that $x - m_i$ may be seen as a measure of the utility of the bidder b_i .

We note that, if we define the set:

$$F = \{i \mid m_i \leq M\} \quad (1)$$

as the feasible set, the problem may have a solution only if $F \neq \emptyset$.

In this case the algorithm is the following:

1. A starts the game with a starting offer $x_0 < M$;
2. bidders b_i may either accept (by calling “stop”) or refuse;
3. if one b_i accepts⁴ the auction is over, go to 5 ;
4. if none accepts and $x_i < M$ then A rises the offer as $x_{i+1} = x_i + \delta_i$ with $0 < \delta_i < M - x_i$, go to 2 otherwise go to 5;

³We call it also the ascending mechanism or the ascending case.

⁴Possible ties may be resolved with a random device.

5. end.

The **best strategy for A** is to use a very low value of x_0 (or $x_0 \simeq 0$) so to be sure to stay lower than the lowest m_i and, at each step, to rise it of a small fraction δ_i with the rate of increment of δ_i decreasing the more x approaches M . Though this strategy may indirectly reveal to the bidders the possible value of M it is of a little harm to A since in any case no bidder is willing to accept the chore for a value lower than his own value m_i .

The **bidder b_i 's best strategy** is to refuse any offer that is lower than m_i and to accept when $x = m_i$ since if he refuses that price he risks to lose the auction in favor of another bidder who accepts that offer.

We have moreover to consider what incentives a bidder may have to act strategically when defining his value m_i . Of course there is no reason for b_i to define a value of m_i lower than the real one (since he has no interest in accepting lower prices). He could be tempted to define a higher value $m'_i > m_i$ so losing the auction in favor of all the bidders who are willing to accept any offer within the range $[m_i, m'_i]$. This means that b_i may use a value higher than m_i only if he is sure that the private values of all the other bidders are still higher. Since no bidder can be sure of this, each of them has a strong incentive to behave truthfully.

In this case, if $F \neq \emptyset$ (see relation (1)), the sum A would expect to pay is equal to m_j where $j \in F$ is such that $m_j < m_i$ for all $i \neq j$, $i \in F$. Of course A does not know such a sum in advance since we are in a game of incomplete information and that value is revealed to A only at the end of the game as an ex-post condition.

5.4 English auction with negative bids

In this section we examine the mechanism⁵ where the **auctioneer** offers a chore and a starting amount of money L .

On their turn **the bidders** start making a succession of lower and lower bids until when a bid is not followed by a still lower bid: the bidder who made this last bid gets both the chore and his bid as a payment for the chore.

As to the **auctioneer** we note that the only parameter he can fix is the value L .

The auctioneer can choose L so that it is the maximum amount of money he is willing to pay (see also section 5.5) but it is neither too low (since in this case the auction could be void) nor too high (since in this case it could also favor the rising of collusions among the bidders (see section 5.6)).

As to the **bidders** we note that if each bidder has an evaluation m_i of the chore as his private information his best strategy is to start bidding at any moment when the current value of the bids is greater than m_i , go on until the current descending price reaches the value m_i and then stop (otherwise he could have a loss $x - m_i$, see further on).

We note, indeed, that if x is the current value of the bids, the bidder b_i has a net

⁵We call it also the descending mechanism or the descending case.

gain equal to $x - m_i$ that is positive for $x > m_i$, null for $x = m_i$ and negative for $x < m_i$ so that the least acceptable outcome is $x = m_i$ with a null net gain.

5.5 The equivalence of the mechanisms

We wish to verify if the two proposed mechanisms are equivalent or not with regard to the values of some parameters and the revenue for the auctioneer.

The first thing we do is a comparison between M and L . We saw that M is the maximum amount of money the auctioneer is willing to pay to sell the chore (see section 5) and the same role is played by L so we can reasonably expect that $L = M$ is true.

We can reason as follows. We suppose to have the same chore and the same set of bidders in the two auction types we examine.

It cannot be $L > M$ otherwise A would risk to pay in the descending case a sum L greater than his maximum willingness to pay M in the ascending case.

On the other hand it cannot be $L < M$ since A in the ascending case would risk to pay a sum higher than the maximum sum he is willing to pay in the descending case.

From all this we see how it must be $L = M$.

We now examine the auction's revenue from the auctioneer/seller point of view. If we suppose that each of the n bidders b_i has the evaluation m_i of the chore we can easily see how the chore is allocated to the bidder b_j where:

$$j = \operatorname{argmin}\{m_i \mid i = 1, \dots, n\} \quad (2)$$

and possible ties are resolved with the use of a properly designed random device. In both cases the revenue for the auctioneer is given by:

$$M - m_j = L - m_j \quad (3)$$

From this we can say that the two mechanisms are equivalent with respect to the seller/auctioneer.

Let us consider things from the bidders point of view.

From their point of view, though they may prefer a descending mechanism to an ascending one, things are equivalent since the chore is allocated to the bidder b_j where j satisfies relation (2).

We underline how in both mechanisms the bidders attend on a voluntary basis so that their individual evaluations m_i represent how much each of them is willing to get to accept the chore. This implies that m_i hides in itself both costs and gains for each b_i from the chore but the relative importance and weight of costs and gains is a private information of each bidder.

5.6 The possible collusions

Up to now we have supposed that the bidders act one independently from the others. Now we examine the possible collusions:

(c_1) among all the bidders and

(c_2) among the auctioneer and some of the bidders.

As to (c_1) we start with an analysis of the collusions in the **descending case**. In this case, indeed, the bidders could agree that one of them (be it b_j) bids L , is compensated with $\hat{m}_j = \max\{\frac{L}{n}, m_j\}$ and all the others $n - 1$ share the resulting surplus $L - \hat{m}_j$ among themselves.

This strategy is potentially fragile since b_j may decide to keep the chore for himself with a net gain of $L - m_j$ since any violation of the binding agreement among the bidders can be hardly punished. Moreover we have that every bidder b_h whose m_h is greater than the share of the surplus may have an incentive to deviate from that strategy.

Of course any coalition not including all the bidders (a limited coalition) is fragile since the excluded bidders are free to make their bids so to incentive the others to leave the coalition.

If a limited coalition includes b_j (see section 5.5) its members may not be sure to get the chore but for a price equal to m_j so that no share of a surplus is possible.

In the **ascending case** the bidders do not know the value of M so collusions are more risky and less profitable.

A possible strategy could be that the bidders keep from bidding until when the price offered by A reaches a minimal ex-ante agreed-on value \hat{m} . At this point one of them, be it b_i who evaluates the chore as $m_i < \hat{m}$, accepts \hat{m} and the chore so that the auction is over and b_i gets the chore and the sum m_i whereas the other $n - 1$ bidders share equally the surplus $\hat{m} - m_i$ among themselves.

The choice of \hat{m} is risky since the bidders may agree on a value that is higher than M so it is never reached in the auction. This risk may be minimized by reducing $\hat{m} > m_j$ (with j defined with the rule (2)) so correspondingly reducing the surplus deriving from the auction.

This strategy is, however, fragile since b_i has strong incentives to deviate unilaterally from it and keep the chore for himself (since any violation of the binding agreement among the bidders can be hardly enforced) with a net gain of $\hat{m} - m_i$ and no surplus to be shared among the other bidders.

In order to maximize the surplus $\hat{m} - m_i$ the bidder b_i should be b_j with j defined by relation (2). On the other hand, this choice maximizes also the temptation for b_j to deviate unilaterally from that strategy (since he is the bidder who gains the most from such a deviation).

As to (c_2) we have that both in the ascending case and in the descending case we can hardly imagine the possibility of collusions between the auctioneer and [part of] the bidders owing to the nature of the proposed mechanisms and to the fact that the auctioneer pays for allocating the chore to one of the bidders that, in his turn, is paid for getting the chore.

5.7 The side effects

Up to now we have supposed that the allocation of a chore to one of the bidders has no harm on the other bidders (hypothesis of the **independent**

bidders) so that the auction ends when either the chore is allocated or a predefined termination condition is satisfied and no allocation occurs (void auction). In some cases the allocation of the chore to one of the bidders may harm some other bidders through side effects so that we may think to introduce correcting or compensatory tools either within the proposed mechanisms or as added steps to such mechanisms. In these notes we are going to follow the latter approach. We can therefore act in the following way.

We can define a three stages mechanism so that we have:

- (1) an auction stage;
- (2) a dispute stage;
- (3) a settlement stage.

Steps (1), (2) and (3) may be repeated more than once until the chore is assigned to one of the bidders.

This three stages mechanism may properly work only if the following conditions hold:

- the bidders are aware of the whole mechanism in advance so they have incentives to include in each own's m_i the sums that each of them may need at steps (2) and (3)
- the bidders act under binding agreements that force each of them to get the chore if he claims for the highest sum in the dispute stage (see further on) or to attend a restricted auction (again see further on) depending on the will of one of them.

At the **auction stage** we have an auction with either an ascending or a descending mechanism that ends with the allocation of the chore and the payment of a sum from the auctioneer to one of the bidders (the so called winning bidder), be it b_j that gets m_j (where j is identified with the rule (2)).

At the **dispute stage** (2) some of the losing bidders (the bidders different from the winning bidder), be them b_i with $i \in I \subset N$, may complain and ask for compensations $c_{i,j}$ to that winning bidder.

If $I = \emptyset$ the auction is over at the stage (1) otherwise it goes on with the stage (3).

So let us suppose, from now on, that $I \neq \emptyset$.

We note that, in general, we have:

$$c_{i,j} < m_i \tag{4}$$

since each $c_{i,j}$ measures an indirect damage for b_i from the chore allocated to b_j and this damage is worth less than the direct damage plus the costs and the gains that form m_i .

Such requests are made from the b_i s to b_j as simultaneous sealed requests.

At the **settlement stage** the bidder b_j (who got the chore at the auction stage) can use one of the following strategies that are known to the complaining bidders:

- (a) may pay the sums $c_{i,j}$ and keep the chore for himself with a gain reduced to:

$$m_j - \sum_{i \in I} c_{i,j} \quad (5)$$

constrained to be positive;

- (b) may sell the chore to the complaining bidder b_h who asks for

$$c_{h,j} = \max c_{i,j} \quad (6)$$

paying him a sum equal to $\min\{m_j, c_{h,j}\}$;

- (c) may start an auction using one of the two proposed mechanisms with the bidders b_i with $i \in I \subset N$ as the participating bidders and with $L = M = m_j$.

The use of the mechanism of point (a) requires that m_j has been properly fixed by b_j so to be able to pay the sums $c_{i,j}$ and, at the same time, maintain a positive gain.

Similar considerations hold also for the mechanism of point (c). In this case the cascaded auction may turn out void so that b_j must plan for it accurately since in that case he must resort to the mechanisms of either point (a) or point (b). We note that the mechanism of point (c) may be repeated a finite number of times since N is finite and so also I that reduces at each repetition up to the empty set when the process stops.

The strategies (a) and (b) for b_j pose some constraints on the strategies of the complaining bidders b_i with $i \in I \subset N$.

Strategy (a) invites the complaining bidders to ask for not too high sums $c_{i,j}$ whereas the strategy (b) invites the complaining bidders to ask higher sums $c_{i,j}$ so that they must seek for a compromise between the two policies.

We underline how the bidders claim their $c_{i,j}$ to b_j before he chooses his strategy at the settlement stage.

If the chore is exchanged from bidder b_j to another bidder b_i (according to one of the strategies (b) and (c)) the stages (2) and (3) are repeated since the new ownership may give rise to damages to a new subset of complaining bidders.

This process goes on until when no exchange occurs and compensations are transferred from the final winning bidder to a set of complaining bidders.

5.8 The possibility of coalitions or consortia

Up to this point we have seen the bidders as individual entities $b_i \in \mathcal{B}$ that attend the auction so to get the chore and a sum of money.

We can relax such an assumption and consider the following two cases:

- (1) we partition \mathcal{B} as $\mathcal{B} = \cup_{i=1}^n B_i$ with $B_i \cap B_j = \emptyset$ for every $i \neq j$,
- (2) we cover \mathcal{B} as $\mathcal{B} = \cup_{i=1}^n B_i$ with $B_i \cap B_j \neq \emptyset$ for some pairs $i \neq j$.

Both coalitions (1) and consortia (1) attend the auction on a voluntary basis so that their participation signifies the will to bid at the best so to get the auctioned chore.

In both cases (1) and (2) it is possible to introduce the side effects we discussed in section 6.11 has a way to analyze negative interactions among either coalitions or consortia.

In the case (1) we speak of the coalitions B_i that attend the auction as individual players.

In this case the coalition B_i assigns to the chore a value M_i as:

(1_a) $M_i = \sum_{b_i \in B_i} m_i$ in an additive way,

(1_b) $M_i > M = \sum_{b_i \in B_i} m_i$ in an superadditive way owing to the presence of synergies so there is a net gain $M_i - M$ for the coalition.

We note how subadditivity is a nonsense since in that case the single bidders would be better off by not joining any coalition.

In both cases (1_a) and (1_b) the coalition B_i attend the auction and can either get the chore (and the sum M_i) or not. If B_i gets the chore and the sum M_i its members must share it among themselves.

If B_i gets the chore it gets the sum M_i to be shared among its members.

In the case (1_a) that sum is shared as m_i to each $b_i \in B_i$.

In the case (1_b) there is a net gain $M_i - M$ so there may be more than one way to share the sum M_i .

In the simplest case the sum M_i is shared as:

$$m_i \frac{M_i}{M} \quad (7)$$

so that every bidder gets a share proportional to his contribution m_i .

Another way is to assign to each bidder a sum equal to m_i and share the net gain $M_i - M$ as:

$$m_i \frac{M_i - M}{M} \quad (8)$$

In the case (2) we speak of the consortia B_i that attend the auction as individual players with individual evaluations M_i .

To understand why a bidder may wish to join more than one set B_i we must specify how such sets form.

Consortia (but the same holds also for coalitions) form before the auction starts according to a blind mechanism so to avoid conflicts of interests. Bidders submit their intention to join a consortium knowing only the identity of the other members but without knowing their private evaluations so that none of them is able to know the value $M = \sum_{b_i \in B_i} m_i$ until the consortia are formed.

When the consortia formation phase is over each bidder b_i knows the value M of each consortium to which he belongs so he can form an expectation on the outcome of the auction.

In this case each bidder may join more than one consortium so to rise his chances to get the chore.

When the consortia have been formed we have the auction phase and when the auction is over we have the allocation of the chore and of the corresponding sum of money M_i to one of the consortia B_i so that M_i is shared among the members of B_i as we have seen in the case of the coalitions.

5.9 Some possible applications

In this section we list some possible applications of the proposed auction mechanisms.

In the **independent bidders** case we can use the proposed mechanisms to define the allocation or **localized** or **point wise** chores to one of the bidders. This is the case of incinerators, solid wastes disposal sites, chemical plants and the like. The main point is that the carrying out of the chore requires the assent of a single bidder. In this case we try to account for the presence of collateral damages among bidders through the mechanism of the side effects (see section 6.11).

In the case of the **coalitions** we can use the proposed mechanisms for the allocation of chores with contiguity constraints among disjoint alternatives.

With this we mean that each alternative solution is represented by a coalition. This is the case of alternative layouts of the same connection line (be it a railway or a highway) or the case of different types of a connection line (a railway versus a highway) that do not share anything but possibly the starting and the ending nodes.

In the case of the **consortia** we can use the proposed mechanisms in similar contexts but with not disjoint alternatives (such as connection lines that share some of the nodes beyond the starting and the ending nodes or some of the arcs).

5.10 The properties of positive auctions

We now verify if the mechanisms we have introduced satisfy the performance criteria we have introduced in section 4.1.

We have that:

- **guaranteed success** is verified if the auctioneer fixes the values L and M in a proper way;
- **individual rationality** is guaranteed since the bidders bid only if they want and none of them pays any penalty bigger than his damage for not attending the auction;
- **simplicity** is verified since the rules of the two auctions are very simple to understand and implement;
- **stability** is guaranteed since each bidder has a simple best strategy to follow;

- **[Pareto] efficiency** is verified since the chore is allocated to the bidder who value it less and all the others are left out whereas the auctioneer pays the least amount of money to get the chore allocated.

In this way we have verified that the proposed mechanisms satisfy the chosen criteria.

6 The negative auction

6.1 Introduction

We present here a mechanism for the allocation of a chore (a disagreeable task with a negative value) from an actor A (the **auctioneer**) to another one from a distinct set of actors \mathcal{B} (the **bidders**). The members of \mathcal{B} are arbitrarily selected by A .

We call such a mechanism a **negative auction**.

According to this mechanism the auctioneer A aims at allocating a chore to one of the bidders $b_i \in \mathcal{B}$ by using a mechanism where the bidders bid for not getting the chore whence the denomination of **negative auctions**.

The final aim of such auctions is the transfer of the chore from the auctioneer to one of the bidders (the losing bidder) together with a monetary compensation from the other bidders (the so called winning bidders).

In the paper we present and discuss a **base model** and **two extended models**.

6.2 The features of various models

The proposed models differ either for the relations among the bidders or for the relations among the bidders and some special actors that we call supporters or for both (see section 6.9 for further details). In all the models there is a filtering phase through which we get the effective set of bidders $\hat{\mathcal{B}}$ as the set of the bidders that effectively attend the auction.

In the **base model** the bidders are one independent from the others and all behave accordingly. We speak of **independent bidders** since the allocation of the chore to one of the bidders causes a damage only to that bidders and to nobody else.

In the **first extension** we drop the hypothesis of the independence of the bidders and suppose the presence of cross damages among the bidders so that each bidder suffer a damage not only from the allocation of the chore to himself but also from the allocation of it to some of his neighboring bidders.

