A negative auction

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Abstract

In this paper we describe a type of auction mechanism where the auctioneer **A** wants to auction an item ζ among a certain number of bidders $b_i \in B$ $(i = 1, \ldots, n)$ that submit bids in the auction with the aim of not getting ζ.
Owing to this feature we call this mechanism a **negative auction.**
The main motivation of this mechanism is that both the bidders and the auctioneer give a negative value to the auctioned item (and so they see it as a bad rather than a good).
The mechanism is presented in its basic simple version and with some possible extensions that account for the payment of a fee for not attending the auction, the interactions among the bidders and the presence of other supporting actors.

1 Introduction

In this paper we describe a type of auction mechanism[1] where the auctioneer **A** wants to auction an item ζ among a certain number of bidders[2] $b_i \in B$ $(i = 1, \ldots, n)$ that submit bids in the auction with the aim of not getting ζ.
Owing to this feature we call this mechanism a **negative auction** ([4]).
The main motivation of this mechanism is twofold ([7] and [8]):

[1]In this paper we are going to use the term mechanism in a rather informal sense as a set of rules, strategies and procedures. For a more formal use of the term we refer, for instance, to [7] and [9].

[2]In what follows we identify a bidder $b_j \in B$ also by the index $j \in N = \{1, \ldots, n\}$.

- both the bidders and the auctioneer give a negative value to the auctioned item (and so they see it as a bad rather than a good),
- the auctioneer has an imperfect knowledge of the bidders and so cannot contact any of them directly.

The mechanism[3], at least in its basic version, is simple and will be described in detail in section 5 whereas the needed details will be presented in the sections 3 and 4.

Algorithm 1.1 *The basic mechanism is based on the following steps.*

- ***A*** *selects the bidders b_i according to some private criteria that depend on the nature of ζ;*
- *the b_i submit their bids in a sealed bid auction;*
- *once the bids have been submitted they are revealed so that:*
  - *the bidder who made the lowest bid is the* ***losing bidder*** *and gets*[4] *ζ;*
  - *the other bidders are termed* ***winning bidders*** *and get the benefit of having avoided the allocation of ζ;*
- *the losing bidder*[5] *b_1 gets ζ and, as a compensation, a sum equal to his bid x_1;*
- *each winning bidder b_i pays to the losing bidder a properly defined fraction of x_1.*

This simple mechanism will be described in some detail in the following sections together with the possible strategies of the bidders and some possible extensions.
The extensions include a pre auction phase, where some of the bidders pay a fee for not attending the auction, and a post auction phase that can assume three forms and that aims at the reallocation of ζ depending on criteria that are different from those that drove the auction phase itself.

[3]The proposed mechanism is loosely inspired by the Contract Net Protocol ([6, 14]).

[4]Possible ties among two or more losing bidders are resolved through a properly designed random device.

[5]We assume that after the bids have been revealed we renumber the bidders so that the losing bidder is the bidder b_1 whereas all the other bidder b_i (with $i \neq 1$) are the winning bidders.

2 Pre auction and post auction phases

In the **pre auction** phase the bidders are allowed to pay to **A** a fee f (that **A** fixed and made common knowledge among the bidders) for not attending the auction. In this case, depending on the amount of the fee, we can have that:

- m bidders prefer to pay the fee in order to not attend the auction;
- $k = n - m$ bidders prefer to attend the auction.

In this case, at the end of the auction phase, **A** has collected an extra compensation equal to $e_c = mf$ that is awarded to the losing bidder.
The value e_c (see also section 8) may be either a public knowledge among the bidders that therefore know m but not necessarily k (since the value n is not necessarily a common knowledge among the bidders) before the auction phase or it may be a private knowledge of **A** to be revealed only after the execution of the auction phase.
As to the last point we note how this feature may be guaranteed or at least enforced through the design of the structure of the pre auction phase that can be designed so to make the communication among the bidders either too difficult or too costly. The easiest solution is to have the bidders, at least in this phase, to be unaware one of the others so to make any inter bidders communication impossible.
In the present paper we consider only the private knowledge case so that the value e_c has no influence on the behavior of the k attending bidders that do not have such information when they submit their bids (see section 8).
We note indeed how even the m bidders who paid the fee can attend the possible post auction phase. This requires that in that phase the full set of bidders is revealed and becomes a common knowledge.
In the **post auction** phase we introduce some mechanisms that try to correct a simplifying assumption that we have made in the basic mechanism.
The basic mechanism is, indeed, based on the assumption that the various b_i are independent one from the others (in the sense that the allocation of ζ to one of the bidders has effect only on that bidder) and, similarly, do not influence any other actor[6].
The mechanisms of the post auction phase aim, indeed, at accounting for the following facts:

