Iterative procedures for the selection of competing projects

Lorenzo Cioni
Computer Science Department, University of Pisa
largo B. Pontecorvo n^o 3, 56127 Pisa, Italy
e-mail: lcioni@di.unipi.it
tel: (+39) 050 2212741, fax: (+39) 050 2212726

Abstract

This paper presents one possible way in which a set of decision makers can perform the choice of a project from a set P of m competing projects. The choice is performed through the use of an iterative procedure that makes use, as internal procedures, of a refinement procedure, that allows the deciders to assign to certain projects finer rankings, and a tie-breaking procedure that allows the deciders to select one project among a set of tied competing projects.
The paper also discusses some possible ways of ranking the projects and presents a filtering procedure that uses the concepts of Pareto optimality/efficiency and Pareto dominance.

1 Introduction

This paper refers to one of the topics I dealt with in my PhD thesis ([5]) and presents one possible way in which a set of decision makers or **deciders** can perform the choice of a project from a set P of $m = |P|$ competing projects.
The choice is performed through the use of an iterative procedure that is similar to the ones we presented in [4] and in [5]. Such a procedure has the high level structure that we present in section 2 and makes use, as internal procedures, of the following procedures:

- an evaluation procedure through which the deciders assign to each project of P a set of n numerical values that represent the associated benefits and costs;

- a procedure for the identification of the Pareto optimal projects within the current set of projects;

- a refinement procedure that allows the deciders to assign to certain projects finer rankings;

- a tie-breaking procedure that allows the deciders to select one project among a set of tied competing projects.

In this paper we assume a **static setting** so that:

- the set D of the deciders is fixed and does not vary during the process;

- the set of the criteria (and so of the benefits and the costs) is fixed by the deciders from the start of the procedure and cannot be changed in the course of the process;

- the set P of the competing projects is seen as an exogenous parameter and can only be reduced during the process up to the point when possibly only one project is left so that it can be chosen as the best project, at least among the projects of the set P.

The overall process is based on the following assumptions:

- in oder to rank the projects we[1] use a set of n criteria that represents the various types of benefits and costs that are associated to each project;

- each criterion is described through a numeric quantity but we assume that the corresponding quantities must be seen as dimensions that are orthogonal among themselves so they cannot be added or averaged or compared in any way;

- in this way each project $p_i \in P$ is associated to a vector $x_i \in \mathbb{R}^n$ and so to a point in an $n-$ dimensional space;

- the positions of the various points allow us to identify the Pareto dominant projects, the Pareto equivalent projects and the Pareto dominated projects;

- for each project $p_i \in P$ we have that the vector $x_i = (x_i^j)_{j=1,\dots,n}$ contains the values that describe that project according to a common set of n_1 benefits and a common set of n_2 costs where $n = n_1 + n_2$;

[1] In many cases we use the term “we” for simplicity but it should be read as meaning “the deciders”.

- for each project $p_i \in P$ we denote with b_i^h the various benefits or the corresponding values, depending on the context, and with c_i^k the various costs or the corresponding values, again depending on the context.

As a shorthand we may use the notation $x_i = (b_i, c_i)$ where b_i is the sub-vector of the values b_i^h and c_i is the sub-vector of the values c_i^k.
We remark how the hypothesis to use numeric quantities to describe the various criteria in many cases is not neutral and may require the quantification of qualitative criteria in order to produce values on cardinal scales that, in any case, are never compared or added or averaged among themselves ([14]). The main reason why we make this assumption is that such numbers may refer to qualitative evaluations turned into numerical quantities for practical purposes so that they correspond to incommensurable quantities.

2 The proposed procedure

In this section we present the high level structure of the procedure that can be used by the deciders of the set D to select the **best project** $\tilde{p}$ from the initial set of m competing projects P. Further details will be presented in the following sections of the paper and especially in the sections 7, 8 and 9.
The procedure essentially works as follows:

(1) it starts with a set P to which the deciders assign a matrix X of evaluations, one row vector $x_i \in \mathbb{R}^n$ for each project $p_i \in P$;

(2) from the set P it derives the set $\hat{P}$ of the Pareto optimal projects through the use of the matrix X;

(3) it tries to reduce at the most the set $\hat{P}$ through a certain number of refinement steps;

(4) it produces ad irreducible set of Pareto optimal projects;

(5) it requires the selection of the best project from such irreducible set as the outcome of the overall process.

Such steps require, in some way, the explicit intervention of the deciders that use ancillary procedures that are designed for those purposes.
We note that at the end of the step (2) we can have the following cases:

(a) $|\hat{P}| = 1$;

(b) $|\hat{P}| > 1$.

In the case (a) the procedure ends and produces as its outcome a single Pareto optimal project that is the natural best selection for all the deciders. On the other hand, in the case (b) the projects of the set $\hat{P}$ undergo a certain number of refinement and filtering steps until either we fall in the (a) case or the procedure produces an irreducible set of Pareto optimal projects. In section 10 we are going to show how the deciders can perform the final selection in this case so to determine the best project from the initial set P. At this point we present the procedure. It uses the following data structures:

- the initial set of the projects P as an exogenous parameter;
- a counter i that indexes the various sets that are defined by the deciders;
- a boolean variable FP that signals the occurring of a fixed point condition when the set of the Pareto optimal projects cannot be further refined.