In this case we imagine that the bidders support each other in avoiding the chore and analyze if such a mechanism is profitable for the bidders so that they are better off by adopting it.

In the **second extension** we introduce the supporters that form a set \mathcal{S} . The supporters are special actors that once the members of the set $\hat{\mathcal{B}}$ are known decide to support some of them so that they have higher probabilities of not getting the chore. Such a support depends heavily on the cross damage that

each supporter suffers from the allocation of the chore to one of the bidders. All the models are analyzed through a presentation of their features and the possible strategies available to the bidders.

6.3 The fee and its meanings

All the proposed models include a pre-auction phase, the so called fee payment phase, through which the set \mathcal{B} of n bidders reduces to the set $\hat{\mathcal{B}}$ of k bidders that effectively attend the auction phase whereas the others $m = n - k$ pay a properly fixed (by A) fee f .

When A chooses the members of \mathcal{B} they do not know each other and are informed of the amount of the fee and may submit a sealed payment of two amounts: 0 or f . In the former case they attend the auction and enter the set $\hat{\mathcal{B}}$ whereas the others exit the auction.

When this phase is over the k members of the set $\hat{\mathcal{B}}$ are made publicly known and the auction phase may start.

Before going on with section 6.5 and those that follow we comment a little bit more on the concept of the fee.

The main reason for introducing the fee is to implement, within the mechanisms, the principle of **individual rationality** since the bidders are chosen by the auctioneer at his will and do not attend the auction on voluntary basis. We can see the fee as the analogous of the reserve price in classical auctions.

The auctioneer A therefore fixes a fee f and must properly choose its amount given his interest of not having a void auction.

From this we can argue that:

- he must fix $f > 0$;
- if A can guess the values \tilde{m}_i of the m_i he can choose $f > \min \tilde{m}_i$ so to force at least some of the less damaged bidders to attend the auction;
- if f is fixed too high then all the members of \mathcal{B} have a strong incentive to attend the auction hoping in a certain number of payments and therefore in a substantial additional compensation.

We note that fixing a fee $f = 0$ is different from not having a fee so as having a free ticket is different from no ticket: a fee $f = 0$ allows the bidders to escape the auction at no cost whereas if no fee is fixed all the bidders are forced to attend the auction and submit to its rules.

We recall that during the fee payment phase the bidders do not know either each others identities or their number so that the decision of either paying or not the fee is up to each bidder.

Let us consider what could happen if such hypothesis should prove false so that the bidders of \mathcal{B} may guess how many bidders are willing to pay the fee and even may agree on some common strategies (see further on).

At the offset A contacts the n members of \mathcal{B} and it may happen that:

- (1) m decide to pay the fee,

(2) $k = n - m$ decide to attend the auction.

In this way the auctioneer collects a sum mf to be given to the losing bidder as an extra compensation.

A critical case may occur if $m = n$ so that all the bidders of \mathcal{B} pay the fee, $\hat{\mathcal{B}} = \emptyset$ and the auction is void. In this eventuality we may think to adopt one of the following solutions:

- (s_1) the fees are given back to the bidders, since no auction occurs, so that each of them has a null utility (since he pays f but gets back f);
- (s_2) the total fee nf are used by the auctioneer to pay an external player P for having him carrying out the chore.

In this paper we disregard the solution (s_2) essentially because:

- it should appear in the description of the game and be known by the other bidders from the offset;
- P should be considered as a member of \mathcal{B} so to be submitted to the same mechanism as the others;
- the case $m = n$ does not represent a Nash Equilibrium so it never occurs;
- the case (s_2) does not represent a Nash Equilibrium so it never occurs;

We indeed may state that:

- in the all pay case with repayment the all pay strategy where the bidders have 0 utility is fragile if there is at least one bidder b_i such that:

$$(n - 1)f \geq m_i \tag{9}$$

- in the all pay case without repayment the strategy where all the bidders have an utility $-f$ is fragile if there is at least one bidder b_i such that:

$$(n - 1)f - m_i > -f \tag{10}$$

In both cases the deviating bidder gets a payoff equal to $(n - 1)f$ whereas all the other get a payoff equal to $-f$.

We note that the existence of such a bidder is guaranteed by the conditions we have made on the value of f .

In the descriptions of the single models we disregard the fee payment phase (that is common to all the models) and suppose to start with a reduced set of k bidders $\hat{\mathcal{B}}$ and with a total fee, equal to mf (with $m = n - k$), that is revealed to the k bidders only when the auction phase is over.

6.4 The phases

The proposed models are characterized by a succession of phases, some common to all the models and some that enter only in some model but not in others. The common phases are:

- fee payment phase,
- auction phase,
- allocation and compensation phase.

The particular phases (that are present only in one or more extensions) are:

- cross-bidders support phase,
- supporters-bidders support phase.

The general succession of the phases is the following (particular phases are enclosed within square brackets):

- (1) fee payment phase,
- (a) [supporters-bidders support phase],
- (b) [cross-bidders support phase],
- (2) auction phase,
- (3) allocation and compensation phase.

Each particular phase will be analyzed in the proper section of this paper. For the moment we simply note that phase (b) is present in the first extension whereas phases (a) and (b) are present in the second extension.

The **auction phase** has a simple structure: the bidders of the set $\hat{\mathcal{B}}$ submit their bids in a sealed bid one shot auction and the lowest bidding bidder⁶ loses the auction and gets the auctioned chore with a compensation c that is given by:

- the total fees mf ,
- the sum x_1 that the losing bidder bid in the auction phase,
- the sums Σ_1 that derive him from the two phases (a) and (b) (see the sections 6.10 and 6.11)

as $c = mf + x_1 + \Sigma_1$.

At the end of this phase possible ties are resolved with the aid of a properly designed random device.

During the **allocation and compensation phase** the losing bidder gets the

⁶We denote conventionally such bidder as b_1 , see further on, and in a similar way all the quantities that concern him.

chore and is compensated with the sum c and the compensation is partly gathered from the non attending bidders (the sum mf) and partly (the sum $x_1 + \Sigma_1$, see further on) from the winning bidders.

The enforcement of the agreement is guaranteed by the winning bidders that are interested in the fact that the agreement is honored by the losing bidder.

6.5 The base model: the lose and the win cases

In the base model we have, therefore, a set $\hat{\mathcal{B}}$ of k independent bidders b_i where each bidder has:

- a private evaluation m_i of the auctioned chore;
- a bid x_i that is publicly revealed by A at the end of the auction phase,
- a compensation x_i that is granted him for sure, whenever he is the losing bidder, from the winning bidders;
- a sum he has to pay to the losing bidder whenever he is one of the winning bidders.

For the moment we disregard the compensation represented by the fees since it is known only ex-post and cannot influence the strategies of the bidders so that we put the bidder b_i in a sort of worst case situation where $m = 0$.

By bidding x_i the bidder b_i may either lose or win the auction.

In the **losing case** b_i gets x_i for a loss of m_i so to get $x_i - m_i$.

In the **winning case** b_i has a gain m_i (the missed damage) and pays a sum equal to:

$$x_1 \frac{x_i}{X} \quad (11)$$

where x_1 is the bid⁷ of the losing bidder b_1 and:

$$X = \sum_{j \neq 1} x_j \quad (12)$$

In this case b_i gets:

$$m_i - x_1 \frac{x_i}{X} \quad (13)$$

as the difference between the missed damage and the sum he has to pay to the losing bidder.

If we denote with p the probability that b_i has to lose the auction (and $(1 - p)$ is the probability that he has of winning it) we can define his expected gain as:

$$E_i(x_i) = p(x_i - m_i) + (1 - p)(m_i - x_1 \frac{x_i}{X}) \quad (14)$$

or as:

$$E_i(x_i) = pl_i + (1 - p)w_i \quad (15)$$

with:

⁷When the auction phase is over we may renumber the bidders so that b_1 is the losing bidder and the other $k - 1$ bidders are winning bidders.

$l_i = (x_i - m_i)$, if he loses;

$w_i = (m_i - x_1 \frac{x_i}{X}) = (m_i - x_1 \frac{x_i}{\alpha - x_i})$ (with $\alpha = \sum_{j \neq 1, i} x_j$), if he wins.

It is easy to see how:

- the term l_i increases with x_i from a negative value ($-m_i$) to a positive value ($M - m_i$);
- the term w_i decreases from a maximum value for $x_i = 0$ to a minimum and potentially negative value for $x_i = M$, see section 6.6 and the Appendix B.

6.6 The base model: the strategies

Every bidder b_i can use one of the following bidding strategies:

- (1) $x_i < m_i$
- (2) $x_i = m_i$
- (3) $x_i > m_i$

In correspondence of such strategies the elements l_i and w_i (see section 6.5) assume the signs that we specify in Table 1 where we have:

– denotes a negative value or a loss;

0 denotes a neutral condition, neither a loss nor a gain;

+ (with subscripts) denotes a positive value or a gain;

$+ \succ 0 \succ -$ defines a strict preference ordering where \succ denotes a strict preference relation between two elements;

$+/-$ denotes a gain that can turn into a loss⁸;

$+_1 \succ +_2 \succ +/-$ defines a strict preference ordering.

. We see, therefore, how the “pieces” of relation (15) have opposing effects⁹.

Strategies	l_i	w_i
$x_i < m_i$	-	$+_1$
$x_i = m_i$	0	$+_2$
$x_i > m_i$	+	$+/-$

Table 1: Signs of the terms l_i and w_i

From this analysis we can label:

⁸See the Appendix B for further details.

⁹See the Appendix B for further details.

- the strategy $x_i < m_i$ as a **risky strategy**,
- the strategy $x_i = m_i$ as a **conservative strategy**,
- the strategy $x_i > m_i$ as an **aggressive strategy**.

We recall that the exact balance between the terms l_i and w_i depends on the value of $p = p(x_i)$ so that the exact value of the expected gain for bidder b_i (as given by expression (14)) depends on that value (see section 6.7).

Without any knowledge of p we can say that:

- a risk adverse bidder b_i prefers a conservative strategy and so a truthful bidding $x_i = m_i$;
- a risk seeking bidder b_i prefers either a risky strategy $x_i < m_i$ or an aggressive strategy $x_i > m_i$ depending on whether he is more convinced to win or to lose.

Under the hypothesis that the bidders are risk adverse we may say that the conservative strategy is the best strategy for the bidders.

6.7 The base model: assessing the value of p

In order for any b_i to assess the value of p he must have some knowledge of the random variables x_j for $j \neq i$. Possible properties of the x_j are that they are **independent** and **identically distributed** over the interval $[0, M]$ for a suitable value $M > 0$.

Under these hypotheses, b_i can evaluate the probability of losing the auction $p = p(x_i)$ as (see the Appendix B for further details):

$$p(x_i) = P(x_i = \min_j x_j) = P(\cap_j x_i \leq x_j) \quad (16)$$

If we use the independence among the random variables we get:

$$p(x_i) = \prod_j P(x_i \leq x_j) = \prod_j (1 - P(x_j < x_i)) \quad (17)$$

The next step is to assume a particular distribution among the possible ones. In the simplest case b_i may assume that the x_j are **uniformly distributed** over the interval $[0, M]$. Under this assumption we have that:

$$F_{x_j}(x) = P(x_j < k) = \frac{x}{M} \quad (18)$$

so that relation (17) (or the expression of the probability for b_i of losing the auction) becomes:

$$p(x_i) = (1 - \frac{x_i}{M})^k \quad (19)$$

whereas the complementary probability of winning it becomes:

$$1 - p(x_i) = 1 - (1 - \frac{x_i}{M})^k \quad (20)$$

From relations (19) and (20) we see that:

- if $x_i = 0$ we have, as expected, $p = 1$ and $1 - p = 0$;
- if $x_i = M$ we have, as expected, $p = 0$ and $1 - p = 1$.

Relations (19) and (20) have been derived under rather strong assumptions so that, in real cases, they can hardly be used by real bidders that, therefore, can rely only on the strategies we have seen in section 6.6

6.8 The base model: the properties and the applications

In order for the description of the basic mechanism to be completed we have:

- (1) to verify whether the proposed basic mechanism satisfies or not a minimal set of basic performance criteria;
- (2) to describe the possible applications of the proposed basic mechanism.

The performance criteria that we use include: guaranteed success, [Pareto] efficiency, individual rationality, stability and simplicity.

The property of **guaranteed success**, from the auctioneer point of view, is satisfied whenever the auction is not void. This happens for sure if A does not fix any fee (and therefore the auction cannot be void and a losing bidder exists for sure) so conflicting with the requirement of **individual rationality** but can occur also if he fixes a fee on condition that the value f is fixed at the right level (see section 6.3).

From the bidders point of view the property of **guaranteed success** is satisfied since the auctioned item is assigned to the bidder who makes the lowest bid and the others compensate him for this according to a simple rule of cost sharing.

As to **[Pareto] efficiency** we may state that:

- the bidder with the lowest bid gets the chore and a proper compensation;
- all the other bidders may pay a sum that is lower than the missed damage from not having the chore allocated to each of them so may have a positive gain.

If the chore is allocated to another bidder with a higher bid all the winning bidders would pay a higher fraction of the compensation so they would be worse off. This is enough to say that the allocation to the lowest bidding bidder is [Pareto] efficient.

The satisfaction of the property of **individual rationality** is guaranteed by the presence of the fee f that allows some bidders to escape the auction by paying it.

As to **stability** we have argued in section 6.6 that for each risk adverse bidder the truthful bidding is the best strategy. Though this arguing must be justified on more solid grounds we think that the intuition we have given should be enough to assure the satisfaction of this property.

Last but not least **simplicity** is assured by the fact that the rules of the auction are simple enough so to be implemented even by bidders with a bounded

rationality.

As to the **applications** of the basic model we note that it is well suited in all the cases where an auctioneer has to allocate a chore with the following distinguishing features:

- the chore involves a single bidder or it can be carried out by a single bidder though it may benefit not only the auctioneer but also other bidders;
- the influences on other bidders, once the chore is allocated, is negligible;
- the influence on actors distinct from the bidders, once the chore is allocated, is negligible;
- all the costs can be summarized within the self damage parameter m_i for each bidder b_i ;
- there is no time dependent parameter nor time dependent costs.

Such features allow the allocation of a chore from the auctioneer to one of the bidders from an equivalent (with respect to the chore) set of bidders.

As a first approximation we may consider the basic model as suitable for the allocation of:

- solid waste disposal plants;
- hazardous waste disposal plants;
- incinerators;
- energy production plants;
- chemical plants.

The mechanism guarantees the allocation of a chore of such types under the hypothesis of no external intervention or support and of the absence of cross-support among the bidders themselves (see section 6.9). We note that the absence of time dependent values and parameters as well as the use of a single parameter to summarize all the costs remains also in the extensions.

6.9 The need of the extensions

In the base model we have assumed that the bidders are independent one from the others so that the allocation of the chore to one bidder is a damage for that bidder only.

This hypothesis, in real cases, is both impractical and unrealistic since the bidders of the set $\hat{\mathcal{B}}$:

- (1) influence each other,
- (2) influence other actors that are not chosen by the auctioneer by that are, anyway, damaged by the allocation of the chore to one of the bidders.

- (1) To account for the reciprocal influences among the bidders we introduce the concept of **cross-damage** that is at the root of the first extension that we analyze in section 6.10.

With the term **cross damage** we denote a damage $d_{i,j}$ that derives to b_i if the chore is allocated to b_j for any pair of bidders of $b_i, b_j \in \hat{\mathcal{B}}$.

The set of the cross damages form the $k \times k$ matrix D (see Table 2) of the cross damages where $\forall d_{i,i} = m_i$, the self damage for bidder b_i .

$d_{1,1}$	\dots	$d_{1,i}$	\dots	$d_{1,k}$
\dots	\dots	\dots	\dots	\dots
$d_{i,1}$	\dots	$d_{i,i}$	\dots	$d_{i,k}$
\dots	\dots	\dots	\dots	\dots
$d_{k,1}$	\dots	$d_{k,i}$	\dots	$d_{k,k}$

Table 2: *Matrix D*

The values $d_{i,j}$ determine, through a function f_i for each bidder b_i , the highest monetary contribution $f_{i,j}^* = f_i(d_{i,j})$ that b_i is willing to give to the bidder b_j (if and only if he will result the losing bidder) so to induce him to accept the allocation of the chore.

We note that, in general, the functions f_i are concave in their independent variable.

In this way from the matrix D and the functions f_i we derive the matrix F of the financings from every bidder to all the other bidders. We note how, obviously, we have $f_{i,i} = 0$ for every i .

- (2) The presence of further actors (the supporters, see section 6.2), on the other hand, is at the root of the the second extension that we briefly analyze in section 6.11.

The basic idea is the following: once the composition of the set $\hat{\mathcal{B}}$ is known each of the supporters (that form the set \mathcal{S}) may evaluate the damage that derives him from the allocation of the chore to each bidder b_j . In this case we denote such a damage for the supporter s_i from the bidder b_j as $d_{i,j}$. Such damages form h (where $|\mathcal{S}| = h$) additional rows for the matrix D of the cross-damages for the bidders of $\hat{\mathcal{B}}$. In this case, therefore, the matrix D (see Table 3) is of type $(k + h) \times k$.

Also in this case each value $d_{i,j}$ determine, through a function f_i for each supporter s_i , the highest monetary contribution $f_{i,j}^* = f_i(d_{i,j})$ that s_i is willing to give to b_j so to induce him to accept the allocation of the chore. In this case, similarly to what happens to the matrix D , we have h additional rows to the matrix F that account for the financings from the supporters to the various bidders.