[6]With the term actor we denote a figure that is distinct from both **A** and the Bs but that wants to attend the auction since he thinks to be damaged from the allocation of ζ to one of the bidders. Such actors are termed **supporters** and form the set S.

(pa_1) the bidders b_i are interdependent and so they may influence each other so that, for any pair of bidders (b_i, b_j), we can define as $d_{i,j}$ the damage caused to b_i from the allocation of ζ to b_j;

(pa_2) the bidders b_i may influence the actors of the set S (see footnote 6) so that, for any actor $s_i \in S$, we can define as $D_{i,j}$ the damage caused to s_i from the allocation of ζ to b_j.

We may assume in general that $d_{i,j} \neq d_{j,i}$ so the cross damages between pairs of bidders are not symmetrically distributed.
In the (pa_1) case we assume that the bidders are interdependent but $S = \emptyset$. In this case the bidders can try to negotiate an allocation to another bidder that is more preferred by all the bidders depending on the values $d_{i,j}$ (for $i \neq j$) and not on the values $m_i = d_{i,i}$ that drive the auction phase.
In the (pa_2) case, we assume that the bidders are independent but $S \neq \emptyset$. In this case the members of S may try to obtain a reallocation depending on the values $D_{i,j}$.
Last but not least the two cases (pa_1) and (pa_2) can be merged in a single case where we have both interdependent bidders and $S \neq \emptyset$.
In all the post auction cases the starting point is the allocation of ζ to one of the bidders on the basis of the outcome of the auction phase where each bidder is guided only by his self damage $m_i = d_{i,i}$.
At the end of the auction phase we can have two cases:

- the resulting allocation is satisfactory[7];

- the resulting allocation is unsatisfactory.

In the former case no reallocation is required whereas in the latter case either the bidders of the set B or the supporters of the set S may try to renegotiate it, within the proposed mechanisms in order to identify a new bidder as the more preferred allocation.
We underline how such reallocation may require the raising of a further compensation for the new bidder in order to have him accept the allocation of ζ.

3 The defining parameters

Both the auctioneer **A** and the bidders of the set B are characterized by some parameters that depend heavily on the nature of the item ζ but also on their individual characteristics.

[7]The concept of satisfaction will be defined for each post auction phase. For the moment we say that an allocation is satisfactory if there are no incentives for its modification either from the members of B or from the members of S or from both.

Definition 3.1 *For what concerns* ***A*** *we have only one parameter: the value* m_A *that* ***A*** *assigns to* ζ *as a measure of his utility since the only benefit that* ***A*** *receives from the auction is the allocation of* ζ*.*
With m_A *we denote:*

- *the damage or the negative utility that* ***A*** *will receive from* ζ *if the auction is void so the allocation fails;*

- *the benefit or the positive utility that* ***A*** *receives from the allocation of* ζ *to one of the* $b_i \in B$*.*

Observation 3.1 *In the former case* m_A *has a negative value whereas in the latter it has a positive value.*

Definition 3.2 *Every* $b_i \in B$ *is characterized by the following parameters (see also [7, 8]):*

- *a value* m_i *that he assigns to* ζ*;*

- *the amount* x_i *he is willing to bid;*

- *the random variables* X_j *that describe the bids of the other bidders;*

- *the interval of the values* $[0, M_i]$ *to which the* m_i *belong;*

- *the intervals of the values* $[0, M_j]$ *to which the* X_j *belong;*

- *the differentiable cumulative distributions* F_j *of the values* X_j*;*

- *the corresponding density functions* $f_j = F_j'$ *of such values.*

Observation 3.2 *We note that:*

(1) *the parameter* m_i *has a dual meaning in the sense that:*

  - *it represents the damage that* b_i *receives from the allocation of* ζ*;*
  - *it represents the benefit that* b_i *gets from the allocation of* ζ *to some other bidder;*

(2) *the parameter* x_i *has a dual meaning in the sense that:*

  - *it represents the sum that* b_i *asks as a compensation for the allocation of* ζ*;*
  - *it defines the fraction* c_i *of the compensation that* b_i *has to pay to the losing bidder.*

We can also define the following probabilities:

- the probability p_i for b_i of losing the auction;
- the dual probability $q_i = 1 - p_i$ for b_i of winning the auction.