It also uses the following procedures:

- a procedure *evaluate*, see section 7, that assign a vector of numerical evaluations to each project;
- a procedure PO, see section 8, that evaluates the new set of the Pareto optimal projects from the current set through the use of the output of the previous procedure;
- a procedure RPO, see section 9, that refine the evaluations of the projects contained in the current set of the Pareto optimal projects;
- a procedure FS, see section 10, that is used by the deciders to perform the final selection from an irreducible set.

The high level structure of the procedure follows.

procedure project_selection(P)

begin

$i{=}0;$

$FP = false;$

$X = evaluate(P);$

$\hat{P}_i = P;$

$c_i = |\hat{P}_i|$;

while $c_i > 1$ and[2] $\neg FP$ do

begin

$\hat{P}_{i+1} = PO(\hat{P}_i, X)$;

$X = prune(X, \hat{P}_{i+1})$;

$c_{i+1} = |\hat{P}_{i+1}|$;

if $c_{i+1} > 1$ then

begin

$X' = RPO(\hat{P}_{i+1}, X)$;

if $X' == X$ then $FP = true$;

else $X = X'$;

end

$i = i + 1$;

end

if $c_i == 1$ then return $\hat{P}_i$;

if $FP == true$ then return $\hat{P}_i = FS(\hat{P}_i)$;

end

It is easy to see how the procedure has a simple structure that is essentially based on a main *while* loop that is executed until one of the following conditions holds:

(a) a single project has been identified so that we have $c_i = 1$;

(b) a fixed point condition is verified so that we have $FP = true$.

We note that the condition $X' = X$ (that causes $FP = true$) can be forced by the deciders whenever they see that no real refinement through the RPO procedure can be obtained. In this case we should get, of course, that the set $\hat{P}$ would not vary so the two conditions coincide.
In the (a) case the selected project is provided as the output of the procedure.
In the (b) case the deciders must execute the **Further Selection** or FS procedure (see section 10) so to identify the best project among those of the set P.
At each iteration of the *while* loop the deciders:

[2]With the notation $\neg a$ we denote the complement of a that is false when a is true and true when a is false.

- define with the procedure PO the set of the currently Pareto optimal projects;

- if they get a single project they are done otherwise they use the procedure RPO (see section 9) to "perturb" the set $\hat{P}$ of the currently Pareto optimal projects;

- if the "perturbation" defines a modified set of the evaluations of the currently Pareto optimal projects they iterate the procedure on the new set;

- otherwise they are in a fixed point condition so they proceed as we have already seen.

3 The structure of the rest of the paper

The rest of the paper is structured as follows. It goes on with a brief examination of the concepts of Pareto optimality, Pareto efficiency and Pareto dominance firstly in the classical case and then in presence of both benefits and costs expressed as independent numerical quantities.
Then we examine the possibility to use two separate rankings:

- one based on the benefits only,

- one based on the costs only.

Under these assumptions we show the possible inconsistencies between the two rankings and examine some way to solve them.
Then we examine the possible uses of lexicographic orderings on the criteria (benefits and costs) according to which the projects are evaluated with the aim of filtering out the set P and possibly produce an outcome containing only one project.
The paper goes on with a description of the structure of the procedures *evaluate*, PO and RPO (see also sections 1 and 2) then it examines how the deciders can behave when they face an irreducible set of Pareto optimal projects $\hat{P}$ among which they have to choose one project to get it implemented.
The paper closes with a short section devoted to the conclusions and to the description of the future research plans.

4 About Pareto optimality

Whenever we have to rank or to order two or more numerical unidimensional quantities $x_i \in \mathbb{R}$ we can use the ordering over $\mathbb{R}$ through relations such as $>$ or $<$ or the like. Things are undoubtedly more difficult if we have to rank or to order two or more numerical multidimensional quantities $x_i \in \mathbb{R}^n$. In this case we can safely imagine the values x_i as points in an $n-$dimensional space and use the concepts of Pareto efficiency that, according to [1] and [2], coincides with the Pareto optimality. In this way we define the so called Pareto efficient solutions that, see [18] and [13], are on the so called Pareto frontier.
For instance, we may have points $(x, y) \in \mathbb{R}^2$ that represent the shares of two players[3] so that each player prefers to get more than less under some constraints. In this case we may have a region of that plane defined by the following relations:

$$x + y \leq k$$

$$x \geq 0$$

$$y \geq 0$$

The points on the Pareto frontier $x+y = k$ are the Pareto efficient or optimal solutions wheres those in the region $x+y < k$ are said to be Pareto inefficient since both players can be better off without damaging each other. This does not occur for the points on the Pareto frontier since a switch from one point (x_1, y_1) on the frontier to another (x_2, y_2), again on the frontier, is such that if one player is better off the other is obligatorily worse off.
We underline the fact that the concept of Pareto efficiency rules out consideration of equity or fairness ([1], [2], [18]). According to such definitions all the points $(x_1, x_2) \in \mathbb{R}^2$ such that $x_1 + x_2 = k$ are Pareto efficient solutions and so even the extreme points $(k, 0)$ and $(0, k)$.
In addition to these definitions we can introduce the concepts of Pareto dominant and Pareto dominated solutions.
In we consider the preceding example and we take a point (x, y) we have that:

- every point (x', y') such that $x' \geq x$ and $y' \geq y$, with at least one strict inequality, is Pareto dominant;

[3]We use the term **player** with the classical meaning it has in Game Theory ([11], [8], [10]).

- every point (x', y') such that $x' \leq x$ and $y' \leq y$, with at least one strict inequality, is Pareto dominated.

We have, therefore, that a solution may be Pareto dominant and inefficient The concept of **Pareto efficiency** ([11], [8], [10]), therefore, can be used to characterize the allocation of the quantities x_i to the n playersp_1, $i = 1, \ldots, n$. In this case we say that an allocation is Pareto efficient if and only if there is not another allocation where at least one of the player may be better off (and so may obtain a higher quantity) without any other player being worse off (so obtaining a lower quantity).
From this definition it is easy to see how we can have any number of Pareto efficient solutions.
If, for instance, two players have to share a quantity a any solution $x = (x_1, x_2)$ such that $x_1 + x_2 = a$ is Pareto efficient whereas any solution such that $x_1 + x_2 < a$ is Pareto inefficient since, from the current perspective, both players can be better off by sharing in some way the surplus $a - x_1 + x_2$.
If an allocation is not Pareto efficient we can, therefore, obtain at least one allocation where one ore more players are better off without damaging any of the remaining players. If all the players can be better off we say that the original allocation is **Pareto dominated** by the new allocation that is termed **Pareto dominant**.
Last but not least we note how, if the switch from an allocation $x = (x_1, \ldots, x_n)$ to another allocation $x' = (x'_1, \ldots, x'_n)$ requires that it exists some i such that we have $x_i > x'_i$ and some j such that we have $x_j < x'_j$, we say that x and x' are **Pareto equivalent**. A set of Pareto equivalent allocations is termed an **irreducible set**.
We remark how, if we switch from $\mathbb{R}^2$ to $\mathbb{R}^n$ for a generic $n > 2$, the above definitions continue to hold though (especially for $n > 3$) we lose any visualization aid.
In this paper we want to use such concepts in the case of the evaluation of competing projects that are characterized by both benefits and costs so we need to adapt them a little bit to this new context. We recall how, according the static setting approach (see section 1), all the projects of the set P are described through the same categories of benefits and costs though each project has its own vector of values.
As a preliminary remark we underline that:

- the benefits are put in an increasing order since the higher a benefit is the better it is;

- the costs are put in an decreasing order since the lower a cost is the better it is;

- a cost is not seen as a negative benefit since, in many cases, this change of sign has no meaning.

In this case (see also section 1) we have that each project p_i is described by an $n-$valued vector $x_i \in \mathbb{R}^n$ where each x_i^j represents a numeric quantity that describe, on a proper cardinal scale, either a benefit or a cost associated with that project.
In order to make things more concrete we assume to have one benefit b and one cost c. According to our approach we consider the two values as incommensurable so we cannot evaluate ratios or other numerical quantities involving such values.
In this case if we have a project p_1 with the associated vector (b_1, c_1) we have that:

- every other project p_j with values (b_j, c_j) is **Pareto dominated** if $b_j \leq b_1$ and $c_j \geq c_1$ with at least one strict inequality;

- every other project p_k with values (b_k, c_k) is **Pareto dominant** if $b_k \geq b_1$ and $c_k \leq c_1$ with at least one strict inequality;

- all the projects p_h with values (b_h, c_h) such that $b_h \geq b_1$ and $c_h \geq c_1$ or $b_h \leq b_1$ and $c_h \leq c_1$ are termed **Pareto equivalent**.

For what concerns the Pareto equivalent projects we can follow three alternative approaches:

- we can consider them as forming an irreducible set of projects;

- we can rank them by assigning different weights to the various benefits and costs (see section 6);

- we can rank them through the use of separate rankings (see section 5).

All such approaches will be briefly described and implicitly compared in section 8.

5 Use of separate rankings

In section 4 we have seen how to represent the various projects of a set P in an $n-$dimensional space. In this case to each project $p_i \in P$ we assign a point $x_i \in \mathbb{R}^n$ and so we define the various categories of projects that we have described in section 4.
Another possibility is to consider separately the n_1 benefits and the n_2 costs. In this case we can proceed in two ways:

(a) in a punctual way;

(b) in an cumulative way.

In the (a) case we represent each project of the set P in two spaces:

- an n_1-dimensional space of the benefits;

- an n_2-dimensional space of the costs.