As a common rule to both extensions we may state that through the relations $f_{i,j} = f_i(d_{i,j})$ the bidders and the supporters define monetary contributions for the other bidders so to push the chore towards a bidder that causes to [at least

$d_{1,1}$	\dots	$d_{1,i}$	\dots	$d_{1,k}$
\dots	\dots	\dots	\dots	\dots
$d_{i,1}$	\dots	$d_{i,i}$	\dots	$d_{i,k}$
\dots	\dots	\dots	\dots	\dots
$d_{k,1}$	\dots	$d_{k,i}$	\dots	$d_{k,k}$
\dots	\dots	\dots	\dots	\dots
$d_{k+h,1}$	\dots	$d_{k+h,i}$	\dots	$d_{k+h,k}$

Table 3: *Extended matrix D*

a majority of] the other actors the least damage. Such a bidder is identified by each other bidder or supporter by using the values of the cross-damages (see next sections for details) with regard to the other bidders.

6.10 The first extension

In this case, during the **cross-bidders support phase**, the k bidders of $\hat{\mathcal{B}}$ exchange among themselves proposals of additional compensations so to induce one of them (the losing bidder) to accept the chore.

We have, indeed, that the $k \times k$ matrix F has the j -th column F_j that defines the sums that a bidder b_j will receive, in the compensation phase, from all the other bidders if and only if he will result the losing bidder. Such sums amount to $\Sigma_j = \sum_{i \neq j} f_{i,j}$.

We note how the elements of such j -th column generally satisfy the constraints:

$$f_{i,j} \leq f_i(d_{i,j}) \quad (21)$$

for all the b_i distinct from b_j and can induce b_j to lower his bid x_j so to lose the auction and get an extra compensation for accepting the chore. Relation 21 expresses the fact that no bidder wants to pay to another bidder a contribution that is higher than the equivalent damage that he suffers from that bidder.

In this way, during the **cross-bidders support phase**, we have an exchange of the proposed values $f_{i,j}$ from every bidder b_i to all the others (and with the constraint $f_{i,i} = 0$) so that every bidder receives the overall proposal Σ_j and may decide to lose the auction and so to lower his bid x_i to a level that still guarantees him a gain and so to a level such that $\Sigma_j + x_i > m_i$ is satisfied. Such values are to be effectively transferred only to the losing bidder as an extra compensation.

In this case the bidders may therefore convince one of them to a piloted loss that is compensated with an additional sum Σ_j .

The best bidder b_j to hold that role should be the one with the lowest $d_{i,j}$ for all i , if it exists, or, at least, one which satisfies $\min_i d_{i,j}$ for the highest number of bidders b_i . In case of ties the best bidder is the one with the lowest value of the self damage m_j . If we again have ties we select one of the equivalent bidders with a properly designed random device.

For the losing bidder b_1 we have that he gets:

$$x_1 + \Sigma_1 = x_1 + \sum_j f_{j,1} \quad (22)$$

and suffers a damage equal to m_1 so to have a balance of $x_1 + \Sigma_1 - m_1$ that we impose to be strictly positive.

On the other hand, each winning bidder b_j pays:

$$x_1 \frac{x_j}{\alpha + x_j} + f_{j,1} \quad (23)$$

(where $\alpha = \sum_{i \neq 1,j} x_i$) and suffers a damage $d_{j,1}$ from the allocation of the chore to b_1 but also has a gain from the missed damage m_j .

With the proposed mechanism we have that b_j may try to pilot the allocation of the chore to the bidder b_i that causes to him the least damage but he has no guarantee of success. Moreover we can state that if b_j knows that the chore will be allocated anyway to such b_i he would be better off by not paying to that bidder the contribution $f_{j,i}$ but this eventuality may be discovered for sure only after the auction has been carried out since it depends also on the actions of all the other bidders.

6.11 The second extension

In this case we build over the first extension so that, before the **cross-bidders support phase** (see section 6.10), we have a **supporters-bidders support phase** during which, once the structure of the set $\hat{\mathcal{B}}$ is publicly known, each of the members s_i of the set \mathcal{S} may support each bidder b_j with sums $f_{i,j}$ (to be paid if and only if the bidder b_j will result the losing bidder) so to induce a bidder b_j to accept the chore.

In this case the only thing that change for each bidder b_i is the fact that the matrix D has $k + h$ instead of k rows so that all the summations that involve the columns must be extended on $k + h$ elements and the same holds also for the column vectors.

Another difference concerns the balance of the supporters s_j . Each of them pays to the losing bidder¹⁰ b_1 the sum $f_{j,1}$ that has been defined during the **supporters-bidders support phase** and suffers a damage $d_{j,1}$ that is hopefully the least damage for that bidder. For such a damage we can consider as valid the considerations we have made at the end of section 6.10.

6.12 The application of the extensions

The proposed extensions¹¹ try to make more realistic the application of the base model to the cases we have seen in section 6.8.

¹⁰Also in this case when the auction is over and before the allocation and compensation we can renumber the bidders so to denote the single losing bidder as b_1 .

¹¹In the case of the extensions we do not consider the satisfaction of the performance criteria and do not repeat the analysis we made in section 6.8 since the basic structure of the model is unchanged and so that analysis could be repeated here without any meaningful modification.

The first extension tries to link the bidders among themselves through the introduction of the concepts of **cross-damage** and **financing**: in this way each bidder may induce the more favorable (for him) bidders to accept the chore by promising each of them an additional sum of money to be paid effectively only to the losing bidder.

In this way each bidder b_i , by offering the sums $f_{i,j}$ and bidding x_i independently from the others, tries to push the chore towards a subset of bidders so that the set of bidders as a whole ends with identifying one bidder as the losing bidder that can be the best for the other $k - 1$.

The second extension, on the other hand, tries to account for the damages from the allocation of a chore also to actors that are excluded by the auctioneer from the auction and that, in the base model but also in the first extension, cannot have their voice heard.

In this way, by defining the role of supporter, we allow such actors to attend, at least indirectly, to the auction and obtain that some bidders may push harder the chore towards some other bidders by promising those bidders more additional financings to be paid, also in this case, only to the single losing bidder. From these considerations we see how the range of applications is unchanged but is made more realistic.

6.13 Some comments about the relations between positive and negative auctions

As a closing point we try to make a comparison between the proposed positive auctions and negative auctions.

From a superficial analysis we could conclude that the two types have nothing in common owing to their different structures and methods of payment.

As to the first point we note how the positive auctions are based on repetitive bids that enforce truthful bidding whereas the negative auction is a one shot sealed bid auction where players can bid untruthfully and strategically.

As to the second point we note, indeed, how in a positive auction the auctioneer pays the bidder who gets the chore whereas in the negative auction the winning bidders pay the losing bidder since he got the chore.

If, however, we disregard the problem of the fee (that may be paralleled with a reservation price) and the feature of individual rationality we may see how the things are from the point of view of who gets the chore and for what a price.

In a positive auction the chore goes to the bidder who evaluates it less than the others and the price he gets is that evaluation.

In the corresponding negative auction (with the same bidders and the same auctioned chore) the chore is allocated to the bidder who bids less than the others at a price equal to that bid. We therefore have that the price for the losing bidder is the same only in the case of truthful bidding (or if he adopts a conservative strategy and makes a bid equal to his evaluation of the chore) otherwise it can be either higher or lower depending on the chosen strategy.

From the losing bidder point of view we may state therefore that the proposed methods are equivalent in case of truthful bidding.

On the other hand, for what concerns the auctioneer it is obvious how a negative auction is better for him than a positive one since in the former case he can allocate a chore at no cost for him whereas in the latter he has to pay for allocating it.

7 Some remarks about the explicit barter models

7.1 The basic motivation

The basic motivation of the models we propose in the next section 8 is the need to describe how an exchange of goods can happen without the intervention of any transferable utility such that represented by money or by any other numerary good. In this way the involved actors do not need to share anything, such as preferences or utilities as common information, but the will to propose a pool of goods that they present each other so to perform a barter.

All the barter are in kind and are essentially based on the following very simple basic scheme (see section 8.2): we have two actors that show each other the goods, each of them chooses one of the goods of the other and, if they both assent, they have a barter otherwise some rearrangement is needed and the process is repeated until either a barter occurs or both agree to give up.

7.2 Goods and chores as services

The key point of the proposed models is that each of the two players owns a set of items that enters it the barter process, I for A and J for B .

In the technical report we suppose that both I and J contain goods or elements that have a positive value for both players.

From this point of view a good may also be a service that one player is willing to perform on behalf of the other.

In this case, for instance, player A asks to player B for one of the available B 's services in exchange for one of the available A 's services that player B asks to player A . Of course this occurs in the one-to-one barter case.

Another perspective sees the two sets I and J as containing chores or items that have a negative value for both players.

In this latter case the two players try to allocate each other their chores so that a chore allocated from A to B can be seen as a service performed by B on behalf of A to solve a problem of A . In this way we can unify the two perspectives and consider the goods case as a general case.

7.3 Some definitions

With the term **barter**, as we use it in section 8, we mean an exchange of goods for other goods without any involvement of money or any other numerary good.

We can have either a **one shot barter** or a repeated or **multi shot barter**. In the **one shot case** the two actors execute the barter only once by using a potentially multi stage process that aims at a single exchange of goods and can involve a reduction of the sets of goods to be bartered. In the **multi shot case** they repeatedly execute the barter process, every time either with a new set of goods or with a possibly partially renewed set of goods but usually excluding previously bartered goods. In section 8 we are going to examine only the one shot barter case between the two actors so that there is no possibility of retaliation owing to repetitions of the barter. We discuss the multi-shot case in section 8.9. In order to avoid interpersonal comparisons and the use of a common scale we let the two players show each other their goods and ask separately to each of them if he thinks the goods of the other are worth bartering. If both answer affirmatively we are sure that such interval exists otherwise we cannot be sure of its existence. Anyway the bartering process can go on, though with a lower possibility of a successful termination. In this way we describe the absence of a common market (as a place where goods have common and exogenously fixed evaluations in monetary terms) between the two players as well as the absence of any outer evaluator that can impose or even only suggest to both players common evaluations according to a common numeraire quantity.

8 The explicit barter models

8.1 Introduction

We suppose the actor A with his pool $I = \{i_1, \dots, i_n\}$ of n heterogeneous¹² goods and the actor B with her pool $J = \{j_1, \dots, j_m\}$ of m heterogeneous goods. The sets I and J represent all the goods that both players are willing to barter on that occasion so that there is no “hidden good” that can be added at later stages. This is a design choice that qualifies the proposed models as models of **explicit barter**. If we imagine that the players have “hidden goods” that can be revealed and added to the sets at later stages we deal with what we may define an **implicit barter** (see section 9). In the present section we deal only with barter of the former type.

In this case A assigns a **private** (i.e. known only by him) vector v_A of n values to his goods of the set I , one value $v(i)$ for each good $i \in I$.

Also B assigns a private vector v_B of m values to her goods of the set J . These vectors are fixed before the barter begins and cannot be modified during the barter. From these hypotheses, for any subset $K \subseteq I$, player A once for all can

¹²To define the concept of **heterogeneity** we let the players show each other their goods and separately ask each of them if he thinks whether the goods of the other can be considered enough heterogeneous or not: in the affirmative case they can go on with the barter whereas in the negative case they must decide what to do, if to go on or give up. See further on for the possibilities to go on.

evaluate, by using a property of **additivity**, the quantity:

$$v_A(K) = \sum_{i_k \in K} v_X(i_k) \quad (24)$$

A similar quantity may be independently evaluated by player B for any subset $H \subseteq J$.

In a similar way we can define a private vector s_A of m values of the appraisals of the goods of B from A and a vector s_B of n values of the appraisals of the goods of A from B . In this case, A can evaluate:

$$s_A(H) = \sum_{j_h \in H} s_X(j_h) \quad (25)$$

for any subset $H \subseteq J$. A similar quantity may be independently evaluated by player B for any subset of I .

These assignments reflect the basic hypotheses that A can see the goods of B but does not know v_B (the values that B assigns to her goods) and the same holds for B with respect to A .

In this way we can define four types of barter:

1. **one-to-one** or one good for one good;
2. **one-to-many** or one good for a basket of goods;
3. **many-to-one** or a basket of goods for one good;
4. **many-to-many** or a basket of goods for a basket of goods.

The second and the third case are really two symmetric cases so they will be examined together in a single section.

8.2 One-to-one barter

Even in this simple type of barter there must be a pre-play agreement between the two actors that freely and independently agree that each other's goods are suitable for a one-to-one barter. The barter can occur either with **simultaneous** (or "blind") requests or with **sequential** requests.

In the case of **simultaneous requests**, at the moment of having a barter we can imagine that the two actors privately write the identifier of the desired good on a piece of paper and reveal such information at a fixed time after both choices have been made. In this case we have that A requires $j \in J$ and B requires $i \in I$ so that A has a gain $s_A(j)$ but suffers a loss $v_A(i)$ and B has a gain $s_B(i)$ but suffers a loss $v_B(j)$.

The two actors can, therefore, evaluate privately the two changes of value of their goods (that we may slightly improperly call **utilities**):

$$u_A(i, j) = s_A(j) - v_A(i) \quad (26)$$

$$u_B(i, j) = s_B(i) - v_B(j) \quad (27)$$

since all the necessary information is available to both actors after the two requests have been devised and revealed.

Equations such (26) and (27) are privately evaluated by each player that only explicitly declares **acceptance** or **refusal** of the barter, declaration that can be verified to be true by an independent third party upon request. We note that a possible strategy for both players is to maximize the value they get from the barter (and so $s_A(j)$ and $s_B(i)$). Owing to the simultaneity of the requests this is not a guarantee for each player of maximizing his own utility since in equations (26) and (27) we have a loss due to what the other player asks for himself (and so $v_A(i)$ and $v_B(j)$) (see section 8.4).

The basic rule for A is the following¹³:

$$\text{if}(u_A \geq 0) \text{ then } \text{accept}_A \text{ else } \text{refuse}_A \quad (28)$$

and a similar rule holds also for B .

We have therefore the following four cases:

1. both players accept, accept_A and accept_B ,
2. player A refuses and B accepts, refuse_A and accept_B ,
3. player A accepts and B refuses, accept_A and refuse_B ,
4. both players refuse, refuse_A and refuse_B .

that we are going to describe in detail in section 8.3.

In the case of **sequential requests** we can imagine that there is a chance move (such as the toss of a fair coin) to choose who moves first and makes a public request. In this way both A and B have a probability of 0.5 to move first.

If A moves first (the other case is symmetric) and requires $j \in J$, B (since she knows her possible request $i \in I$) may evaluate her utility in advance using equation (27) whereas the same does not hold for A that, when he makes the request, does not know the choice $i \in I$ of B and so cannot evaluate $v_A(i)$. At this level B can either explicitly refuse (if $u_B < 0$) or implicitly accept (if $u_B \geq 0$).

In the **refuse case** B can only take the good j off her set so that the process restarts with a new deliberation of the possibility of the barter and a new chance move.

In the **accept case** the implicit acceptance is revealed by the fact that also B makes a request. In this case B may be tempted to chose $i \in I$ so to evaluate:

$$\max u_B(i, j) = \max (s_B(i) - v_B(j)) = \max s_B(i) \quad (29)$$

where the quantity $v_B(j)$ is fixed (since it depends on the already expressed choice of A) and cannot be modified by B .

¹³In the general case we have $u_A \geq \varepsilon$ with $\varepsilon > 0$ if there is a guaranteed minimum gain or with $\varepsilon < 0$ if there is an acceptable minimum loss.

Acting in this way, B may harm A by causing $u_A < 0$ and this would prevent the barter from occurring at this pass. Roughly speaking we can say that since B choses after A she can act accommodatingly or in an exploitative way: in the first case the probability that the barter occurs is higher than in the second case. Anyway B makes a request of $i \in I$ so that also A can evaluate his utility through equation (26).

Now, using rules such as (28), we may have the cases we have already seen except for the case of double refusal since the case where who choses as the second refuses is handled at a different stage of the algorithm (see section 8.3). All this goes on until both accepts (so the barter occurs) or one of them empties his set of goods or both decide to give up since no barter is possible, how it will be clear from the description of the algorithm that we are going to make in section 8.3.

8.3 Formalization of the models

In this section we present a concise but fairly detailed listing of the two models of the one-to-one barter, starting from the case of **simultaneous or “blind” requests**.

In this case the algorithm is based on the following steps:

- (1) both A and B show each other their goods;
- (2) both players decide if the barter is [still] possible or not;
 - (a) if it is not possible then go to step (6);
 - (b) if it is possible then continue;
- (3) both simultaneously perform their choice (so A chooses $j \in J$ and B chooses $i \in I$);
- (4) when the choices have been made and revealed both A and B can make an evaluation (using equations (26) and (27)) and say if each accepts or refuses (using rules such as (28));
- (5) we can have one of the following cases:
 - (a) if ($accept_A$ and $accept_B$) then go to step (6);
 - (b) if ($refuse_A$ and $accept_B$) then \\\at A 's full discretion
 - i. either A executes $I = I \setminus \{i\}$ and if $(I \neq \emptyset)$ then go to step (2) else go to step (6);
 - ii. or A only executes a new choice and then go to step (4);
 - (c) if ($accept_A$ and $refuse_B$) then \\\at B 's full discretion
 - i. either B executes $J = J \setminus \{j\}$ and if $(J \neq \emptyset)$ then go to step (2) else go to step (6);
 - ii. or B only executes a new choice and then go to step (4);

- (d) if ($refuse_A$ and $refuse_B$) then
 - i. A executes either $I = I \setminus \{i\}$ or a new choice; \\\at A 's full discretion
 - ii. B executes either $J = J \setminus \{j\}$ or a new choice; \\\at B 's full discretion
 - iii. if (both A and B make a new choice) then go to (4);
 - iv. if (only one of A and B makes a new choice and the reduced set of the other is not empty) then
 - if (the barter is still possible) then go to (4);
 - if (the barter is not possible) then go to (6);
 - v. if (only one of A and B makes a new choice and the reduced set of the other is empty) then go to step (6);
 - vi. if (both reduce each one's set and $I \neq \emptyset$ and $J \neq \emptyset$) then go to step (2) else go to step (6);

(6) end of the barter.