We recall that the **losing bidder** is the bidder who gets ζ and a compensation from the other bidders, the **winning bidders**.

4 The basic assumptions

In this section we introduce the basic assumptions that we make on the parameters that characterize both the auctioneer and the bidders and that will be maintained through the rest of the paper.

Assumptions 4.1 *The only assumption we can make on* ***A*** *is that his value* m_A *is a private information of the auctioneer so that it is not known to the bidders.*
If we relax this assumption so that m_A *becomes a common knowledge among the bidders we may assume that such a knowledge may influence the evaluations of the bidders since they may derive form this knowledge hints on the real nature of the auctioned item.*

Assumptions 4.2 *The basic assumptions that involve the characteristic parameters of the bidders may be summarized as follows*[8]*:*

- *the bidders are assumed to be* ***risk neutral*** *so that their utility is linearly separable ([7]) and can be expressed as the difference between a benefit and a damage and so as* $x_i - m_i$ *if the bidder* b_i *loses the auction or as* $m_i - c_i$ *if he wins it;*
- *the random variables* X_j *are assumed to belong to a common interval* $[0, M]$ *for a suitable* $M > 0$*;*
- *the random variables* X_j *are assumed to be independent random variables;*
- *the valuations* m_i *are assumed to be* ***private values*** *of the single bidders;*
- *the bidders* b_j *are assumed to be* ***symmetric*** *so they are characterized by the same* F *and by the same corresponding* f*;*

[8]See [7, 8]

- *the random variables X_j are assumed to be uniformly distributed on the interval $[0, M]$ so that we have, for $x \in [0, M]$:*

$$P(X_j \leq x) = F(x) = \frac{x}{M} \tag{1}$$

and, correspondingly:

$$f(x) = \frac{1}{M} \tag{2}$$

From the foregoing assumptions we derive that the probability p_i for each bidder b_i of losing the auction is the same for all the bidders so we can denote it as p and use $q = 1 - p$ to denote the dual probability of winning the auction.

Observation 4.1 *Possible relaxations of the foregoing assumptions involve:*

- *the possibility that the bidders are risk adverse*[9] *so that his utility is no more linearly separable but it is a convex function of x_i;*

- *the possibility that the evaluations are either common or interdependent among the bidders;*

- *the possibility that the bidders are asymmetric so that we can have different intervals $[0, M_j]$ and different functions F_j and f_j for each bidder b_j as well as the possibility to have different distributions (such as a Gaussian or a triangular distribution) also under the symmetry assumption.*

Such relaxations can be introduced either one at a time or in combinations. Their treatment, that makes the analysis more complex, is out of the scope of the present paper and is the subject of further research efforts (see section 8 for further details).

5 The basic mechanism and its strategies

The **basic mechanism** has only the auction phase among independent bidders with $S = \emptyset$.

[9] We recall that, in classical terms, a player is **risk neutral** ([5]) if he is indifferent between attending a lottery and receiving a sum equal to its expected monetary value whereas he is **risk averse** if he prefers the expected value to attending the lottery. We can also say that a player is risk neutral if his utility function is linearly separable in gain and loss whereas, if he is risk averse, it can be seen as a concave function. In our context we have to consider the opposite perspective and so we consider the utility function of risk averse bidders as a convex function of its meaningful parameters.