Once we have done this we can evaluate the set $\hat{P} = \hat{P}_1 \cap \hat{P}_2$ where the set $\hat{P}_1$ contains the Pareto optimal projects according to the n_1 benefits whereas the set $\hat{P}_2$ contains the Pareto optimal projects according to the n_2 benefits. The main problem in this case is that the set $\hat{P}$ may be empty so that we have no project that is Pareto efficient for both the benefits and the costs. If, for instance, we have two projects p_1 and p_2 and four criteria, two benefits and two costs, we may have that:

- p_1 dominates p_2 according to the benefits;

- p_2 dominates p_1 according to the costs.

In this case we have $\hat{P} = \emptyset$. The same may occur, of course, in presence of more than two projects and of a higher number of criteria, both benefits and costs.
In the (b) case[4] we consider again benefits and costs separately but we proceed as follows in the case of the benefits, for the costs we proceed in a similar way:

- we map the benefits on a scale from 0 to 100 so to uniform their range of values and represent them as percentages;

- we assign different weights w_i to the benefits;

- we evaluate the weighted sum of the benefits so that to each project p_i we assign a single cumulative benefit b_i.

We note how we have:

- $w_i \in [0, 1]$ for each i;

- $\sum_i w_i = 1$.

[4]We note how this case may represent a solution to the difficulties we have seen may occur in the previous case.

In a similar way for the project p_i we may evaluate a single cumulative cost c_i. In this way we are in a bidimensional case where we assign a pair of values (b_i, c_i) to each project p_i. Once we have defined a pair of values for each project we represent the projects as points in $\mathbb{R}^2$ in order to define the set $\hat{P}$ of the Pareto efficient projects.
In this case we have evaluated the weighted sums of both benefits and cots since we assume that the benefits, once they have been expressed according to numerical values, can be considered as commensurable among themselves and we assume that the same holds also for the costs.
The critical step in this case is represented by the evaluation of the values of the weights for the benefits and for the costs so that, in absence of any strong indication, the best solution is to assume for the weights identical values. In this case the weighted averages turn into arithmetic averages. With this we mean that, in the case of the benefits, we have:

$$w_i = \frac{1}{n_1} \tag{1}$$

so that:

$$b_i = \frac{\sum_j b_i^j}{n_1} \tag{2}$$

where b_i^j is the value for the project p_i according to the $j-$th benefit type. The same holds for the costs.

6 Possible use of lexicographic orderings

Up to now we have either used the criteria (or the benefits and the costs) as forming a single set or as forming two separate sets, one of the benefits and the other of the costs. In the latter case (see section 5) we have examined the possibility to use two separate rankings (one based on the benefits and the other based on the costs) but also the possibility to merge the benefits and the costs in two distinct values to be used in a bidimensional space.
In this section we examine the possibility to assign to the various criteria different weights so to define a total strict ordering of the criteria and to use such ordering to perform a lexicographic ordering of the projects.
We recall that we assume to have n_1 benefits and n_2 costs. If we assign different weights to the various benefit types we can order them in a strict decreasing order as:

$$b_1 \succ b_2 \cdots \succ b_{n_1-1} \succ b_{n_1} \tag{3}$$

by possibly renumbering such criteria and where $\succ$ denotes a strict preference relation endowed with classical properties.

In a similar way we may proceed with the costs so to get:

$$c_1 \succ c_2 \cdots \succ c_{n_2-1} \succ c_{n_2} \tag{4}$$

At this point we consider the two criteria b_1 and c_1 and define the set of the Pareto optimal projects $\hat{P}$ from the set P according to these criteria. If we have $|\hat{P}| = 1$ we are over otherwise we consider the two criteria b_2 and c_2 and repeat the procedure on the set $\hat{P}$. This procedure goes on until:

- we find a set $\hat{P}$ such that $|\hat{P}| = 1$ so that it ends with success;
- we find a set $\hat{P}$ that cannot be further reduced through the use of other criteria (that may have been even used up) so that the procedure produces a set that requires a further selection to be performed (see section 10).

We note that at each step we use two criteria b_i and c_i so that we are guaranteed to get a non empty subset $\hat{P}$. If we have $|\hat{P}| > 1$ we disregard such pair of criteria as non discriminating so to switch to the following pair. If this occurs for all the possible pairs of criteria and we have $n_1 = n_2$ then we must conclude that this method is non deciding on the current set of projects. If, on the other hand, we have $n_1 \neq n_2$ we can proceed by ranking the left out projects according either to the remaining benefits (if $n_1 > n_2$) or to the remaining costs (if $n_1 < n_2$).
If, for instance, we have three projects p_1, p_2 and p_3, two benefits $b_1 \succ b_2$ and three costs $c_1 \succ c_2 \succ c_3$ we may proceed as follows.
We consider b_1 and c_1 and represent the three projects as points in $\mathbb{R}^2$. In this way we get the set $\hat{P}$. If we have $|\hat{P}| = 1$ we are over so we do not need to consider the other criteria. If we have $|\hat{P}| > 1$ we must switch to b_2 and c_2 so to repeat the procedure, for instance, on the two remaining projects. If we have $|\hat{P}| = 1$ we are over whereas if we have $|\hat{P}| > 1$ we must use the last criterion c_3. Also in this case we have no guarantee that there will not be a tie as it can occur if, for instance, the two remaining projects have equal values according to c_3. In this case the procedure ends with a set $\hat{P}$ such that $|\hat{P}| > 1$. In section 10 we are going to show how we can deal with these cases.
The main problem with this approach is that we may have a project that wins[5] according, for instance, to the criteria b_1 and c_1 but that would lose[6]