The solution we have adopted at point (5.d) is the most flexible since it mixes the two cases (5.b) and (5.c) and gives the two players the full spectrum of possibilities at the same time remaining simple enough to be understood and implemented by even not full rational players .

We remark how at the very beginning of the process we suppose that the barter is possible though this is not necessarily true at every successive interaction.

We now give the description of the model with **sequential requests**. We denote the player who moves first as 1 (it can be either A or B) and the player who moves second as 2 (it can be either B or A) and for both we use male forms. With a similar convention we denote as I_1 the set of goods and i_1 a single good of player 1 whereas for player 2 we have respectively I_2 and i_2 :

- (1) both players show each other their goods;
- (2) both players decide if the barter is [still] possible or not;
 - (a) if it is not possible then go to step (10);
 - (b) if it is possible then continue;
- (3) there is a chance move to decide who moves first (be it 1) and makes a choice;
- (4) 1 reveals his choice $i_2 \in I_2$;
- (5) 2 can now perform an evaluation of all his possibilities;
- (6) if 2 refuses he takes i_2 off his barter set then go to (2);
- (7) if 2 accepts he can reveal his choice $i_1 \in I_1$;
- (8) both 1 and 2 can make an evaluation (using equations such as (26) and (27)) and say if each accepts or refuses (using rules such as (28));

- (9) we can have one of the following cases:
- (a) if ($accept_1$ and $accept_2$) then go to step (10);
 - (b) if ($refuse_1$ and $accept_2$) then \\\at 1's full discretion
 - i. either 1 performs $I_1 = I_1 \setminus \{i_1\}$ and if ($I_1 \neq \emptyset$) then go to step (2) else go to step (10);
 - ii. or 1 only performs and reveals a new choice and then go to step (8);
 - (c) if ($accept_1$ and $refuse_2$) then \\\at 2's full discretion
 - i. either 2 performs $J_1 = J_1 \setminus \{j_1\}$ and if ($J_1 \neq \emptyset$) then go to step (2) else go to step (10);
 - ii. or 2 only performs and reveals a new choice and then go to step (8);
- (10) end of the barter.

We note that the case (9.c) ($accept_1$ and $refuse_2$) can occur as a consequence of the case (9.b) and the jump to (8).

8.4 Possible strategies in the one-to-one barter

We now make some comments on the possible strategies that the players can adopt in the case of the algorithms we have shown in section 8.3.

In the case of **simultaneous requests** both players perform their choice without knowing the choice of the other. If they evaluate their utilities according to equations such as (26) and (27) their best strategy would seem to choose the good of the other that each value at the most.

In this case we have that:

A requires $\hat{j} = \operatorname{argmax}_{j \in JS_A}(j)$ and causes B a loss that A may only roughly estimate;

B requires $\hat{i} = \operatorname{argmax}_{i \in IS_B}(i)$ and causes A a loss that B may only roughly estimate.

Acting in this way each of them may have the other player to refuse the barter. As we have seen a refusal may turn into the withdrawal of a good from one of the sets I or J . This fact is surely unfavorable for each player.

Both players therefore have strong incentives to devise better strategies.

In what follows we introduce one possible strategy under the hypothesis the both players use a more slack rule than rule (28) so that acceptance or refusal are rather discretionary than linked to a condition satisfaction criterion.

We devise a strategy for player A whereas for player B we have two possibilities:

- (1) B follows a generic non systematic strategy,
- (2) B follows a similar strategy.

The strategy for A is the following.

A orders the set J of B in increasing order (from the lowest to the highest) according to the values he gives to its elements.

In the case (1) B uses a generic strategy of selection whereas in the case (2) she uses an analogous strategy over the set I of A : we can state that (2) is the best response for B if A uses that strategy.

The process of choice and request involves a certain number of pass until an agreement is reached either in a positive or in a negative sense. At the generic l -th pass (with $l = 1, \dots$) A requires the current item of higher value $j_l \in J$ whereas B chooses $i \in I$.

After the $(l - 1)$ -th choice from both A and B at pass l we may have:

- (a₁) A accepts so that everything depends on the decision taken by B ,
- (a₂) A refuses so that both goes at pass $l + 1$ -th independently from the choice made by B .

In this way A (but a similar argument holds also for B) scans the vector J from higher to lower values goods looking for the right opportunity to perform a barter and having as the last choice the remaining good of lowest value. We recall indeed that at any pass, in case of a refusal, each player may decide to prune his/her own sets of goods.

In the case of **sequential requests** we have that the two players make the choice one after the other according to an order that, at each step, depends on a random device. In this case, therefore, the players can adopt strategies similar to those we have seen for the simultaneous requests case but can try to exploit the advantage of being second mover.

Let us suppose we are at a generic step where A moves as first and B as second. We consider B 's point of view but similar considerations hold also for A 's point of view. B has ordered the goods of I in increasing order of value. In this case we have:

- A chooses $j \in J$ so that B is able to evaluate $v_B(j)$
- B can choose $i \in I$ so to get a high value of his utility $u_B(i, j) = s_B(i) - v_B(j)$ but
- without hurting A since in that case A could refuse the barter.

We recall that a refusal may turn into the pruning of a set and so in a unfavorable outcome for the requesting player that had requested the good that is going to be pruned. From these considerations we derive that the step-by-step strategy that we have seen in the simultaneous requests case can be profitably used also in this case.

Similar strategies can be conceived, with the proper modifications and adaptations, for the other three models of barter that we are going to describe in the next two sections.

8.5 One-to-many and many-to-one barterers

In these two symmetric cases one of the two actors has the possibility to require one good whereas the other has the possibility to require a basket of goods (that can even contain a single good) and so any subset of the goods offered by the former. This kind of barter must be agreed on by both actors and can occur only if one of the two actor agrees to be offering a pool of “light” goods whereas the other agrees to be offering a pool of “heavy” goods.

The meaning of the terms “light” and “heavy” may depend on the application and must be agreed on during a pre-barter phase by the actors themselves. We remark how the adopted perspective (lack of any quantitative common scale) turns into qualitative evaluations of the goods so that they are termed **light** if they are assigned **qualitatively low values** whereas they are termed **heavy** if they are assigned **qualitatively high values**.

The aim of this preliminary phase is to give one of the two actors the possibility of asking for any set of goods whereas this same possibility is denied to the other. If there is no agreement during this phase, three possibilities are left: they may decide either to give up (so the barter process neither starts) or to switch to a one-to-one barter (see section 8.2) or to a many-to-many barter (see section 8.6).

If there is a pre-barter agreement we may have two symmetrical cases. In this section we are going to examine only the “one-to-many” case. In this case we have that A owns “light” goods and may require only a single good $j \in J$ but B owns “heavy” goods and may require (at her free choice) a subset $\hat{I}_0 \subseteq I$ of goods with $|\hat{I}_0| \leq n$ and the two requests may be either simultaneous or sequential.

In the case of **simultaneous requests** both actors can evaluate their respective utilities, soon after the requests have been revealed, by using equivalent relations to (26) and (27):

1. $u_A(\hat{I}_0, j) = s_A(j) - v_A(\hat{I}_0)$
2. $u_B(\hat{I}_0, j) = s_B(\hat{I}_0) - v_B(j)$

where both players use equations like (24) and (25) and the additivity hypothesis.

Also in this case we can have the four cases we have seen in section 8.2. We note, however, how in this case if, for instance, A refuses, using a rule such as (28), he can either repeat his request (with B keeping fixed her request) or can act as we are going to show in section 8.6. In the latter case indeed A can partition his goods in subsets that he is willing to barter, possibly updating these subsets at every refusal. Except for this fact the barter goes on as in the *one – to – one* case with simultaneous requests.

In the case of **sequential requests** the procedure does not use a chance move to assign one of the two actors the right to move first but gives this right to the actor that owns the pool of “light” goods. After this first move the barter goes on as in the *one – to – many* case with sequential requests but without

any chance move and with the modification we have introduced for the case of the refusal (see section 8.6).

8.6 Many-to-many barter

In this case A may choose and require any subset $\hat{J}_0 \subseteq J$ with $1 \leq |\hat{J}_0| \leq m$ of the goods of B whereas B may choose and require any subset $\hat{I}_0 \subseteq I$ with $1 \leq |\hat{I}_0| \leq n$ of the goods of A and the two requests may be either **simultaneous** or **sequential**.

Also this kind of barter must be agreed on by both actors in a pre-barter phase during which they both agree that in the course of the barter each of them can ask for any subset of the goods of the other player.

Since also in this case we can have either simultaneous or sequential requests the algorithms are basically the same that in cases of one-to-one barter. The main differences are about

- (1) the use of the subsets and
- (2) the way in which every case of refusal is managed.

As to the point **(1)** we note that in the algorithms we must replace single elements with subsets of the pool of goods so that the evaluations must be performed on such subsets by using equations (24) and (25) and so the additivity hypothesis.

As to the point **(2)** in the algorithms for the one-to-one barter the solution we adopted was the possible pruning of the set of the goods from the refusing actor (see the points 5 or 9 (b), (c) and (d) of the algorithms of section 8.3).

This solution cannot be applied in the present case since this policy could empty one of the two initial pools or both in a few steps. To get a solution in this case we can devise an independent partitioning strategy of the two sets of goods from both actors A and B .

In this case at the very start of the barter the two players show each other their sets of goods so to hide their preferences that are partially revealed only after each refusal. After every (possibly double) refusal the player who refuses (be it A) uses the procedure $partitioning_A(I)$ to split I in labeled disjoint subsets so to make clear to B which are the subsets of goods that he is inclined to barter at that stage. The case of B is fully symmetric. We note that under the additivity hypothesis the sets I and J can be partitioned at will by their respective owner. This solution is implemented by replacing all the occurrences of the assignment instructions $I = I \setminus \{i\}$ and $J = J \setminus \{j\}$ respectively with the following assignments:

$$I = partitioning_A(I) = \{I_i \mid \cup_i I_i = I \ I_i \cap I_j = \emptyset \ \forall i \neq j\} \quad (30)$$

$$J = partitioning_B(J) = \{J_i \mid \cup_i J_i = J \ J_i \cap J_j = \emptyset \ \forall i \neq j\} \quad (31)$$

so to replace a flat set with a set of disjoint labeled subsets.

In this case, referring to A , we have that if A refuses the barter proposed by

B he can either repeat his request with B keeping fixed her request or he can partition his set in subsets as collective goods that he is willing to barter with subsets of the goods of B . The case of B is fully symmetrical.

We recall how the barter in this case may evolve as follows. At the very start the two players propose each other their sets of goods. Then we can have the following cases. (1) Both players make a request and both accept. In this easy case the barter is successful and ends. (2) Both players make a request but one accepts whereas the other refuses. The refusing player has the possibility to rearrange his set of goods. This rearrangement is a partitioning of the player's set of goods according to the rules (30) or (31) so that the other player, at the next step, knows which are the subsets that can enter successfully into a barter. (3) Both players make a request and both refuses. The rearrangement is performed by both players at the same time.

For further details we refer to section 8.3.

8.7 The basic criteria

In this section we refer to the criteria we introduced in section 4. Such criteria, in order to be used in our context of two players without either any common scale or any numerary good, must be adapted or must be redefined somehow so to be in agreement either with the essence of their classical definitions or with intuition or with both. In what follows we are going to make use of a general notation that must be specialized in the single models we have already presented in the proper past sections.

We start with **envy-freeness**. If we denote with¹⁴ $a_A(\cdot)$ and $l_A(\cdot)$ the values in A 's opinion and evaluation, respectively, of what A obtains and loses from the barter (and with $a_B(\cdot)$ and $l_B(\cdot)$ the same quantities for player B ¹⁵) we say that the allocation deriving from a barter (or a barter tout court) is **envy-free** if we have for A :

$$\frac{a_A(\cdot)}{l_A(\cdot)} \geq 1 \quad (32)$$

and for B :

$$\frac{a_B(\cdot)}{l_B(\cdot)} \geq 1 \quad (33)$$

As we have already seen from section 8.1 on, if a barter actually occurs it is guaranteed to be envy-free. Relation (32) means that the value that A assigns

¹⁴With \cdot we denote a generic set of bartered goods. This set may contain, depending on the case, also a single element.

¹⁵Such quantities need to be specialized case by case. In the one-to-one barter model, for instance, we have that:

1. $a_A(\cdot) = s_A(j)$
2. $l_A(\cdot) = v_A(i)$
3. $a_B(\cdot) = s_B(i)$
4. $l_B(\cdot) = v_B(j)$

whereas in the other cases the single elements must be replaced by the properly defined subsets.

to what he gets from the barter is at least equal to the value that A assigns to what he loses from the barter. We assign a similar meaning to relation (33) with regard to B .

Since, in the case of two players, we want to maintain the equivalence between proportionality and envy-freeness we must give a definition that mirrors the classical definition of proportionality and reflects this equivalence.

For player A we may define a barter as proportional if it satisfies the following condition:

$$\frac{a_A(\cdot)}{a_A(\cdot) + l_A(\cdot)} \geq \frac{l_A(\cdot)}{a_A(\cdot) + l_A(\cdot)} \quad (34)$$

so that the fractional value of what A gets from the barter is at least equal to that of what he loses from it. We remark that $a_A(\cdot) + l_A(\cdot)$ represents the value that A assigns to the bartered goods.

A similar condition holds also for B :

$$\frac{a_B(\cdot)}{a_B(\cdot) + l_B(\cdot)} \geq \frac{l_B(\cdot)}{a_B(\cdot) + l_B(\cdot)} \quad (35)$$

We say that a barter is proportional if both (34) and (35) hold.

It is easy to see how from equation (34) it is possible to derive equation (32) and vice versa. The same holds also for equations (35) and (33).

As to **equitability** we must adapt its definition to our framework in the following way. We need firstly some definitions. We define (with respect to the occurrence of the barter itself) I and I' , respectively, as the ex-ante and ex-post sets of goods of A and J and J' , respectively, as the ex-ante and ex-post sets of goods of B . If (i, j) denotes the bartered goods (i from A to B and j from B to A) in a one-to-one barter, we have:

$$I' = I \setminus \{i\} \cup \{j\} \quad (36)$$

$$J' = J \setminus \{j\} \cup \{i\} \quad (37)$$

In the case of other kind of barter involving also subsets of goods we must appropriately replace single goods with subsets.

On the sets I' and J' we define, for the player A , the quantities that represent the values for A himself, after the barter, of his goods and B 's goods, respectively, as $a_A(I')$ and $l_A(J')$. We therefore define a barter as **equitable** for A if the fractional value of what he gets is at least equal to the fractional value he gives to what he loses from the barter or:

$$\frac{a_A(j)}{a_A(I')} \geq \frac{l_A(i)}{a_A(I')} \quad (38)$$

On the other hand the barter is equitable for B if, using the corresponding quantities we used in equation (38) but referred to player B , we have:

$$\frac{a_B(i)}{a_B(J')} \geq \frac{l_B(j)}{a_B(J')} \quad (39)$$

If both relations hold we say that the barter is **equitable**. We remark that we are under an additivity hypothesis where the value of a set is given by the sum of the values of its elements so that the value that a player assigns to a set, such as I' or J' , is the sum of the values that the player assigns to the elements of that set.

As to **efficiency** we say that a barter of the two goods (i, j) (or of the one-to-one type) is efficient if there is not another pair of goods (i', j') that gives at least to one player a better result without hurting the other.

For players A and B this means that there is no barter (i', j') that satisfies the following inequalities:

$$\frac{a_A(j)}{l_A(i)} \leq \frac{a_A(j')}{l_A(i')} \quad (40)$$

$$\frac{a_B(i)}{l_B(j)} \leq \frac{a_B(i')}{l_B(j')} \quad (41)$$

with at least one of them satisfied with the $<$ relation.

In such relations the pairs $l_A(i)$, $a_A(j)$ and $l_A(i')$, $a_A(j')$ are related to A and are associated respectively to (i, j) and to (i', j') . Similar quantities are defined also for player B .

We remark how we are under the hypothesis that at least one of the following inequalities hold:

1. $i' \neq i$

2. $j' \neq j$

Also in this case if the barter involves subsets of goods such relations must be modified by replacing single goods with properly defined subsets of goods. We note that if the barter is such that both players attain:

$$\frac{a_{A_{max}}}{l_{A_{min}}} \quad (42)$$

and

$$\frac{a_{B_{max}}}{l_{B_{min}}} \quad (43)$$

we are sure to have an **efficient barter** whereas if both attain:

$$\frac{a_{A_{min}}}{l_{A_{max}}} \quad (44)$$

and

$$\frac{a_{B_{min}}}{l_{B_{max}}} \quad (45)$$

we are sure that the barter is surely inefficient. In (42) and (44) with respectively $a_{A_{max}}$ and $a_{A_{min}} \leq a_{A_{max}}$ we denote the maximum and minimum values that A assigns to the goods he can get from the barter and with $l_{A_{max}} \geq l_{A_{min}}$ and $l_{A_{min}}$ we denote the maximum and minimum values that A assigns to the goods he may lose from the barter. In (43) and (45) we have the same quantities

assigned to the corresponding goods by player B .

We remark how conditions (42), (43), (44) and (45) are sufficient conditions of efficiency and do not represent effective strategies for each player since condition (42), for instance, has a quantity that depends on the choice of A at the numerator but a quantity that depends on the choice of B as denominator. Last but not least, we note, from the equations (42) and (43), how **efficiency** of a barter cannot be always guaranteed and must be verified case by case.