Algorithm 5.1 *We can describe the basic mechanism with the following algorithm*[10]*.*

(ph_1) ***A*** *auctions* ζ*;*

(ph_2) *the* b_i *make their bids* x_i *in a sealed bid one shot auction;*

(ph_3) *the bids are revealed;*

(ph_4) *the lowest bidding bidder*[11] b_1 *gets* ζ *and* x_1 *as a compensation for this allocation;*

(ph_5) *each of the other bidders* b_i *pays to* b_1 *a fraction* c_i *of* x_1 *such that:*

$$\sum_{i \neq 1} c_i = x_1 \tag{3}$$

Observation 5.1 *For what concerns the values* c_i *we assume a* ***proportional repartition*** *among the bidders so we have:*

$$c_i = x_1 \frac{x_i}{X} \tag{4}$$

where $X = \sum_{j \neq 1} x_j$*. In this way we account for the fact that the bidders who receive a bigger advantage from the allocation of* ζ *to* b_1 *pay the higher fractions of the compensation.*

At this point we state and prove the following proposition.

Proposition 5.1 (Weakly dominant strategy) *From the assumptions we made in section 4 we derive that it is a* ***weakly dominant strategy*** *for each bidder to submit a bid* x_i *equal to his evaluation* m_i *of the auctioned item* ζ*.*
Proof
From what we have stated in sections 3 and 4 we derive easily that the expected utility from the auction for every bidder b_i *when he faces the phase* (ph_2) *can be expressed as:*

$$E(b_i) = p(x_i - m_i) + (1 - p)(m_i - x_1 \frac{x_i}{X}) \tag{5}$$

[10] Also in this section we assume that, when the phase (ph_3) is over we can renumber the bidders so that b_1 is the losing bidder whereas the b_i (with $i \neq 1$) are the winning bidders.

[11] Possible ties are resolved with the random selection of one of the tied bidders.

as the sum of the utility if he loses the auction multiplied with the probability of losing it and the utility if he wins it multiplied with the probability of winning it.
Relation (5) can be rewritten as:

$$E(b_i) = (1 - \frac{x_i}{M})^{n-1}(x_i - m_i) + (1 - (1 - \frac{x_i}{M})^{n-1})(m_i - x_1\frac{x_i}{X}) \quad (6)$$

by using the following equalities:

$$p = (1 - \frac{x_i}{M})^{n-1} \quad (7)$$

$$q = 1 - p = 1 - (1 - \frac{x_i}{M})^{n-1} \quad (8)$$

that have been derived by using the hypotheses of independence and identical and uniform distribution of the X_j and by imposing that the x_i is lower than any of the X_j for $j \neq i$.
Since in relations (5) and (6) we want to impose that in any case each bidder b_i has a non negative utility we get the following constraints[12]*:*

- $y_1 = x_i - m_i \geq 0$
- $y_2 = m_i - x_1\frac{x_i}{X} = m_i - x_1\frac{x_i}{x_i+X'} \geq 0$

where y_1 is the utility for b_i if he loses and y_2 is his utility if he wins.
From the former constraint we derive:

$$x_i \geq m_i \quad (9)$$

For what concerns the latter constraint, from the definition of y_2 and by performing the derivations with respect to x_i, we easily derive that:

- $y_2' < 0$
- $y_2'' > 0$

so y_2 is ***concave decreasing*** *with:*

- *a maximum value equal to $y_2(0) = m_i$ for $x_i = 0$,*
- *a minimum value equal to:*

$$y_2(M) = m_i - x_1\frac{M}{M + X'} \quad (10)$$

  for $x_i = M$.

[12] We note how we can write $X = x_i + X'$ where X' accounts for the bids of the bidders distinct from b_1 and b_i.

It is easy to verify that we have $y_2(m_i) > 0$ whereas we cannot exclude that $y_2(M)$ may assume negative values though this is a rather unlikely event. From relations (7) and (8) we can easily see how:

- *p has a maximum value of 1 for $x_i = 0$, decreases for x_i increasing and attains a null value for $x_i = M$;*

- *q has dual behavior since it has a minimum value of 0 for $x_i = 0$, increases for x_i increasing and attains the maximum value of 1 for $x_i = M$;*

At this point we want to find the value $\bar{x}_i$ where we have

$$p = q \tag{11}$$

so that for $x_i < \bar{x}_i$ we have that p dominates q whereas we have the opposite for $x_i > \bar{x}_i$. From relation (11) and relations (7) and (8) we get:

$$(1 - \frac{x_i}{M})^{n-1} = 1 - (1 - \frac{x_i}{M})^{n-1} \tag{12}$$

From relation (12), with some easy algebra, we derive:

$$\bar{x}_i = (1 - (\frac{1}{2})^{\frac{1}{n-1}})M \tag{13}$$

We note that $\bar{x}_i \to 0$ as $n \to \infty$ so that q tends to dominate p for any x_i. According to all this we have that b_i should maximize y_2 so to bid no less than m_i and so (given the constraint we have imposed on y_1) he should bid a sum equal to m_i.