[5]With this we mean that it will be the only project in the set $\hat{P}$ as it has been defined by the selected criteria.

[6]With this we mean that it would be a dominated project according to all the other pairs of criteria or single criterion rankings.

according to all the other criteria. This is a limiting case, of course, but is enough to state that this method should be applied only in particular cases and with a warning to consider how a selected project would behave with the remaining criteria. In this paper we do not deal with this feature of this ranking method any further.

7 The use of the *evaluate* procedure

In the present static setting the deciders of the set D receive the set of m projects P that they must evaluate and from which they have to extract the best project. Their very first step is, therefore, represented by the definition of a set of n criteria, benefits types and costs types, through which they can evaluate each project $p_j \in P$ by assigning to it n numerical values.
Once such criteria have been defined they form a set of n elements, part of which (n_1) are benefits b_i and part of which (n_2) are costs c_i.
At this point the deciders:

- examine one project $p_j \in P$ at a time;
- assign to each project a vector $x_i \in \mathbb{R}^n$.

Such steps require the deciders to both take individual decisions ([13], [7]) and to negotiate ([12], [15]) since ([4], [5]) for each project they must:

- individually asses the values to be associated to the benefits and to the costs for the current project (decision step);
- negotiate with the others so to assign an agreed on value to every benefit and to every cost for the current project.

If the negotiation step fails the deciders can enter in a new decision step having gained some, possibly strategic, knowledge of the evaluations of the others. The procedure goes on until a compromise is reached so that each criterion gets associated a single value. If the deciders are able to define only intervals of values we assume that the procedure produces the corresponding mean values as a sort of decision through an arbitration ([12]).
The outcome of the procedure *evaluate* is therefore the matrix $X \in \mathbb{R}^{m \times n}$ of m rows, one for each project, and n columns, one for each criterion.

8 The structure of the *PO* and the *prune* procedures

What we have seen in the sections 4 and 5 gets a meaning in the implementation of the *PO* procedure.
If we follow what we have seen in section 4 we have that each project p_i is represented as a vector $x_i \in \mathbb{R}^n$ of benefits and costs with $x_i = (b_i, c_i)$. In this case the deciders start from the project p_1 and:

- try to find all the projects that are dominated by p_1 (and so that have not higher benefits and not lower costs with at least one strict inequality);
- discard such dominated projects;
- if they find a dominant project p_i (that has not lower benefits and not higher costs with at least one strict inequality) they discard p_1 (since it is dominated) and repeat the procedure with p_i.

When the deciders have removed all the dominated projects and are not able to find any new dominant project they are left with the set $\hat{P}$ of Pareto equivalent projects that is the outcome of the procedure.
If we use what we have seen in section 5 we consider the benefits as separated form the costs. In this way the procedure *PO* produces:

- a set $\hat{P}_1$ of Pareto optimal projects according to the n_1 benefits,
- a set $\hat{P}_2$ of Pareto optimal projects according to the n_2 costs,
- the global set $\hat{P} = \hat{P}_1 \cap \hat{P}_2$ of the Pareto optimal projects.

We note that $\hat{P}$ may be empty (see section 5). If, for instance, we have two projects (p_1 and p_2), two benefits (b_1 and b_2) and two costs (c_1 and c_2) we may have that:

- $\hat{P}_1 = \{p_1\}$,
- $\hat{P}_2 = \{p_2\}$,
- $\hat{P} = \hat{P}_1 \cap \hat{P}_2 = \emptyset$.

We note how similar considerations hold in presence of more than two projects and of a higher number of benefits and costs.
As we have seen in section 6 the deciders can also use a lexicographic ordering of the criteria in order to rank the projects of the current set. Since this

approach seems to be flawed we do not deal with it any longer in this paper. In this case we may consider the set P as coincident with the set $\hat{P}$ so that all the projects are seen as Pareto equivalent.
Last but not least, for what concerns the *prune* procedure we note that, once the set $\hat{P}$ has been derived from a set P through the use of the PO procedure, it extracts from the matrix X the rows that correspond to the discarded projects so to produce a reduced matrix X that contains only the required rows.