8.8 Fairness of the proposed solutions

In this section we aim at verifying if the solutions we have proposed in the previous sections satisfy the criteria we stated in section 8.7 so that we can say whether they produce fair barter or not.

We start with **envy-freeness** in the one-to-one barter. In this case a barter occurs if and only if both A and B get a non negative utility from it or if both players think each of them gets no less than one loses. This turns, in the simplest case, in the following conditions (involving strictly positive quantities):

$$(b_1) \quad s_A(j) - v_A(i) \geq 0 \text{ or } \frac{s_A(j)}{v_A(i)} \geq 1$$

$$(b_2) \quad s_B(i) - v_B(j) \geq 0 \text{ or } \frac{s_B(i)}{v_B(j)} \geq 1$$

so that (b_1) coincides with relation (32) and (b_2) coincides with relation (33).

In this way we can derive that if a barter occurs then it is guaranteed to be **envy-free** (and therefore **proportional**, since we have maintained the equivalence between the two concepts in the current case of two players).

In more complex settings things may be more tricky to prove but, following similar guidelines, it is possible to show that whenever a barter occurs it is guaranteed to be envy-free.

We recall that in every case where a set of goods is involved we can evaluate its worth by using the additivity hypothesis.

As to **equitability** (see relations (38) and (39)) we refer only to player A since the case of B is completely analogous. In this case we remark that:

$$(eq_1) \quad a_A(j) < a_A(I')$$

$$(eq_2) \quad l_A(i) < a_A(I)$$

From (eq_1) and (eq_2) we can easily derive $a_A(j)l_A(i) < a_A(I')a_A(I)$ or:

$$\frac{a_A(I')}{a_A(j)} > \frac{l_A(i)}{a_A(I)} \quad (46)$$

On the other hand from (eq_1) it is possible to derive:

$$\frac{a_A(I')}{a_A(j)} > \frac{a_A(j)}{a_A(I')} \quad (47)$$

If we compare relations (46) and (47) with relation (38) we can easily see that there may be possibilities to have an equitable barter for A and, in a similar way, an equitable barter for B so to get an **equitable barter**.

For A this occurs if we get:

$$\frac{a_A(I')}{a_A(j)} > \frac{a_A(j)}{a_A(I')} > \frac{l_A(i)}{a_A(I)} \quad (48)$$

since the rightmost inequality is equivalent to relation (38).

In order for this to happen we must have:

$$a_A(j)a_A(I) > a_A(I')l_A(i) \quad (49)$$

or:

$$a_A(j)a_A(i) = a_A(j)l_A(i)\alpha > a_A(I')l_A(i) \quad (50)$$

so that we need to find the minimum value $\alpha > 1$ such that:

$$\alpha a_A(j) > a_A(I') \quad (51)$$

holds. Instead than using (eq_1) we could have used (eq_2) so to derive the corresponding necessary value for another proportionality coefficient β .

In this way, since we do not use at all the condition of envy-freeness, we establish an independence between the two concepts but for the fact that if a barter is not envy-free it does not occur so that it is not possible to evaluate its degree of equitability.

Last but not least we deal with the verification of the **efficiency** of a barter (i, j) in the case of a one-to-one barter. In this case we must verify that there is not another barter (i', j') such that the relations (42) and (41) hold.

Even if A choses \hat{j} (see section 8.4) B could have chosen i' such that $l_A(i') < l_A(i)$ so that relation (42) (with $j = \hat{j}$) would be verified implying that the current barter (i, \hat{j}) is not efficient.

Similar considerations hold also for B . From these considerations we derive that **efficiency** for both players can be verified only a posteriori. If it is violated we derive **inefficiency** from which both actors may derive a **regret** that could be (at least partially) compensated through repeated barterers (see section 8.9).

Summing up, we can say that, in the case of one-to-one barter:

- envy-freeness is guaranteed every time a barter occurs,
- equitability may be guaranteed at every barter,
- efficiency must be verified a posteriori at every barter,

so that the **fairness** of a barter is a by-product of the barter process itself and is not a-priori guaranteed by its structure.

Similar considerations hold also for the other three models by replacing single elements with properly defined subsets of elements.

8.9 Extensions

The planned extensions include the possibility of (1) **repeated barter** involving (2) even **more than two** players and (3) the **relaxing of additivity**. If we allow the execution of **repeated barter**s we must introduce and manage the possibility of the retaliations between the players from one barter session to the following sessions and how the pool of goods are defined and/or modified between consecutive barter sessions. In the proposed algorithms (currently stateless) we can deal with the presence of the **retaliation** through **state variables** that account for past attitudes of the players (Axelrod (1985) and Axelrod (1997)).

If we allow the presence of **more than two players** we must introduce the mechanisms for the execution of parallel and concurrent negotiations.

If, for instance, we have three actors A , B and C we can have (in the case of one-to-one barter with simultaneous requests) the following possibilities:

- (1) circular one-to-one requests where, for instance, A makes a request to B , B to C and C to A ;
- (2) one-to-many requests so that A makes a request to B and C , B makes a request to A and C and C makes a request to B and A .

In the former case there can be no conflict/concurrence whereas in the latter it can occur that two actors ask the same item to the third causing a conflict that must be resolved in some way.

In both cases we have that:

- the barter occurs if and only if every actor accepts what is proposed by the others;
- if all actors refuse the others' proposals a rearrangement (that depends on the nature of the barter) of the respective pools occurs followed by a repetition of the barter;
- in all the other cases the procedure must allow the refusing actors (two at the most) to repeat their request.

Obviously in all the other cases the interactions tend to be more and more complex. Analysis of such extensions can be carried out using the tools suggested in Myerson (1991), section 9.5 where *graphical cooperation structures* are introduced and used.

As a last extension we mention the **relaxing of additivity**. Additivity is undoubtedly a simplifying assumption and is based on the hypothesis of the relative independence of the goods that the actors want to barter among themselves. This hypothesis in many cases is not justified since functional links, for instance, make the goods acquire a value when and only when they are properly combined. In such cases the goods must be bartered as dynamically chosen subsets and cannot enter properly in a one-to-one barter. The issue is very complex (so complex that Brams and Taylor (1996) and Brams and Taylor (1999)

deal with it only marginally) and here we only make some basic comments and considerations and present a toy example.

We recall that player A choses among the goods of B and vice versa. What A loses, owing to the choice performed by B , belongs to the set I and is evaluated according to the values of v_A and what he gets belongs to J and is evaluated according to the values of s_A . Similar considerations hold also for player B .

Up to now we have supposed that A evaluates subsets of the goods involved in the barter with additive rules and that the same holds also for B . From this point on we are going to consider both subadditivity and superadditivity for player A but similar considerations hold also for player B .

We note that as to s_A subadditivity (or the case where the value of the set is lower than the sum of the values of its composing elements) is meaningless since in this case A would be better off by simply asking for a single good from B (possibly in a repeated barter). On the other hand subadditivity on v_A is highly implausible since there is no reason to believe that A would bring to the barter goods that taken as sets are worth less than the single goods.

From these considerations we derive that A sees :

- (1) J in a superadditive way by hypothesis and
- (2) I in a superadditive way as his worst case

and similar considerations hold also for the player B .

As to (1) this means that $\forall K \subseteq J$:

$$s_A(K) \geq \sum_{j_k \in K} s_A(j_k) \quad (52)$$

A is of course more interested in subsets $K \subseteq J$ such that:

$$s_A(K) > \sum_{j_k \in K} s_A(j_k) \quad (53)$$

We call the subsets for which relation (52) holds the **superadditive subsets** of J and those for which relation (53) holds the **strictly superadditive subsets** of J .

As to (2) we recall that I contains the goods that A loses in the barter so that the condition:

$$v_A(H) \geq \sum_{i_h \in H} v_A(i_h) \quad (54)$$

(for $H \subseteq I$) represents a worst condition for A with regard to the additive case in the evaluation of his utility in the one-to-many and many-to-many barter cases.

At this point we have the cases of Table 4 where we show the possible typologies of the players with regard to the values s_A for A and s_B for B .

From this perspective, the fact that A is superadditive means that at least relation (52) holds and the same is true for B if she is superadditive.

From that Table we see that if both players are superadditive they are more

A vs. B	additive	superadditive
additive	one-to-one	one-to-many
superadditive	many-to-one	many-to-many

Table 4: *Possible types for the ways in which each player evaluates their requested goods*

willing to agree on a many-to-many barter, if they are both additive they may prefer a one-to-one barter whereas if one is superadditive and the other is additive they may agree on either a many-to-one or a one-to-many barter depending on which is the superadditive player.

In the closing part of this section we are going to deal only with the **many-to-many** barter case with **simultaneous requests** where A asks for the goods of the set $J_0 \subseteq J$ and loses the goods of the set $I_0 \subseteq I$ whereas B asks for the goods of the set $I_0 \subseteq I$ and loses the goods of the set $J_0 \subseteq J$.

Also in this case the core of the algorithms (see sections 8.3 and 8.6) is composed by the four cases that may occur at each pass:

- (a) both A and B accepts the proposed barter so that the process ends with a success;
- (b) A accepts but B refuses;
- (c) A refuses whereas B accepts;
- (d) both A and B refuse.

In the symmetric cases (b) and (c) the accepting player keeps his request fixed while the refusing player has two possible mutually exclusive strategies:

- can repeat his choice;
- can partition (on the first refusal) or rearrange a partitioning (on successive refusals) his set of goods so that another round may occur.

In the case (d) each player has both the repeater and the modifier strategies at his disposal.

The fact that a player rearranges in some way his goods through the definition of variable partitions interfere with the superadditive evaluations of the other player and this may cause both players agree that there is no possibility for the process to go on (see the step (2.a) of the simultaneous requests algorithm of section 8.3).

To make things more concrete we now make a toy example. We suppose to have a player A with his set of goods $I = \{i_1, i_2, i_3, i_4, i_5\}$ and another player B with her set of goods $J = \{j_1, j_2, j_3, j_4, j_5, j_6\}$.

We suppose that both A and B have [strictly] superadditive evaluations of the involved goods (so that relation (52) and possibly relation (53) hold) and both

are interested in a many-to-many barter and agree to carry it on.
Such a barter may therefore involve not all the possible subsets of I and J but only some of them so that:

- B can see I as made of the following set of [possibly strictly] superadditive subsets without A knowing this:

$$I = \{I_1, I_2, I_3, I_4\} = \{\{i_1, i_2\}, \{i_1, i_3, i_4\}, \{i_4, i_5\}, \{i_2, i_3, i_5\}\} \quad (55)$$

- A can see J as made of the following set of [possibly strictly] superadditive subsets without B knowing this:

$$J = \{J_1, J_2, J_3, J_4, J_5\} = \{\{j_1, j_2, j_3\}, \{j_1, j_3, j_4\}, \{j_4, j_5\}, \{j_3, j_5, j_6\}\}, \{j_2, j_3, j_5, j_6\}\} \quad (56)$$

At the very start A and B agree on a many-to-many barter with, for instance, simultaneous requests. Both A and B make their requests and evaluate their utilities so to decide if each accepts or refuses.

If both players accept the barter ends with a success.

If only one player refuses, only the refusing player can either reiterate the request or partition his set of goods so to make clear to the other which subsets he is willing to barter. The players have the same possibilities also in the case of double refusal.

In this way the partitioning may conflict with the way in which each player sees the goods of the other so that either they are able to fix this mismatch, during the next phases, in order to attain the barter or both declare that no barter is possible and so the barter ends with a failure. For the fine grain structure of the algorithm we refer to the sections 8.3 and 8.6.

In our example we could have:

- (1) A asks for J_1 and B asks for I_2 so the currently tentative barter is (I_2, J_1) ;
- (2) A refuses and B accepts;
- (3) A may change his request as J_3 so the currently tentative barter becomes (I_2, J_3) ;
- (4) B refuses and partitions J as $J = \{J', J''\} = \{\{j_1, j_2, j_4\}\{j_3, j_5, j_6\}\}$;
- (5a) A asks for J' and both accepts so the barter occurs;
- (5b) A refuses so that both refuse and decide that no barter is possible;
- (5c) A may change his request as J_5 so the currently tentative barter becomes (I_2, J_5) ;
- (5d) B refuses and partitions J as $J = \{J', J''\} = \{\{j_1, j_3, j_4\}\{j_2, j_5, j_6\}\}$ and so on until either (5a) or (5b) occur.

We remark how at step (3) we have $s_A(J_1) = s_A(J_3)$ but at step (4) we have $v_B(J_1) < v_B(J_3)$ so that A accepts but B refuses.

8.10 The applications of the explicit barter models

In an explicit barter we have two players A and B that truthfully reveal the sets of items they are willing to barter.

We remark how the items of each set can simply be listed without being shown and this is particularly true in the case where the sets contain also services.

If one of the players acts strategically, so to misrepresent his set of items, whereas the other acts truthfully we have that the strategic player may force the other to refuse the barter so that both suffer a missed gain.

If both act strategically they may agree to barter low value items whereas they could have at their disposals much more valuable (within the barter) items.

We have therefore determined that both actors have strong incentives to truthfully reveal the sets of items they are willing to barter. We now see some possible applications of the proposed models.

In the **one-to-one** case we may use it for the exchange of:

- one good for one good,
- one good for one service,
- one service for one chore
- one chore for one chore,
- one service for one service,

that the two players see as [at least qualitatively] equivalent in the pre-barter phase. In the **one-to-many** and **many-to-one** cases we have similar situations but for the fact that one player must allow the other to choose a subset of his items depending on a common agreement during a pre-barter phase. This means that, in the one-to-many case we have, for instance, the following cases:

- a set of goods for one good,
- a set of goods for one service,
- a set of services for one chore
- a set of chores for one chore,
- a set of services for one service.

Last but not least, in the **many-to-many** case we have that the foregoing possibility must be conceded by each player to the other so that, from an applicative point of view, this last case is coincident with the first one but for the fact that where we have one we must read one set.

9 Some remarks about the iterative barter models

9.1 Introduction

In section 10 we present¹⁶ two types of models of barter that may be seen as an extension of the models we presented in Cioni (2008c) and Cioni (2008d) and that we term as iterative since they are based on iterative algorithms through which either one or both reveal the composition of the sets of items they are willing to barter.

Both models involve indeed a pair of actors/players A and B that aim at bartering a pair of items.

In the former model (the so called **pure model**) neither actor reveals to the other the set of items he is willing to barter but such a revelation occurs incrementally during the process since by exchanging proposals and counter proposals the two players reveal each other the composition of such sets. The bargaining process goes on until an agreement is reached and a bargaining occurs or both players agree that no bargaining is possible so that the process ends with a failure.

In the latter or **mixed model**, on the other hand, we have an asymmetric situation where one of the players, be it A , shows to B his set of items, be it I' , on which the bargaining process may start with a proposal from B . Also in this case the process goes on with a series of proposals and counter proposals from both players until one of the foregoing cases occurs. In this case player B reveals the composition of his set of items during the course of the process.

We note that, in both cases, each player can be said to know which are the items he is willing to bargain in the barter process.

This knowledge may be verified in advance by asking to each player if a given proposal would be or not in his bargaining set (or the set of the acceptable proposals).

From this perspective we can say that each player is characterized by a bargaining set, I_A for A and J_B for B , whose structures and whose preference orderings are private knowledge of each player and can be only partly revealed during the barter process. The main difference between the two models that we propose is that in the latter model the set I_A is, at least partly, a common knowledge of the two players under the form of the set I' .

9.2 Some notes about the barter

In this section with the term **barter** we denote a process through which two players A and B can exchange a pair of items¹⁷ (i, j) where both items are evaluated according to each player's private evaluation system that determines either his rejection or his acceptance of the proposed items.

¹⁶The section 10 is based on Cioni (2009a).

¹⁷In what follows we sometimes refer to the former element of each pair (i, j) as the i -item and to the latter as the j -item.

The exchange, if it occurs, is in kind so that the items (i, j) are the only involved items and there is no parallel or compensatory exchange of money or any other numerary good between the players (see section 9.3).

In any generic pair (i, j) the identifier i identifies what passes from A to B either under the form of a good or a bad or a service in exchange of the item identified by j , of the same types, from B to A .

In Table 5 we show the possible ownership of the items of each pair depending on their types.

The admissible types are:

- (a) **good** (or g) or an item that has a positive value for both players;
- (b) **bad** (or b) or an item that has a negative value for both players;
- (c) **service** (or s) or an item that has a instrumental value for one or both players and may represent a task that a player carries out for the other.

Each ownership is provided under the form of a pair (p_1, p_2) (p_1 for i and p_2 for j) where both identifier may be A or B . In this way we identify the provider or owner of an item so that, for instance, in correspondence of g, g we have (A, B) to mean that the former good is owned by A and provided to B whereas the latter is owned by B and provided to A so that, as receivers, we have (B, A) . We now examine the entries of Table 5:

- in the case g, g we have A, B since A gives to B a good in exchange for another good of B ;
- in the case g, b we have B, B since B gives to A both a good and a bad;
- in the case g, s we have A, B since A gives to B a good in exchange for a service from B ;
- in the case b, g we have A, A since A gives to B both a bad and a good as a compensation;
- in the case b, b we have A, B since A gives to B a service and B gives to A another service in exchange;
- in the case b, s we have A, A since A gives to B both a bad and a service;
- in the case s, g we have A, B since A gives a service to B and receives from B a good as a compensation;
- in the case s, b we have B, B since B gives to A both a bad and a service;
- in the case s, s we have A, B since A gives to B a service and B gives to A another service in exchange.