Observation 5.2 *We note that we have:*

$$\frac{p}{p'} \to 0 \, as \, n \to \infty \tag{14}$$

where p' is the derivative of p as a function of x_i whereas:

$$\frac{q}{q'} \to \infty \, as \, n \to \infty \tag{15}$$

where q' is the derivative of q as a function of x_i.

Observation 5.3 *It is obvious that at phase (ph_3) each b_i knows if he is the loser or one of the winners.*
In the former case he has a utility:

$$x_1 - m_1 \tag{16}$$

whereas in the latter he has a utility:

$$m_i - x_1 \frac{x_i}{X} \tag{17}$$

Observation 5.4 *We have in this way verified how the truthful bidding is a weakly dominant strategy for each bidder in the basic mechanism of the negative auction.*

Observation 5.5 *The proposed mechanism has a strong analogy with a First Price Sealed Bid auction ([7]). In the auctions of this type the winning bidder is the highest offering bidder who pays his bid. Under hypotheses similar to the ones we made in sections 3 and 4 we have that in a First Price Sealed Bid auction the best strategy for each bidder is to bid a little less than one's own evaluation or to bid* $x_i = m_i - \delta$ *with* $\delta \to 0$ *for* $n \to \infty$*.*
If we suppose to use negative prices our mechanism is analogous to a First Price Sealed Bid auction so, in our case, the best strategy for each bidder is to bid a little more than one's own evaluation or to bid $x_i = m_i + \delta$ *with* $\delta \to 0$ *for* $n \to \infty$*.*

6 The use of the fee

In this section we present the pre auction phase where:

- m bidders pay the fee f in order to not attend the auction;
- $k = n - m$ bidders prefer to attend the auction.

We make the hypothesis that the sum $e_c = mf$ is a private information of **A** so it is unknown to the other k bidders that neither know n. For the k attending bidders we can repeat what we have said in section 5.
In this case the losing bidder, at the end of the auction phase, gets the following final compensation:

$$f_c = x_1 + e_c = x_1 + mf \tag{18}$$

If the mechanism has a post auction phase then all the initial n bidders can attend to it, as we will show in the following sections.
At this point we define the following profiles:

(ne_1) all the n bidders pay the fee f,

(ne_2) none of the n bidders pays the fee f.

We want to see if such profiles are Nash Equilibria[13] (NE) or not.
In the case (ne_1) we have that if the bidders collude among themselves and decide that they all pay the fee f they collect the sum $e_c = nf$. In this case, every bidder would have a utility equal to[14] $m_i - f$. If only one bidder b_j

[13] A Nash Equilibrium is a profile of strategies for the bidders where none of them has a gain from an individual deviation ([1, 2, 9, 10]).

[14] This requires $f < m_i$ for every b_i. We comment on this assumption shortly.

individually violates the collusive agreement he gets a utility equal to:

$$(n-1)f - m_j \tag{19}$$

since no further compensation from the auction phase is possible. The individual deviation is profitable (so that (ne_1) is not a NE) if we have:

$$(n-1)f - m_j > m_j - f \tag{20}$$

or if:

$$m_j < f\frac{n}{2} \tag{21}$$

So if the fee f is such that the constraint (21) is satisfied for at least one b_j the collusive agreement is not a NE and the auction cannot be void since **A** is able to find a bidder to which to allocate ζ with a compensation paid by the other bidders.
We note that if **A** fixes f as:

$$f > \frac{2M}{n} \tag{22}$$

we have:

$$\frac{n}{2}f > M \geq m_i \, \forall b_i \tag{23}$$

and so relation (21) is surely verified.
In the case (ne_2) the individual deviation depends on the possible policies of the single bidders since we have that $e_c = 0$ so from this condition we cannot derive any incentive for the bidders to deviate.
In order to understand under which conditions the case (ne_2) can occur we therefore examine a more general case and so under which conditions a bidder is better off if he pays the fee than if he attends the auction.
A bidder b_i has indeed the following possibilities[15]:

(1) he pays the fee f and has an utility[16] $u_i^p = m_i - f$;

(2) he does not pay and attend the auction and so:

  (2a) he has an utility $u_i^l = x_i - m_i$ if he loses the auction,

  (2b) he has an utility $u_i^w = m_i - x_1 \frac{x_i}{x_i + X'}$ if he wins the auction.