9 The structure of the RPO procedure

The PO procedure has no guarantee to produce a single valued set. In many cases it may produce a set $\hat{P}$ of more than one Pareto optimal project. In this case the deciders can only refine the analysis of such projects by using the same set of criteria.
They cannot modify the set of the criteria (in a static setting) but even if they could modify it (by adding new criteria) they should verify the effect that the new criteria have on the previously discarded projects. In this paper we do not deal with this issue, that contribute to the definition of a dynamic setting (see section 11), any further.
The RPO procedure takes as input:

- the current set $\hat{P}$ of the Pareto optimal projects,

- the corresponding matrix X of the evaluations of such projects.

It produces, as its output, a matrix X'. If $X' = X$ we are in a fixed point condition so that the deciders must resort to the FS procedure. If, on the other hand, we have $X' \neq X$ the deciders can use again the PO and the *prune* procedures.
The main aim of the deciders is therefore to revise the evaluations of each project of the current set $\hat{P}$ so to obtain more realistic, though agreed on by them all, numeric values. They perform a refinement of such values, since the criteria do not change, with the aim to have the representative points of some projects to move out of the Pareto frontier either in the inside, so to become possibly dominated, or on the outside so to become possibly dominant.
If, for instance, we consider two benefits and the projects as represented by points in the $\mathbb{R}^2$ space we have that the points on the line $x_1 + x_2 = k$ (with $x_1 \geq 0$ and $x_2 \geq 0$) are on the Pareto frontier. If, through a refinement of the values associated to the projects on the Pareto frontier, we move the points representative of some projects inside the area $x_1 + x_2 < k$ such projects

may become dominated whereas if we move the point representative of one project in the area $x_1 + x_2 > k$ that project may become dominant.
To obtain this shift of the position of the representative points of the projects the deciders must examine with care every project ([3], [5]) so to assess with higher and higher precision the values associated to each criterion, be it a benefit or a cost.

10 The selection among the Pareto optimal projects

In this section we outline the structure of the procedure FS that is used by the deciders whenever a condition of fixed point occurs. In this case the deciders face a set $\hat{P}$ of irreducible Pareto optimal projects but they are requested to select one of them as the best project and so the project they think to be worth of being implemented.
In order to make things more concrete let us assume to have two projects left in the set $\hat{P}$ and that the deciders have used five criteria, two benefits (b_1 and b_2) and three costs (c_1, c_2 and c_3) but similar considerations hold also in the general case of m projects and n_1 benefits and n_2 costs.
Once the deciders have defined the set $\hat{P}$ of the two irreducible Pareto optimal projects the two projects are seen as equivalent so the final selection must be a political one though possibly based on some rules shared among the deciders.
In the **first approach** that we propose the deciders evaluate the sum of the benefits for each project so to get:

$$B_1 = b_1^1 + b_1^2 \tag{5}$$

for p_1 and:

$$B_2 = b_2^1 + b_2^2 \tag{6}$$

for p_2. In the above equations the subscripts stand for the project whereas the superscripts stand for the benefit. In a similar way they evaluate C_1 (as the total cost for p_1) and C_2 (as the total cost for p_2). So doing the deciders assign to each project a point (B_i, C_i) in the plane $\mathbb{R}^2$ so that they can see if either p_1 dominates p_2 or vice versa. Unfortunately we can have that neither of such conditions hold since we may have that one of the following conditions is instead satisfied:

(1) $B_1 > B_2$ and $C_1 > C_2$

(2) $B_1 < B_2$ and $C_1 < C_2$

so that the two projects are seen as Pareto equivalent. A variant of this approach could involve the use of arithmetic means instead of the sums but this modification does not guarantee the dominance of one project over the other. In any case both methods may represent a first attempt to find a Pareto dominant project within a set of irreducible Pareto optimal projects. Another approach could be for the decider to resort to procedures inspired by voting methods such as the Borda or the Condorcet methods (see [6], [16], [17], [9]).
We recall that the **Borda method** with h candidates and n voters is based on the following procedure:

- each voter assign to each candidate a value from the interval $[1, n]$ without repetitions;

- the values are added for each candidate over all the voters so that each candidate receives a single numerical value;

- the candidate who gets the highest score is the Borda winner;

- we can have tied candidates so we need a method to break ties among the top ranked candidates, see further on within our context.

The main problem with the Borda method is that its outcome can be manipulated by the voters through a strategic voting that requires the full knowledge of the preferences of the voters. In our context we think that this feature is a minor problem since the various projects (the candidates in our world) are evaluated according to the various criteria (the voters in our world) through rankings (with possible ties) over $\mathbb{R}$.
The **Condorcet method**, on the other hand, is based on the execution of pairwise rankings of the candidates for all the voters according to their preferences. In this case for a pair of candidates (a, b) we count the number of times where a beats b and those where b beats a. We say that a beats b (and we write $a \succ b$) for the whole set of the candidates if the former number is higher than the latter whereas b beats a for the whole set of the candidates if the former number is lower than the latter and that the two candidates are tied if the two numbers are equal. Also in this case we may have tied candidates and, moreover, we may have a loss of transitivity since, in the case of three candidates (a, b, c) we may have $a \succ b \succ c \succ a$. In this case we are not able to identify the so called Condorcet winner (or a candidate that defeats all the others in the pairwise comparisons) neither its dual, or the Condorcet loser.
To use approaches inspired by these methods we therefore consider:

- the h remaining projects $p_1, \ldots, p_h$ as the candidates,

- the n criteria $c_1, \ldots, c_n$ as the voters.