By using the content of Table 5 we can define another Table that, for each element of each pair (i, j) , defines which player is the giver and which is the receiver. In Table 6 for each pair of items (i, j) we define:

i and j	g	b	s
g	A,B	B,B	A,B
b	A,A	A,B	A,A
s	A,B	B,B	A,B

Table 5: *Ownership of the elements of the pair (i,j)*

id	case	i giver	i receiver	j giver	j receiver
1	g,g	A	B	B	A
2	g,b	B	A	B	A
3	g,s	A	B	B	A
4	b,g	A	B	A	B
5	b,b	A	B	B	A
6	b,s	A	B	A	B
7	s,g	A	B	B	A
8	s,b	B	A	B	A
9	s,s	A	B	B	A

Table 6: *Pairs (giver, receiver) for each item of each pair (i,j)*

- who is the giver of the item i and who is the corresponding receiver;
- who is the giver of the item j and who is the corresponding receiver.

For instance in correspondence of the pair $(i, j) = (g, g)$ we have A the giver of i that is received by B and vice versa for the item j .

In this way we can identify the items each player gets or loses in a barter so to assign a value to such acquisitions and losses. In section 9.3 we are going to use such values in the definition of the evaluation criteria that we are going to apply in section 10.5 to the models we are going to describe in section 10.

For player A this means that we imagine he uses two private values $v_A(j)$ and $v_A(i)$ to evaluate, on a private scale, what A gets from the barter and what A loses from it in that order.

In this way A can evaluate the ratio:

$$\rho_A = \frac{v_A(j)}{v_A(i)} \quad (57)$$

as a dimensionless quantity.

For player B , in a similar way, this means that we imagine she uses two private values $v_B(i)$ and $v_B(j)$ to evaluate, on a private scale, what B gets from the barter and what B loses from it in that order.

In this way B can evaluate the ratio:

$$\rho_B = \frac{v_B(i)}{v_B(j)} \quad (58)$$

as a dimensionless quantity.

As to the quantities that are involved in relations (57) and (58) we note that they:

- represent private information of each player;
- are measured according to private scales that may not be common knowledge between the players;
- include possibly independent discount factors for each player so to account for damages occurring to each of them from the passing of time.

Relations (57) and (58) are used respectively by player A and player B to accept or refuse a proposed barter (see section 10).

Before going on we must fix in some way four problematic cases that are contained in Table 6 and that we list again in Table 7.

id	case	i giver	i receiver	j giver	j receiver
2	g,b	B	A	B	A
4	b,g	A	B	A	B
6	b,s	A	B	A	B
8	s,b	B	A	B	A

Table 7: *Problematic cases*

We say that the case $(i, j) = (b, g)$ ($id = 2$) is problematic¹⁸ since it does not present a A, B, B, A structure (a bidirectional transfer) but a structure that defines a transfer of both items from one player to the other (an unidirectional transfer). We can solve this case by imagining that the transfer of a bad from B to A is equivalent to the transfer of a good (of the proper compensating value) from A to B . In this way this case is brought back to the case with $id = 1$. In a similar way we can solve the case with $id = 4$.

For what concerns the case $(i, j) = (b, s)$ ($id = 6$) we can solve it by imagining that the transfer of a bad from B to A is equivalent to the transfer of a service (of the proper compensating value) from B to A . In this way this case is brought back to the case with $id = 9$.

In a similar way we can solve the case with $id = 8$.

9.3 The performance and evaluation criteria

For the evaluation of the proposed barter procedures we use both a set of **performance criteria** and a set of **evaluation criteria**.

As to the **performance criteria** we use: **guaranteed success**, **individual rationality**, **simplicity** and **stability**.

With **guaranteed success** we denote the fact that a procedure is guaranteed

¹⁸Similar arguments hold also for the other three cases.

to end with a success, with **individual rationality** we denote the fact that it is in the best interest of the players to adopt it so that they both use a procedure only if they wish to use it and can withdraw from it without any harm nor penalty higher than the damage they can receive from carrying it on.

Simplicity is a feature of the rules of a procedure that must be easy to understand and implement for the players without being too demanding in terms of rationality and computational capabilities.

Last but not least with **stability** we denote the availability to the players of equilibrium strategies that they can follow to attain stable outcomes in the sense that none of them has any interest in individually deviating from such strategies (Myerson (1991), Patrone (2006)).

As to the **evaluation criteria** we use a set of classical criteria (Brams and Taylor (1999), Brams and Taylor (1996)) that allow us to verify if a barter can be termed **fair** or not.

Such criteria are:

- envy-freeness;
- proportionality;
- equitability;
- [Pareto] efficiency.

We say a barter is fair if they are all satisfied and is unfair if any of them is violated.

In the case of two players (Brams and Taylor (1999), Brams and Taylor (1996)) envy-freeness and proportionality are equivalent, as it will be shown shortly.

Generally speaking, we say that an agreement turns into an allocation of the items between the players that is **envy-free** if (Brams and Taylor (1996), Brams and Taylor (1999) and Young (1994)) none of the actors involved in that agreement would prefer somebody's else portion, how it derives to him from the agreement, to his own. If an agreement involves the sharing of benefits it is considered envy-free if none of the participants believes his share to be lower than somebody's else share, whereas if it involves the share of burdens or chores it is considered envy-free if none of the participants believes his share to be greater than somebody's else share. In other words a procedure is envy-free if every player thinks to have received a portion that is at least tied for the biggest (goods or benefits) or for the lowest (burdens or chores).

If an allocation is envy-free then (Brams and Taylor (1999)) it is **proportional** (so that each of the n players thinks to have received at least $1/n$ of the total value) but the converse is true only if $n = 2$ (as in our case). If $n = 2$ proportionality means that each player thinks he has received at least an half of the total value so he cannot envy the other. If $n > 2$ a player, even if he thinks he has received at least $1/n$ -th of the value, may envy some other player if he thinks that player got a bigger share at the expense of some other player.

As to **equitability** in the case of two players (and therefore in our case) we say (according to Brams and Taylor (1999)) that an allocation is equitable if

each player thinks he has received a portion that is worth the same in one's evaluation as the other's portion in the other's evaluation. It is easy to see how equitability is generally hard to ascertain (Brams and Taylor (1996) and Brams and Taylor (1999)) since it involves inter personal comparisons of utilities. In our context we tried to side step the problem by using a definition that considers both utilities with respect to the same player.

Last but not least, as to **[Pareto] efficiency**, we say (according to Brams and Taylor (1999)) that an allocation is efficient if there is no other allocation where one of the players is better off and none of them is worse off. In general terms efficiency may be incompatible with envy-freeness but in the case of two players where we have compatibility.

Such criteria, in order to be used in our context of two players without either any common scale or any numerary good, must be adapted or must be redefined somehow so to be in agreement either with the essence of their classical definitions or with intuition or with both.

We start with **envy-freeness**.

If we denote with $v_A(j)$ and $v_A(i)$ the values in A 's opinion and evaluation, respectively, of what A obtains and loses from the barter (and with $v_B(i)$ and $v_B(j)$ the same quantities for player B) we say that the allocation deriving from a barter (or a barter tout court) is **envy-free** if we have for A :

$$\rho_A = \frac{v_A(j)}{v_A(i)} \geq 1 \quad (59)$$

and for B :

$$\rho_B = \frac{v_B(i)}{v_B(j)} \geq 1 \quad (60)$$

Relation (59) means that the value that A assigns to what he gets from the barter is at least equal to the value that A assigns to what he loses from the barter. We assign a similar meaning to relation (60) with regard to B .

Since, in the case of two players, we want to maintain the equivalence between proportionality and envy-freeness we must give a definition that mirrors the classical definition of proportionality and reflects this equivalence.

For player A we may define a barter as proportional if it satisfies the following condition:

$$\frac{v_A(j)}{v_A(j) + v_A(i)} \geq \frac{v_A(i)}{v_A(j) + v_A(i)} \quad (61)$$

so that the fractional value of what A gets from the barter is at least equal to that of what he loses from it. We remark that $v_A(j) + v_A(i)$ represents the value that A assigns to the bartered items.

A similar condition holds also for B :

$$\frac{v_B(i)}{v_B(i) + v_B(j)} \geq \frac{v_B(j)}{v_B(i) + v_B(j)} \quad (62)$$

We say that a barter is proportional if both (61) and (62) hold.

It is easy to see how from equation (61) it is possible to derive equation (59)

and vice versa. The same holds also for equations (62) and (60) so that the equivalence of the two definitions has been maintained in the case of two players. We now pass to the criterion of **equitability**.

We must adapt its definition to our framework in the following way. We need firstly some definitions. With respect to the occurrence of the barter of the items (i, j) we define for player A :

- V_A a measure for A himself of his current welfare before the barter occurs;
- V'_A a measure for A himself of his current welfare after the barter has occurred;

and for player B :

- V_B a measure for B himself of his current welfare before the barter occurs;
- V'_B a measure for B himself of his current welfare after the barter has occurred.

With the term **welfare** we denote a personal and private evaluation from each player of his global situation (under the hypothesis of either additivity or superadditivity¹⁹) through a single value that is used to rank the items that are entering the barter process.

We therefore define a barter of the items (i, j) as **equitable** for A if the fractional value of what he gets is at least equal to the fractional value he gives to what he loses from the barter or:

$$\frac{v_A(j)}{V'_A} \geq \frac{v_A(i)}{V_A} \quad (63)$$

On the other hand the barter is equitable for B if, using the corresponding quantities we used in equation (63) but referred to player B , we have:

$$\frac{v_B(i)}{V'_B} \geq \frac{v_B(j)}{V_B} \quad (64)$$

If both relations hold we say that the barter is **equitable**.

We note that if $V'_A \geq V_A$ then relation (63) implies envy-freeness for A whereas if $V'_B \geq V_B$ then relation (64) implies envy-freeness for B .

To make inequalities (63) and (64) of more practical use we may rewrite them, for instance for player A , as follows (for every turn after the first one):

$$\frac{v_A(j)}{\downarrow_A} \geq \frac{v_A(i)}{\uparrow_A} \quad (65)$$

where:

¹⁹With reference to sets with the term **additivity** we denote the fact that the value of a set is given by the sum of the values of its components whereas if this value is at least equal to that sum we speak of **superadditivity** and of **strict superadditivity** if it is strictly greater.

\downarrow_A is the minimum value of all the j –items that A has bargained before the current turn of bargaining;

\uparrow_A is the maximum value of all the i –items that A has bargained before the current turn of bargaining.

We say that (65) is easier to use than relation (63) because keeping track of both a maximum and a minimum value in a multi step process is easier than evaluating at each step the new value of the welfare under the hypothesis that the proposed barter occurs.

A similar relation holds also for player B :

$$\frac{v_B(i)}{\downarrow_B} \geq \frac{v_A(j)}{\uparrow_B} \quad (66)$$

Last but not least we examine the criterion of **[Pareto] efficiency**.

A barter of the items (i, j) is **[Pareto] efficient** if there is not another pair of items (i', j') that gives at least to one player a better result without hurting the other, under the hypothesis that at least one of the following inequalities hold:

1. $i' \neq i$
2. $j' \neq j$

For players A and B this means that there is no barter (i', j') that satisfies the following inequalities:

$$\frac{v_A(j)}{v_A(i)} \leq \frac{v_A(j')}{v_A(i')} \quad (67)$$

$$\frac{v_B(i)}{v_B(j)} \leq \frac{v_B(i')}{v_B(j')} \quad (68)$$

with at least one of them satisfied with the $<$ relation.

In such relations the pairs $v_A(i)$, $v_A(j)$ and $v_A(i')$, $v_A(j')$ are related to A and are associated respectively to (i, j) and to (i', j') . Similar quantities are defined also for player B .

We note that if the barter is such that both players attain:

$$\frac{v_{A_{max_j}}}{v_{A_{min_i}}} \quad (69)$$

and

$$\frac{v_{B_{max_i}}}{v_{B_{min_j}}} \quad (70)$$

we are sure to have an **efficient barter** whereas if both attain:

$$\frac{v_{A_{min_j}}}{v_{A_{max_i}}} \quad (71)$$

and

$$\frac{v_{B_{min_i}}}{v_{B_{max_j}}} \quad (72)$$

we are sure that the barter is surely inefficient. In (42), (43), (44) and (45) we have:

$v_{A_{max_j}}$ is the best j -item that A can get from the barter;

$v_{A_{min_j}} < v_{A_{max_j}}$ is the worst j -item that A can get from the barter;

$v_{B_{max_i}}$ is the best i -item that B can get from the barter;

$v_{B_{min_i}} < v_{B_{max_i}}$ is the worst i -item that B can get from the barter.

We remark how conditions (42), (43), (44) and (45) are sufficient conditions of efficiency but may also be hints for either good or bad strategies for both players.

Last but not least, we note, from the equations (67) and (68), how **efficiency** of a barter cannot be always guaranteed and must be verified case by case.

9.4 Incremental construction/revelation

One of the key point of the proposed models is the fact that either one or both players reveal incrementally the set of pairs of items (i, j) each of them is willing to barter with the other. Each pair is seen as a single element of the set and from two pairs (i, j) and (h, k) we can obtain one of the following pairs:

- (i, h)
- (i, k)
- (h, j)
- (k, j)

and all their possible convex combinations, depending on the nature of the involved items.

It is therefore necessary to understand the ways through which a set is incrementally enlarged from the initial empty set to a maximal set, the so called bargaining set, that include all the possible elements that a player is willing to barter.

The first way we can use is the following (in what follows we consider the case of A , the case of B is analogous and will not be explicitly considered).

A may start with $I_0 = \emptyset$ and add one element at a time according to some insertion criteria until a criterion of stop is met so that the process is interrupted and the final set I_∞ is constructed. In this way A builds up the following succession of sets:

$$I_0 \subset I_1 \subset \dots \subset I_\infty \quad (73)$$

where the set I_∞ may, at least in theory, contain infinitely many elements.

In this way A proceeds **bottom up** since he starts from the empty set and eventually ends with the whole set of the items I_∞ that may coincide or not with I_A (since it contains also elements from the [counter] proposals of B).

This process gives to A the greatest flexibility since it allows him to build up new elements by mixing and or merging the existing ones so that the construction process can adapt better to the course of the barter process.

The main problem with this approach is that the barter may prove a very time consuming process since the number of the possible combinations increase exponentially with the increase of the number of the available elements.

Another way that A can use is the following that we may call **top down**. A starts with his fixed and predefined set I_A of n elements. Each item is initially set as *invisible* (so that again A starts with a publicly known set $I_0 = \emptyset$) and during the process one element at a time is set visible. In this way A builds up the following succession of sets:

$$I_0 \subset I_1 \subset \dots \subset I_n \quad (74)$$

with:

$$I_n = I_A \quad (75)$$

This incremental disclosure may be obtained by using a set of n flags initially set at *invisible* and by setting at each step one flag at a time at *visible* so to reveal the associated element.

The process ends when a barter occurs or when all the elements of the set I_n are revealed without any barter occurring.

In this case we get the lowest flexibility since the elements are fixed from the start but we are sure the process has a fixed bound that must occur when all the elements have been revealed without any barter occurring.

At this point we have to define:

- in the bottom up approach, which criterion can be used by player A (and similarly by player B) to add a new element to the current set I_i ;
- in the top down approach, which criterion can be used by player A (and similarly by player B) to set as visible a new element to the current set I_i .

In both cases the simplest criterion is the following: a player either adds or sets visible an element that is expected to give him an advantage greater than the one deriving him from the current proposal.

We recall that such an element represents the new [counter] proposal and so an advantageous elements for the proposing player (otherwise he would not propose it).

The other point is to clarify how player A (but the same is true also for player B) can derive at step i the element (h, k) such that:

$$I_i = I_{i-1} \cup (h, k) \quad (76)$$

In the bottom up approach such an element can be either an element of I_A or an element composed by using elements from J_{i-1} or a mixture of both cases.

In the top down approach such an element is simply one of the elements of I_A .

10 The iterative barter models

In this section we propose two iterative barter models. In the former model, that we call **pure model**, neither A nor B shows each other the sets of the items they wish to barter.

In the latter model, that we call **mixed model**, we suppose that one player, be it A , shows the items he is willing to barter from the offset of the process whereas the other, in this case B , keeps her items hidden but reveal them during the process by making either proposals or counter proposals aiming at the reaching of an agreement and therefore a barter.

10.1 General remarks

Both models are described by using simple algorithms of which we present the general structure and the various options that the each of two players has at each step. In order to keep the structure of each algorithm simple and readable we may use strings to describe sub procedures that we verbally describe separately.

Since both models are based on a succession of proposals and counter proposals we firstly need to define what do we mean with the terms proposal and counter proposal. We also list which are the moves that each player can use during the process.

A **proposal** is a pair of item identifiers (i, j) that a player proposes to the other as the object of the barter and whose ownership is defined as we have seen in Table 5.

On the other hand, given a proposal (i, j) a **counter proposal** is a pair (i', j') such that either $i' \neq i$ or $j' \neq j$ is true since:

- if $i' = i$ and $j' = j$ we have an implicit acceptance so that the process ends with a success;
- if $i' \neq i$ and $j' \neq j$ we define it as a new proposal.

It is obvious that a proposal in reply to a counter proposal is termed a counter proposal and not a counter counter proposal so to avoid the chaining of counter prefixes.

Within our perspective we have that:

- a counter proposal may follow only a proposal;
- a new proposal may follow either a pass move or a reject move (see further on).

Both a proposal and a counter proposal can be followed by one of the following moves from the listening player:

- **pass**,
- **give up**,

- **accept**,
- **reject**.

A **pass** move is a way through which a player may signal to the other that it is necessary that he shows some more goodwill in order for the process to go on.

A **give up** move is a way for one player to signal to the other that he thinks the process is not worth being carried on any more.

We note that two successive give up moves (one from each player) cause the process termination with a failure.