[15] We use the decorations p, l and w as exponents to denote, in the order, a payment, a loss and a win.

[16] In this case we evaluate the utility under the hypothesis of risk neutrality and so as the difference between the benefit, as represented by the missed allocation of ζ, and the payment as represented by the fee f.

From the case (1) we derive the first constraint since we have that if $u_i^p < 0$ then b_i does not pay the fee and attends the auction. This requires that:

$$u_i^p = m_i - f \geq 0 \tag{24}$$

or:

$$f \leq m_i \tag{25}$$

If condition (25) is violated for every b_i so that we have:

$$f > m_i \tag{26}$$

for every b_i we have that no bidder pays the fee. In this way we have that if $f > max\{m_i\}$ or if f is very high no bidder pays the fee and so they all attend the auction phase. If f is assigned a lower value some of the bidders prefer to pay it whereas others prefer to attend the auction. Lastly, if f gets a very low value we have that all the bidders may prefer to pay it so that the auction phase is void, without any discordance with what we have seen with regard to (ne_1).

Once we have established that relation (24) is satisfied we want to make a comparison with the cases $(2a)$ and $(2b)$ so to understand if a bidder is better off by paying the fee or by attending the auction. We can make the following comparisons:

$$m_i - f \geq x_i - m_i \tag{27}$$

and:

$$m_i - f \geq m_i - x_1 \frac{x_i}{x_i + X'} \tag{28}$$

If such relations are satisfied then b_i is better off by paying the fee and so by not attending the auction.

From relation (27) we derive:

$$f \leq 2m_i - x_i \leq m_i \tag{29}$$

(since we have assumed $x_i \geq m_i$) and so not really a new constraint since it coincides with relation (25).

On the other hand from relation (28) we get:

$$f \leq x_1 \frac{x_i}{x_i + X'} \leq x_1 \frac{x_i}{(n-1)x_1} \leq \frac{M}{n-1} \tag{30}$$

since, by the definition of x_1 and x_i, we get $X = x_i + X' \geq (n-1)x_1$ and $x_1 \leq x_i \leq M$ for every b_i. From relation (30) we derive that if the fee f is small enough then the bidders have incentive to pay it otherwise they have incentives to attend the auction. From this we may derive that if **A** fixes f high enough (for instance $f = M/2$) he can be sure to have a non void auction even if some bidders may prefer to pay the fee f.

7 The post auction phase

7.1 Introductory remarks

In the simplest case, when the auction phase is over, the allocation is performed by the bidders on the basis of the values $m_i = d_{i,i}$ only. This way of proceeding is based on the assumption that the bidders are independent and so that the allocation damages only each individual bidder and neither other bidders nor any other of the actors of the set S (the supporters).
In section 7.2 we see how we can account for the interdependence of the bidders and so for the damages among the bidders. We therefore present an algorithm based on a succession of **push** operations by which a bidder can push ζ towards another more preferred bidder (according to the values attributed to the cross damages $d_{i,j}$). In this case we have no supporters so that $S = \emptyset$.
In section 7.3 we assume that the bidders are independent but $S \neq \emptyset$ and we examine if the supporters can push ζ towards another more preferred bidder (according to the values attributed to the cross damages $D_{i,j}$ by the $s_i \in S$).
Last but not least in section 7.4 we present an attempt to merge the two approaches since we assume to have both interdependent bidders and $S \neq \emptyset$.