If we devise a method inspired by the **Borda method** we may define a matrix B with:

- n rows, one for each voter/criterion c_i;

- h columns, one for each candidate/project p_j.

Every cell $b_{i,j}$ of the matrix contains a value such that the sum of the values on each row is equal to:

$$\sum_{j=1}^{h} b_{i,j} = \frac{(h+1)h}{2} \quad \forall i = 1, \ldots, n \tag{7}$$

We have no unicity constraint on such values since we want to be able to account for tied projects. Once the values $b_{i,j}$ have been assigned to the various entries of the matrix we may evaluate, for each project p_j, the quantity:

$$B_j = \sum_{i=1}^{n} b_{i,j} \tag{8}$$

At this point we use such values to order the various projects. We have two cases:

- no ties among the projects;

- there are tied projects.

In the former easy case we have that the project with the highest value is the selected project or the winning candidate.
In the latter case we have that if the ties do not involve top ranked candidates they can be discarded so we are in the previous case. If the ties involve the top ranked candidates so that, for instance, we have:

$$p_1 \sim p_2 \sim p_3 \succ p_4 \succ \ldots \tag{9}$$

we have that the projects p_1, p_2 and p_3 are equivalent so that the deciders can select one of them at random.
We have now to see how each voter can assign the values $b_{i,j}$ to the various project so to satisfy the constraint imposed by relation (7).
If he has no ties among the projects he proceeds as in the classical Borda

method. If there are ties among the projects the method must be adapted through the use of fractional values. We illustrate the basic ideas in the case of four projects. In this case we have:

$$\frac{(h+1)h}{2} = 10 \qquad (10)$$

since $h = 4$. If one voter has[7]:

$$p_1 \sim p_2 \sim p_3 \sim p_4 \qquad (11)$$

then it must assign a value of 2.5 to each project. Some other cases are the following:

$p_1 \sim p_2 \succ p_3 \succ p_4$ with values equal to, respectively, 3.5, 3.5, 2 and 1;

$p_1 \succ p_2 \succ p_3 \sim p_4$ with values equal to, respectively, 4, 3, 1.5 and 1.5;

$p_1 \succ p_2 \sim p_3 \sim p_4$ with values equal to, respectively, 4, 2, 2 and 2.

From these cases it should be easy to derive the general rule. The basic idea, anyway, is that the values must be assigned so to account for both the ties and the strict orderings between the projects. We underline how the manipulability of the Borda method has no consequence for us since in this case we have that the projects are ranked, according to each criterion, through their associated values and so according to a natural ordering over $\mathbb{R}$ and, moreover, since the values $b_{i,j}$ are assigned by the deciders to the various projects and criteria according to common agreed on (or objective) rules and not according to individual and privately known preferences.
If we devise a method inspired by the **Condorcet method** we may proceed as follows in order to identify the Condorcet winners. We remark that we use the plural form since we must admit ties among projects.
In this case before we can describe the method we have to define what we mean when we say that, given a pair of projects (p_i, p_j) we have that either p_i wins or loses or is tied with p_j.
To state such definitions we consider a pair (p_i, p_j) of projects and all the criteria c_h and evaluate:

- the number x of times where p_i is better than p_j since it gives either a higher benefit or a lower cost;

[7] We use $\sim$ to denote a tie and $\succ$ to denote a strict preference between two projects. Such relations are endowed with classical properties.

- the number y of times where p_i is worse than p_j since it gives either a lower benefit or a higher cost;

- the number z of times where p_i is tied with p_j since it gives either the same benefit or the same cost.

Such evaluations depend, of course, on the nature of each criterion. If we have $x - y > 0$ we have that p_i wins over p_j, if we have $x - y < 0$ we have that p_i loses with p_j and if we have $x - y = 0$ we have that p_i is tied with p_j. At this point we can give a description of the high level structure of the procedure:

(1) we start with p_1 and put $p_c = p_1$ as the current project;

(2) we compare p_c with p_i with $i \in [2, h]$;

(3) until p_c wins with p_i it is the current Condorcet winner;

(4) if p_c loses with p_i we discard the current p_c, replace it with the current p_i (so we put $p_c = p_i$) as the current Condorcet winner and register both the old p_c (only if $i > 2$) and the new p_i as Condorcet winners in an array called Winners;

(5) if p_c ties with p_i we keep on considering p_c as the current Condorcet winner but we register both p_c and p_i in an array called Ties.

At the end of the procedure we have:

- a project as a possible Condorcet winner p_c;

- an array Winners;

- an array Ties.