An **accept** move closes the barter with a success since it signals that a player accepts the last [counter] proposal made by the other player.

A **reject** move means that the received [counter] proposal made by the other player cannot be accepted.

Both an **accept move** and a **reject move** can follow any [counter] proposal but if the answer is a reject move the turn remains to the rejecting player that can make his counter proposal.

A **pass move** gives the turn to the other player and may be answered by either a new proposal, by a pass move or by a give up move. We note that there cannot be more than two consecutive pass move so that a natural succession for a closure with a failure may be: *pass_A, pass_B, give up_A, give up_B*.

Last but not least a give up move may be followed either by a new proposal or by another give up move from the other player: in the former case the process goes on whereas in the latter it is interrupted with a failure and in a way that does not necessarily involve the use of pass moves.

10.2 The pure model

In the case of the **pure model** the situation we are interested in can be described in the following terms.

We have one player that wants to exchange an item with another player but none of them has a knowledge of the items the other is willing to barter.

The only way to proceed is through an iterative process. At each step of the process a pair of items (i, j) is proposed and such a pair may be either accepted or refused in some way.

In the former case the process ends with a success so that the barter occurs. In the latter case we may have:

- a pass move so the next move is up to the other player,
- a reject move so the next move is up to the same player;
- a counter proposal.

At the beginning of the barter we have:

the set of pairs-of-items-to-be-bartered of A is I_0 ;

the set of pairs-of-items-to-be-bartered of B is J_0 .

We therefore have an *initialization phase* where we put:

- $I_0 = \emptyset, J_0 = \emptyset, i = 1$ and $j = 1$;
- we select at random who moves first, be it A . The other case being fully symmetrical will not be examined here.

In the description of the algorithm we use the notation $propose_A$ to summarize the execution of the following steps:

1. A presents a [counter] proposal $p_A = (i, j)$;
2. $I_i = I_{i-1} \cup p_A$;
3. $i = i + 1$;

and symmetrically we use the notation $propose_B$ to summarize the execution of the following steps:

1. B presents a [counter] proposal $p_B = (i', j')$;
2. $J_{j'} = J_{j'-1} \cup p_B$;
3. $j' = j' + 1$;

The main structure of the algorithm in this case is the following:

- (0) *initialization phase*;
- (1) $propose_A$;
- (2) B may:
 - (2_a) *accept*; go to (4);
 - (2_b) *reject*; $propose_B$; go to (3);
 - (2_c) $propose_B$; go to (3);
 - (2_d) *give up*; go to (5);
 - (2_e) *pass*; go to (6);
- (3) A may:
 - (3_a) *accept*; go to (4);
 - (3_b) *reject*; $propose_A$; go to (2);
 - (3_c) $propose_A$; go to (2);
 - (3_d) *give up*; go to (5);
 - (3_e) *pass*; go to (6);
- (4) end;

If we denote with the player i either A or B and with j either B or A we can define the moves that can follow either a *pass* or a *give up* move in the following ways:

- (5) a *give up_i* move may be followed by:
 - (5_a) *propose_j*; go to (3) if $j = B$ else go to (2);
 - (5_b) *give up_i*; go to (4);
- (6) a *pass_i* move may be followed by:
 - (6_a) *propose_j*; go to (3) if $j = B$ else go to (2);
 - (6_b) *pass_j*; *give up_i*; *give up_j*; go to (4);
 - (6_c) *give up_j*; go to (5);

The execution of either an *accept* or a *reject* move on a proposal (i, j) from player A (the case of player B is fully symmetrical and will not be analyzed) is based on the consideration of the values $v_A(j)$ and $v_A(i)$ through a function:

$$eval_A(i, j) = f_A(v_A(j), v_A(i)) \quad (77)$$

where the function f_A synthesizes the procedure of comparison from A between the values $v_A(j)$ and $v_A(i)$. In its simplest form we can express it as:

$$f_A(v_A(j), v_A(i)) = v_A(j) - v_A(i) \quad (78)$$

Such function can be used in rules such as the following²⁰:

$$\mathbf{if}(eval_A(i, j) \geq 0) \mathbf{then} \textit{accept}_A \mathbf{else} \textit{refuse}_A \quad (79)$$

so to establish a strict preference ordering \succ on the proposals. We can indeed say²¹:

$$(i, j) \succ_A (i', j') \Leftrightarrow eval_A(i, j) > eval_A(i', j') \quad (80)$$

and the same holds also for B .

10.3 The mixed model

In the **mixed model** we have an asymmetric situation where one of the players, be it A , shows to the other, B , his set of items I' whereas the first move is up to the other player, B in this case.

The algorithm in this case has the following structure that is very similar to the one we have seen in section 10.2 but for the *initialization phase* and the starting move.

- (0) *initialization phase*;

²⁰We could have $eval_A(i, j) \geq \varepsilon$ with $\varepsilon > 0$ if a minimum gain is required or $\varepsilon < 0$ if a maximum loss is acceptable.

²¹It is obvious that with \succ_A we denote the strict preference relation of player A .

- (1) *propose_B*;
- (2) *A* may:
 - (2_a) *accept*; go to (4);
 - (2_b) *reject*; *propose_A*; go to (3);
 - (2_c) *propose_A*; go to (3);
 - (2_d) *give up*; go to (5);
 - (2_e) *pass*; go to (6);
- (3) *B* may:
 - (3_a) *accept*; go to (4);
 - (3_b) *reject*; *propose_B*; go to (2);
 - (3_c) *propose_B*; go to (2);
 - (3_d) *give up*; go to (5);
 - (3_e) *pass*; go to (6);
- (4) end;

For the points (5) and (6) and for the description of the *propose_A* and *propose_B* moves we refer to the same points and the same description we gave in section 10.2.

The *initialization phase* is unchanged for *B* (so that again we have $J_0 = \emptyset$ and $j = 1$) but for *A* we must account for the presence of the set I' whose content is defined by *A* freely and completely at his will for what concerns both the type and the number of the contained items.

From this we have:

$$I_0 \subset I' \otimes I' \tag{81}$$

where:

$$I' \otimes I' = \{(i, j) \mid i, j \in I' \text{ but } i \neq j\} \tag{82}$$

so that I_0 is rather fuzzily defined for *B* so reducing his seemingly initial advantage over *A*.

10.4 Possible strategies

We now present and analyze the possible strategies of the two players in either the pure model or the mixed model.

In the **pure model** both players act without any knowledge of the other's player set of elements but with each player only knowing his bargaining set, I_A if *A* and J_B if *B*.

In this case the player who moves first, be it *A*, can do no better than choosing an element (i, j) from his bargaining set I_A so to maximize the value:

$$eval_A(i, j) = f_A(v_A(j), v_A(i)) \tag{83}$$

This is possible for A if he chooses one of the two cases $id = 4$ or $id = 6$ of Table 6 where he has full control of both the items of the proposed element $I_1 = \{(i, j)\}$.

In his turn B may use both J_B and I_1 to devise his [counter] proposal to A so defining his set J_1 after his first step.

At the generic step $i > 1$ for A and $j > 1$ for B we have that each player:

- knows the current proposal (i, j) and is able to evaluate it using the function²² $eval_h$;
- can possibly devise a [counter] proposal (i', j') and propose it if it is preferred by the player to the current one according to the preference ordering we introduced with relation (80) or if:

$$(i', j') \succ_h (i, j) \Leftrightarrow eval_h(i', j') > eval_h(i, j) \quad (84)$$

- can frame the new [counter] proposal on the knowledge of both I_i and J_j so to favor at the best also the other player (therefore increasing the probability of an acceptance) by using the common knowledge between them;
- if the [counter] proposal of the foregoing steps cannot be devised the player can either accept the current proposal, press the other for a change of route (with a *pass* move) or signal the will to stop the process (with a *give up* move).

As the last step we note how each player can use also his bargaining set for framing a new [counter] proposal but with a lower probability of if being accepted by the other player.

In the **mixed model** the presence of an asymmetry does not modify very much what we have seen before since the initial knowledge of B of the items of A is rather fuzzy and, moreover, she is the player who must move first. We can, therefore, state that the strategies we have seen for the **pure model** case can be profitably used also in the **mixed model** case.

10.5 Satisfaction of the criteria and applications

We now examine if the proposed models satisfy or not the performance and evaluation criteria we have introduced in section 9.3.

For what concerns the **performance criteria** we have that:

- **guaranteed success** is not always satisfied since a process of barter may end without any effective barter and without any penalty for the player who withdraws from the process;

²²With the subscript h we denote either the player A or the player B depending on the case.

- **individual rationality** is guaranteed to each player that feels he engaged himself in an unfavorable process from the availability of *pass* and *give up* moves;
- **simplicity** is assured by the fact that the frame of both algorithms is a sequence of proposals and counter proposals until an agreement is reached either to have a barter or to give up since no barter is effectively possible;
- **stability** is satisfied since both players have easy to follow and implement strategies (see section 10.4).

We recall that the acceptance or not of a proposed barter is governed by the relations (77), (78) and (79) of section 10.2.

For what concerns the **evaluation criteria** we have that if a barter occurs, according to the previously recalled relations, then it is **envy-free** (see relations (59) and (60)) for both players and therefore it is also **proportional**, from the equivalence of the two concepts in the case of two players.

It is easy to verify, by the same relations, that if a barter is fair then it occurs. The check of the **equitability** requires the check of the relations (63) (or of the corresponding relation (65)) and (64) (or of the corresponding relation (66)). Such a check may be performed by both players at every step of the barter so that, in the best case:

- one player may propose a barter that for him is envy-free and equitable;
- the other may verify such a proposal and accept if he too thinks that for him it is envy-free and equitable.

As to the **[Pareto] efficiency** we need to verify the the existence or not of a pair (i', j') of items that satisfy the relations (67) and (68)). As we have already seen both players have sufficient conditions to attain an efficient barter even if such conditions require a co-operative attitude between the players and such an attitude is not self-enforcing (Myerson (1991)) so that both players may have (more or less plausible reasons) for individually deviating from it. Such conditions, moreover, may be conflicting so that they cannot drive effectively a strategy. Moreover the check of the relations (67) and (68)) may be impossible since not all the possible elements (i, j) are known during the barter so that the inefficiency of a barter may be discovered only when both players have accepted it.

Summing up, we can say that:

- envy-freeness is guaranteed every time a barter occurs,
- proportionality is guaranteed every time a barter occurs,
- equitability may be guaranteed at every barter,
- efficiency may be attained, since the players have sufficient conditions for attaining it, but may also be easily missed.

From these considerations we have that the **fairness** of a barter is a by-product of the barter process itself and is not a-priori guaranteed by its structure.

From all this we derive that each of the two players may judge an occurred barter as either inequitable or inefficient or both and therefore unfair, since not all the fairness criteria are verified.

Last but not least we comment a little on the possible **applications** of the proposed models.

The **pure model** can be applied in all the cases where two players meet to perform a barter but none of them has a knowledge of the items that the other may be willing to barter so that this knowledge must be acquired during the barter process itself through a try-and-error process.

In the current version of the pure model we imagined the two players as peers in the barter, this fact being represented by the random move for the choice of the first time mover. It is easy to remove this feature by giving, for some reason, the right to make the first move to one of the two players. For the range of the possible applications we refer to the tables we provided in this section, mainly table 6, from where it is possible to understand the nature of the items that the two players may wish to exchange.

On the other hand, the **mixed model** can be used in all the cases where one of the two players cannot conceal or is forced to show the initial set of his items so that the other has some information about the possible advantageous (for him) proposals that he can make without even revealing any of the items of his private set.

Also in this case we refer to the tables we provided in this section, mainly table 6, from where it is possible to understand the nature of the items that the two players may wish to exchange.

11 Possible extensions to the iterative barter models

Up to now we have seen the basic models involving a pair of players in a one shot barter for the exchange of a pair of items.

In this section we briefly present the following planned extensions:

- (e_1) the possibility of having more than two players i.e. a plurality of players;
- (e_2) the possibility of performing repeated barter;
- (e_3) the possibility of performing multi-pairs barter.

In the following subsections we examine the various extensions singularly and one independently from the others though it is obvious that they could be combined together in various ways. The treatment is at an introductory level since such extensions represent the core of a future research stream.

11.1 A plurality of players

Instead of a pair of players A and B we may define a set \mathcal{P} of $n \geq 4$ players (with n even²³) that can form $n/2$ pairs (so to have $n/2$ contemporaneous barterers) in the following ways:

- by a random selection,
- by raising up of hands,
- by mutual selection.

In the case of **random selection** we may imagine a procedure that:

- (0) starts with $i = n$;
- (1) at the i -th step it chooses at random one of the $i(i-1)/2$ values so to match a pair of players;
- (2) if $i = 2$ go to (3) else $i = i - 2$ go to (1);
- (3) end;

At step (1) we may imagine to have a dice with i faces so that the outcome of $j \in [1, i]$ corresponds to one of the possible $i(i-1)/2$ pairings. We have to see how to assign the possible pairings to the faces of the dice. To do so we can set up a $n \times n$ matrix V with all the elements at 0 but those above the main diagonal that assume (row by row from left to right) the increasing integer values from 1 (for the element $v(1, 2)$) to $n(n-1)/2$ (for the element $v(n-1, n)$) so that $v(i, j) \neq 0$ for $j > i$.

After each step we remove the two matched players, renumber the remaining players, reduce accordingly the matrix V , reassign the values $v(i, j)$ with the same rule and repeat the procedure.

In the case of **raising hands** we can imagine that one player raises up one of his hands. If more than one player raises then each of them lowers his hand, waits for some random amount of time and then raises again his hand. This goes on until there is only one hand up at a time. At this point one or more of the others may join him to form a pair for a barter. If more than one player express the wish to join there is a random selection of one of them. The process goes on until $n/2$ pairs have been formed, at each step the number of waiting players being decremented by 2.

In the case of **mutual selection** the players are divided, through the use of

²³In abstract term we might have two cases:

- (1) n is even so $n/2$ pairs of players form and no player is left out;
- (2) n is odd so $n/2$ pairs form but one player is left out.

In this section we assume that n is even since the case n odd has no sense but in cases of repeated barterers (see section 11.2) where the player who is left out at one stage gains some precedence at the next stage according to a form of balanced alternation (Brams and Taylor (1999)).

any suitable random device²⁴, in two subsets of $n/2$ elements each: the former P_1 contains the choosers and the latter P_2 the choices.

Each player of P_1 chooses at his will a player from P_2 : if the latter accepts the pair forms otherwise the two players switch from one set to the other. If there are multiple selections the choice is up to the player of P_2 that is forced to select one of his choosers. The matched players are removed from the respective sets. When, according one of the foregoing ways, the $n/2$ pairs are formed each of them may use one of the algorithms we have seen in section 10 to perform a barter.

The basic idea is that the n players are willing to engage each other in a barter so the $n/2$ pairs form even if there is no guarantee that this turns out in $n/2$ effective barter.

11.2 Repeated barter

In this case we may use some concepts from Game Theory (Myerson (1991)) and consider each barter as a single stage of the potentially endless process of repeated barter between two players A and B .

To get this chaining effective we need to make some changes to the models of section 10. Such changes include:

- the addition of moves to implement the chaining;
- the introduction of state variables through which each player records the attitude of the confronting player during the previous stages (Axelrod (1985), Axelrod (1997));
- the definition of a barter as either with memory or memoryless.

As to the **moves** we need:

- to modify the **accept** move in a **accept and stop** move so to signal a lack of will to go on with the barter;
- to add the **accept and repeat** move so to signal to the other player the willingness to the execution of one more stage;
- to add the **repeat?** move so to allow a player to request to the other the execution of one more stage;
- to add a **repeat** move so to answer affirmatively to an explicit or implicit request for one more stage;
- to add a **refuse** move so to answer negatively to an explicit or implicit request for one more stage.

²⁴We can imagine a urn containing n balls, $n/2$ red and $n/2$ blue, and n selections without replacement for the assignment of each extracting player to one of the two subsets according the color of the extracted ball.

so to allow the players to link two stages together in a repeated barter.

The **state variables** represent private information of each player through which he may record, from one stage to another, if the attitude of the other player at the previous stages has been more cooperative or more exploitative. In this way each player may implement long run strategies of retaliation so to punish spiteful attitudes.

Last but not least we define a barter as **with memory** if at the end of each stage the composition of the revealed set is a valid common knowledge (Myerson (1991)) between the players whereas it is defined as **memoryless** if it is lost and has no common knowledge value since each player is free to modify at his will the composition of his set without any notice to the other. This feature of a repeated barter must be agreed upon by both players at the very offset of the process, may be changed only after a new mutual agreement after each stage is over and forms a common knowledge between the two players.

11.3 Multi pairs barter

In this case every player at each step may propose more than one pair of items to be bartered. For instance, A may propose:

$$(i_j, j_l)(i_h, j_k) \quad (85)$$

and B can reply with a counter proposal with even more or less elements or can even accept. The acceptance might be:

- **partial** if the agreement can involve only a subset of a [counter] proposal;
- **global** if the agreement involves a [counter] proposal as a whole.

Partial agreements are fully meaningful only within a repeated barter setting whereas in a one shot barter we usually speak of global multi pairs agreement whenever the two players agree on bartering all the items contained in one proposal.⁶

12 Concluding remarks and future plans

This section presents for the members of both the family F_1 and the family F_2 some concluding remarks and hints of future research plans.

In sections 5 and 6 we presented some auctions mechanisms of two distinct types. Those of the former type are termed **positive auctions** and resembles classical mechanisms but for the fact that they aim at the allocation of chores rather than goods.

The one of the latter type faces the problem of the allocation through an auction from a negative perspective since the chore is allocated to the bidder who proves less capable of avoiding it but that is compensated for this incapability. Both types of mechanisms need a deeper and more formal analysis of their structure, their properties, the possibility of collusions and, most important, their

practical applications in the area of environmental problem solving.