The array Winners contains the identifiers of those projects that have won pairwise comparisons with other projects. If it contains less than three elements we cannot have cycles. If it contains three or more elements we may have cycles among the projects that have the following structure: $p_i \prec p_j \prec p_k \prec p_i$. In this case the method has failed. We note how this is a reason of failure also in the case of the classical Condorcet method.
If we have no cycles among such projects we consider p_c and, by an inspection of the content of the Ties array, all the projects that are tied with p_c. If such projects exist they form a set of equivalent projects among which the deciders can choose one project at random as the best project among those

of the initial set P. In this case we have indeed a situation like the following (where the three projects on the right are the content of the array Ties):

$$p_c = p_i \sim p_j \sim p_l \sim p_k \tag{12}$$

In this case, if we consider all the criteria as having the same weight or the same importance, we cannot but consider such projects as equivalent so that a random selection is the easiest way to get a final selection otherwise we could resort to the methods we have seen in section 6.

11 Conclusions and future plans

In the present paper we have presented an iterative procedure through which a set D of deciders can select the best project out of a set P of competing projects. The procedure has been sketched and described together with all its ancillary procedures but not up to the minimal details since many of them are based on decision and negotiation phases that depend on the nature of the deciders and on the nature of the involved projects. Such phases cannot be fully characterized if we take the descriptive approach that we have used in the present paper ([13]).

The only principle we have adopted in this paper if the principle of Pareto dominance though which we have defined either dominated or dominant projects though we have suggested also the use of methods inspired by the Borda and the Condorcet methods of voting. In this way we have completely disregarded issues such as fairness ([1], [2]) and equity ([18]) since we believe that such criteria can be used only when the deciders have to decide how to share among themselves the benefits and costs associated to a given project ([5]) rather than in the ranking of a set of projects with the aim of choosing the best project.

Future plans involve the adoption of a **dynamic setting** and the evaluation of the introduction of issues of fairness and equity in the process especially in the phase where the deciders evaluate the matrix X.

At this level the deciders may indeed decide to penalize those projects whose benefits and costs can be hardly shared among themselves so to satisfy shared criteria either of fairness or of equity.

For what concerns the dynamic setting we note how in that case:

- the set D of the deciders could vary during the process;

- the set of the criteria could be modified by the deciders during the process;

- the set P of the competing projects could be seen only in part as an exogenous parameter so that new projects can could be either proposed or devised by the deciders during the process.

Future plans include, therefore, an analysis of the possibility that other deciders join the ranking and selection process as well as the analysis of the possibility that new criteria are added by the deciders whereas others are possibly discarded, again by the deciders, since they prove to be non discriminating for the current set of projects.

References

[1] Steven J. Brams and Alan D. Taylor. *Fair division. From cake-cutting to dispute resolution.* Cambridge University Press, 1996.

[2] Steven J. Brams and Alan D. Taylor. *The win-win solution. Guaranteeing fair shares to everybody.* W.W. Norton & Company, 1999.

[3] Lorenzo Cioni. *The roles of System Dynamics in environmental problem solving.* 2008 International System Dynamics Conference, 2008. Athens, Greece, July 20 - 24.

[4] Lorenzo Cioni. *Deciding within a competition.* XL AIRO, 2009. Siena, Italy, September 8-11.

[5] Lorenzo Cioni. *Methods and Models for Environmental Conflicts Analysis and Resolution.* PhD Thesis in "Mathematics for the Economic Decisions", University of Pisa, 2010.

[6] Pietro Grilli di Cortona, Cecilia Manzi, Aline Pennisi, Federica Ricca, and Bruno Simone. *Evaluation and Optimization of Electoral Systems.* SIAM Monographs on Discrete Mathematics Applications, 1999.

[7] David M. Kreps. *Notes on the theory of choice.* Westview Press, 1988.

[8] Roger B. Myerson. *Game Theory. Analysis of conflict.* Harvard University Press, 1991.

[9] Hannu Nurmi. *Voting Paradoxes, and How to Deal with Them.* Springer, 1999.

[10] Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory.* MIT Press, 1994.

[11] Fioravate Patrone. *Decisori (razionali) interagenti. Una introduzione alla teoria dei giochi.* Edizioni plus, 2006.

[12] Howard Raiffa. *The Art & Science of Negotiation.* Harvard University Press, 1982.

[13] Anatol Rapoport. *Decision Theory and Decision Behaviour. Normative and Descriptive Approaches.* Kluver Academic Publishers, 1989.

[14] Fred S. Roberts. *Measurement Theory with Applications to Decision-making, Utility, and the Social Sciences.* Addison-Wesley, 1979.

[15] Jacques Rojot. *Negotiation: From Theory to Practice.* Macmillan Academic and Professional LTD, 1991.

[16] Donald G. Saari. *Decisions and Elections, Explaining the Unexpected.* Cambridge University Press, 2001.

[17] Donald G. Saari. *Disposing Dictators, Demistifying Voting Paradoxes. Social Choice Analysis.* Cambridge University Press, 2009.

[18] G.H. Peyton Young. *Equity. In Theory and in Practice.* Princeton University Press, 1994.