In section 8 we have introduced a family of barter models between two actors that execute a one shot barter through which they exchange, according to one among various mechanisms, the goods of two separate and privately owned pools. The various models have been introduced under the hypothesis of additivity according to which the value of a set is given by the sum of the values of its composing elements.

In that section we presented the basic algorithms for the one-to-one barter, we showed the possible uses of the proposed models, we verified if some criteria of fairness are satisfied by the proposed models or not and we also introduced some extensions.

The main extension we presented is the relaxing of the additivity hypothesis with the adoption of superadditive sets where the value of a set is at least equal to the sum of the values of its elements. In this way we model functional relations among the goods that increase their joint values.

The presentation we made in section 8 is at an introductory level and a lot of formalization is still to be done for what concerns both the presented models, their extensions and the possible uses in concrete cases .

We need indeed to examine more formally the basic models of one shot barter; to improve the proposed algorithms; to examine the properties of such algorithms and their plausibility and, last but not least, to analyze and formalize the extensions we essentially only listed in section 8.9.

In section 10 we have introduced two algorithms that can be used in the case of one shot barter between a pair of players.

In the former algorithm we have asymmetric situation where the two players try to conclude a barter through an incremental revelation of the sets of items each of them is willing to barter.

In the latter algorithm, on the other hand, the situation is asymmetrical since one of the player either does not want to or cannot hide the set of his items to the other player that therefore has an initial advantage and the right to make the starting offer.

The section presented both a description of the algorithms and their evaluation according to well defined classical performance criteria.

In the closing part of that section we also presented some possible extensions whose analysis and formalization are still to be completed. Similarly we need to complete the analysis of the possible applications of the proposed models to real world cases where exchanges of items occur between players that aim at attaining their objectives without sharing any common scales of qualitative or quantitative values.

Appendix A: candle auctions

The scope of the procedure

We define an auction inspired procedure through which an agent A (the so called initial auctioneer) allocates a chore to one agent from a predefined set of sub-agents by letting them arrange the things among themselves including the definition of the compensation for the losing sub-agent.

The compensation is a multiple of the fee f that A fixes at his will and the selected sub-agents are called bidders or players.

The procedure is implemented through a simple algorithm that involves the sub-agents without any further intervention of A .

The motivations of the procedure

The proposed procedure is based on certain assumptions on both the auctioneer side and the bidder side.

On **the auctioneer side** it assumes that the auctioneer does not know very well the features of the bidders (but only knows that each of them can carry out the chore) and therefore prefers that they find an allocation solution among themselves and that such solution may, on one hand, compensate at the most one of them without, on the other hand, charging excessively any of the others.

On **the bidders side** it aims at defining an adaptive sharing of the final contribution to the losing bidder (the one who gets the chore at the end of the auction) from the other bidders that exclude an a-priori defined subdivision (such as a uniform subdivision²⁵ or a proportional subdivision).

In this way we aim at a procedure that is adaptive and potentially more rewarding for the losing bidder without charging too much each of the winning bidders and without hurting in any way the auctioneer

The algorithm

The [initial] auctioneer A :

- chooses the n bidders of B ;
- chooses the chore to be auctioned;
- fixes the fee f that each bidder may pay for not accepting the candle (see further on).

The bidders of the set B are chosen by A from a much bigger set and each bidder may be seen as having associated a **type** (see section 12) that defines his behavior in the course of the game as will be explained in section 12.

As to the entity of the fee f we note that it can be fixed freely by A at any value

²⁵We say a sum x is shared in a **uniform way** among n actors if each of them is charged with a fraction $\frac{x}{n}$ whereas we say it is shared in a proportional way if for a bid x_i its bidder is charged with a fraction $x \frac{x_i}{X}$ where $X = \sum_{i=1}^n x_i$.

since that value has no effect on A himself. We note that if $f = 0$ the losing bidder has no compensation (see further on) so that we pose the constraint $f > 0$. On the other hand the lower is f the more easily the bidders are going to pay it whereas the higher it is the more unlikely they are going to pay it. Since the value of f affects the behavior of the bidders we can imagine, for simplicity, that A extracts it from a urn containing all the integer values in the interval $[f_{min}, f_{max}]$. The extremes of the interval are exogenous values to be fixed in some way. Further details in section 12.

The algorithm is structured as follows:

- initialization phase;
- auction phase;
- compensation phase.

In the **initialization phase** A in zero time:

- initializes or switches on a flag that we call candle;
- performs a random initialization of the maximum value L of a counter on an interval (unknown to the bidders) $[0, M]$;
- performs a random selection of one of the bidders, be it b_i ;
- gives the candle to b_i ;
- starts the counter t (that is incremented one tick at a time up to L ticks) from 0.

The **auction phase** is structured as follows. We underline how the counter t is incremented of one tick as stated in each case. At $t = 1$ b_i has the candle:

- (1) b_i selects another bidder b_j at his will and offers him the candle;
- (2) b_j can:
 - (a) accept it so b_j takes the role of b_i , $t = t + 1$, go to (1);
 - (b) refuse it and put the fee f in the common pot, $t = t + 1$, go to (1).

At the generic step t the content of the pot is kf (with $k \in [0, t]$) and this may influence the behavior of both b_i and the others b_j that are contacted by him (see section 12).

At each step the bidder b_i must chose another bidder b_j so to offer him the candle. The choice can be made either to maximize the probability of keeping the candle or to maximize the probability of giving it away: the former case occurs whenever b_i thinks he has a gain from having the candle when the counter expires whereas the latter occurs whenever b_i thinks he has a loss from having the candle when the counter expires.

For further details we refer to section 12.

The generic bidder b_j to which the candle holder b_i offers the candle can either accept it or refuse it according to strategies we describe in section 12.

If b_j accepts the candle then he takes the role of b_i (the candle holder) and uses the same search and offer procedure.

If b_j refuses the candle then he must put f in the common pot and b_i goes on with his imposed search and offer procedure.

The private information

Each bidder b_j has the following private information:

- d_j as the damage that derives to b_j from having the candle when the counter expires;
- a vector of values c_j whose elements $c_{j,h}$ denote the damage that derives to b_j from having the candle owned by b_h when the counter expires;

and can use them for his strategies either as the candle holder or as a contacted bidder. We note that:

- $c_{j,j} = d_j$;
- the vectors c_j represent the rows of a square cost matrix C whose dimension coincides with the number of the bidders,
- the values $c_{j,h}$ can be used by b_j to partition the set of the other bidders in disjoint subsets and to order such subsets in increasing order of cost (see section 12).

Besides such information each bidder b_j knows his type t_j whereas has a probability distribution for the type of every other bidder.

For the moment we consider only two possible types: green or conservative (g) and selfish or exploitative (s).

From this we have that whenever b_j faces another bidder b_h he knows that with probability p he is facing a g type bidder and with probability $1 - p$ he is facing an s type bidder.

Termination condition

When the counter t reaches L the bidder who has the candle loses the auction and gets the chore and the amount of fees put in the common pot by the bidders during the auction.

Compensation phase

When the auction is over the candle holder b_i gets the chore and a compensation equal to k^*f where with k^* we denote the number of fees paid by the players different from b_i so that b_i obviously does not count as a gain the sums paid by himself.

We can have the following cases:

- (a) $k^* f > d_i$ so that b_i has a gain,
- (b) $k^* f = d_i$ so that b_i is fully compensated,
- (c) $k^* f < d_i$ so that b_i has a loss.

Why we say it is an auction

We define the proposed mechanism as an **auction** since at each step the candle holder b_i can be seen as the “auctioneer” whereas one of the others either “bids” a fixed sum for not getting the candle or gets it and becomes the current “auctioneer”.

In a traditional auction at the end the auctioneer gives away the auctioned good and gets a certain amount of money whereas in this case the final auctioneer gets the candle (i. e. a chore) and a compensation (the “net” common pot²⁶) that some of the others contributed to form in a way that resembles an all pay auction.

The properties

The balances

The balances of the various bidders depend heavily on L since the higher is L the higher may be k^* for every bidder. From this we derive that L has a lower bound. They also depend on the variety of the interactions among the players and so on their strategies.

The losing bidder b_i may be worse off if k^* is not high enough.

The winning bidders b_j with $j \neq i$ pay on the average:

$$\frac{L}{n}f \tag{86}$$

and each has a gain of d_j . For each b_j the balance is²⁷:

$$d_j - k_j f \tag{87}$$

where k_j is the number of times bidder b_j refuses the candle and pays the fee, with $0 \leq k_j \leq L$. Every bidder can control the amount of money that he puts into the pot by controlling his number of refusals.

Collusions

Never occur. Who pays the fee to the pot has not the candle and who, at the end, remains with the candle gets the content of the pot as a compensation.

²⁶We call it “net” since we exclude the sums paid the the final auctioneer himself.

²⁷We use the notation k_j to denote the value k^* for bidder b_j since each bidder has his own value k^* as the number of times the other players paid the fee.

The strategies

At this point we must specify, at each step, the possible strategies for both the current candle holder b_i and the players that may be contacted by him and so b_j with $j \neq i$.

- (1) For what concerns b_i he has two possible strategies and one constraint:
 - (a) **attractive** if he tries to keep the candle for himself by contacting bidders that he is sure will refuse it so paying the fee f so increasing the common pot;
 - (b) **repulsive** if he tries to give away the candle by contacting bidders that is sure will prefer to accept it rather than paying the fee f .

The strategy (a) is worth following if some time has already elapsed since the start of the game and/or the fees in the pot are such that b_i is sure to have a gain from getting the candle. This occurs for sure if $k^*f > d_i$. Such strategy may either succeed or fail. If it succeeds it causes a further increase of the common pot of f at the benefit of b_i . Unfortunately it may fail since the contacted player b_j , using his private information, may find the common pot high enough so to accept the candle without paying the fee f .

On the other hand the strategy (b) is worth following if the game is at the start and/or the common pot is low so the best thing b_i to do is to pass the candle to a bidder b_j such that $d_i - k^*f > c_{i,j}$. In this case the best bidder b_j is the one with the minimum $c_{i,j}$. In order to prevent b_i from continuously contacting the same b_j we introduced a constraint (see further on).

Such strategy may either succeed or fail. If it fails the bidder b_j pays the fee f and this makes b_i better off whereas if it succeeds it makes b_i , at least temporarily, better off.

We note that the common pot may be low for the following reasons:

- the game is in the initial phases,
- the game has been characterized by the passing of the candle among the players so that few of them paid the fee f .

We note that the common pot may be high for the following reasons:

- the game has already been running for some time,
- the game has been characterized by a lot of refusals so that many of the players paid the fee f .

The constraint we introduced imposes b_i to contact at least half of the bidders in turn, according to whatever ordering he chooses, before contacting again the same player so to prevent him from exploiting a refusing bidder by repeatedly contacting him. The easiest ordering is based on the values $c_{i,j}$. Using such values b_i can partition the set B_{-i} of the other bidders

in subsets of equal cost bidders so to get an ordering of such subsets and contact their members according to an increasing costs criterion.

- (2) For what concerns the bidders b_j their repertoire of moves is richer since they can either **accept** or **refuse** the candle.

A player is said to refuse the candle if he does not accept it so we can restrict ourself to the definition of the conditions for the acceptance of the candle.

We recall that we may have either $t_j = g$ or $t_j = s$.

If $t_j = g$ the bidder b_j accepts the candle if he hopes to find a bidder b_h such that $c_{j,h} < c_{j,i}$ even at the risk of remaining with the candle so to suffer the damage d_j independently from the content of the pot. In this case we presumably have $c_{j,h} < d_j < c_{j,i}$ or $c_{j,h} < c_{j,i} < d_j$.

If $t_j = s$ the bidder b_j accepts the candle if he has a gain from getting it so that $k^*f > d_j$.

In the acceptance case b_j becomes the candle holder and, at least for one turn, acts as b_i .

In the refusal case b_j puts the sum f in the common pot and puts himself on waiting for a possible future choice until the counter expiration time.

At this point we have to establish a relation among the two pairs of strategies we have defined:

- attractive and repulsive when the bidder is the candle holder,
- green and selfish when the bidder is not the candle holder,

From the descriptions we have that a $t_j = g$ bidder tends to adopt a repulsive strategy when he plays as the candle holder whereas a $t_j = s$ tends to use an attractive strategy when in the same role.

The timings

The counter starts from 0 and stops at a random value L . The unitary increments occur at every acceptance or refusal so there is no way for the bidders to lose time so to exploit an advantageous position.

The experimental side

The model should be implementable through the use of an agents system by defining two populations of agents with the defined types and strategies and by having them play the game for some random amount of time L with the specified timings.

One possible extension, to be tested experimentally, could be the following: each bidder b_j has a randomly fixed sum of money F_j (integer multiple of f) that he can use to pay the fee f so that when that sum is finished he can only accept the candle without any further possibility of refusal.

Appendix B: mathematical facts and findings

In this Appendix we simply recall or present and discuss some mathematical results we have used in the text.

- (1) A uniform random variable X over the interval $[0, M]$ has a distribution function given by:

$$F_X(x) = P(X \leq x) = \frac{x}{M} \quad (88)$$

and the corresponding density function:

$$f_X(x) = \frac{1}{M} \quad (89)$$

We recall that for independent random variables $X_i, i = 1, \dots, k$ we have:

$$P(\cap_j X_j \leq x) = \prod_j P(X_j \leq x) \quad (90)$$

If the k variables are, in addition, identically distributed we have:

$$P(\cap_j X_j \leq x) = \prod_j P(X_j \leq x) = F_{X_j}(x)^k \quad (91)$$

and if they are uniformly distributed on the same interval $[0, M]$ we finally have:

$$P(\cap_j X_j \leq x) = \prod_j P(X_j \leq x) = F_{X_j}(x)^k = \left(\frac{x}{M}\right)^k \quad (92)$$

- (2) We recall equations (14) and (15) for the bidder b_i and rewrite them here for convenience:

$$E_i(x_i) = p(x_i - m_i) + (1 - p)(m_i - x_1 \frac{x_i}{X}) \quad (93)$$

or as:

$$E_i(x_i) = pl_i + (1 - p)w_i \quad (94)$$

with:

$$\begin{aligned} l_i &= (x_i - m_i), \text{ if } b_i \text{ loses;} \\ w_i &= (m_i - x_1 \frac{x_i}{\alpha + x_i}), \text{ if } b_i \text{ wins;} \end{aligned}$$

where $\alpha = \sum_{j \neq 1, i} x_j$.

In equation (14) we have a term w_i that, in the case of the strategy $x_i > m_i$, may assume a negative value.

Let us verify it. We have:

$$m_i - x_1 \frac{x_i}{\alpha + x_i} < 0 \quad (95)$$

if:

$$m_i(\alpha + x_i) < x_1 x_i \quad (96)$$

Since we have $\alpha + x_i > x_i$ but we may well have $m_i < x_1$ we have that relation (96) may be satisfied so that using strategy $x_i > m_i$ may cause a loss to bidder b_i . This is the reason why, in Table 1, we used the symbol $+/-$. It is easy to verify that in the case of the strategies:

- $x_i = m_i$
- $x_i < m_i$

this cannot occur.

In the case of the former strategy, by using relation (96), we would indeed derive the obviously false inequality $(\alpha + x_i) < x_i$ whereas in the case of the latter strategy it is immediately seen how relation (96) can never be satisfied since we obviously have $\alpha + x_i > x_1$ from the definition of lowest bid x_1 .

From this we have that the value x_i^* of the bid of b_i for which the term w_i changes sign is greater than m_i (see next point and Figure 1 for details).

- (3) In Figure 1 we label as (0) the graph of the term $x_1 \frac{x_i}{\alpha + x_1}$ of w_i . Such a term assumes the value $x_1 \frac{M}{\alpha + M}$ for $x_i = M$ (see the horizontal line above the graph labeled (0)). On the other hand we have:

- (1) denotes the behavior of w_i if $m_i > x_1 \frac{M}{\alpha + M}$,
- (2) denotes the behavior of w_i if $m_i < x_1 \frac{M}{\alpha + M}$.

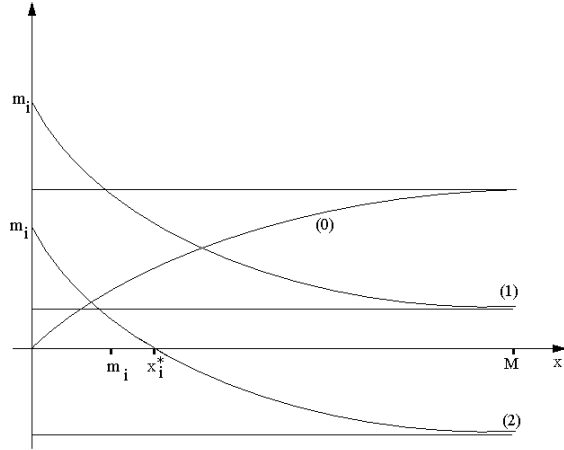


Figure 1: *The possible graphs of w_i*

In the latter case we have the value x_i^* of the change of sign which is, as we have already seen, greater than m_i .

We can evaluate x_i^* if we impose:

$$m_i - x_1 \frac{x_i}{\alpha + x_i} = 0 \quad (97)$$

so to get:

$$x_i^* = \frac{m_i \alpha}{x_1 - m_i} \quad (98)$$

In this way we have $x_i^* > 0$ iff $x_1 > m_i$. From relation (98) we get:

- if $x_i < x_i^*$ then $m_i - x_1 \frac{x_i}{\alpha + x_i} > 0$,
- if $x_i > x_i^*$ then $m_i - x_1 \frac{x_i}{\alpha + x_i} < 0$.

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