7.2 The interaction among the bidders

Definition 7.1 (The added parameters) *In addition to the parameters we have seen in section 3 and the assumptions we have made in section 4 we introduce the following parameters for every bidder b_i:*

- $d_{i,j} \geq 0$ *is the damage that b_i receives if ζ is allocated to b_j;*
- $c_{i,j} \geq 0$ *is the contribution that b_i is willing to pay to b_j to have him accept the allocation of ζ.*

Observation 7.1 *It is obvious that $m_i = d_{i,i}$ and $c_{i,i} = 0$.*

Before going on we recall that the auction phase ends with the allocation of ζ to b_1 who receives a compensation equal to x_1.
We can define the due payment that b_1 receives from every bidder $b_i \neq b_1$ as:

$$\sigma_{i,1} = x_1 \frac{x_i}{X} \tag{31}$$

(with $X = \sum_{j \neq 1} x_j$) so that we have:

$$\Sigma_1 = \sum_{i \neq 1} \sigma_{i,1} = x_1 \tag{32}$$

We can also define:

$$\Sigma_j = \Sigma_1 - \sigma_{j,1} \tag{33}$$

to be used shortly.

Mechanism 7.1 *In this case the mechanism has the following structure:*

- *possible pre auction phase,*
- *auction phase,*
- *allocation and compensation phase,*
- *reallocation phase.*

From the allocation and compensation phase b_1 would get, from the members of $N_{-1} = N \setminus \{1\}$, the commitments of payment $\sigma_{i,1}$ that form the compensatory sum Σ_1 whereas the **reallocation** phase depends on the values $d_{i,j}$. When the allocation phase is over, b_1 orders the $d_{1,j}$ $\forall j \neq 1$ with regard to $d_{1,1} = m_1$. We can have two cases:

- $d_{1,1} < d_{1,j}$ $\forall j \neq 1$ so b_1 is satisfied and no reallocation is required;
- $\exists J_1 \subseteq N_{-1}$ such that $\forall j \in J_1$ $d_{1,j} < d_{1,1}$.

In the former case the mechanism **ends** and b_1 receives the commitments at payment as effective compensations from the other bidders.
In the latter case b_1 may **negotiate a reallocation** with the members of J_1 that he orders in increasing order of the damages $d_{i,j}$. We note that for any b_j with $j \in J_1$ we define as $\bar{c}_{1,j} = d_{1,1} - d_{1,j}$ the maximum contribution that b_1 is willing to pay to b_j to have him accept ζ whereas with $c_{1,j} < \bar{c}_{1,j}$ we denote the current value of this contribution.

Algorithm 7.1 *The attempt of reallocation may proceed along the following steps:*

(1) b_1 *defines* J_1*;*

(2) we have two cases:

  (2a) $J_1 = \emptyset$ *so go to (5);*

  (2b) $J_1 \neq \emptyset$ *so go to (3);*

(3) b_1 *contacts (in the order) a* b_j *with* $j \in J_1$ *and offers him a further compensation* $c_{1,j} < \bar{c}_{1,j}$ *so that* b_j *would get* $\Sigma = \Sigma_j + c_{1,j}$*;*

(4) at this point we have two cases:

(4a) b_j accepts and so becomes the new b_1 with $\Sigma_1 = \Sigma_j + c_{1,j}$; go to (1);

(4b) b_j refuses so we have two cases:

($4b_1$) *there is one more b_j that can be contacted so go to (3);*

($4b_2$) *there is no b_j to contact so the procedure ends with a failure; go to (5);*

(5) end;

The operation at step (3) is a **push** operation through which the current b_1 tries to allocate ζ to some other bidder b_j having a benefit equal to $d_{1,1} - d_{1,j} - c_{1,j}$. Such procedure may either succeed or fail. For it to succeed the current b_j must accept the proposal of b_1. It is easy to see that b_j accepts if the following conditions are verified:

(ac_1) $\Sigma \geq m_j$

(ac_2) $d_{j,1} \geq d_{j,j}$

If condition (ac_1) is violated b_j surely refuses the push proposal whereas if the condition (ac_2) is violated b_j can accept ζ, with a risky decision, if he is sure he can push it to some other bidder b_h such that $d_{j,h} < d_{j,1} < d_{j,j}$. The procedure has the following termination conditions:

- when no bider accepts a push proposal from the current b_1;
- when for a bidder b_1 we have $J_1 = \emptyset$ so the currently losing bidder is satisfied with the allocation;
- when there would be a cycle.

The last case deserves some more comments. If we have, avoiding to rename the successive losing bidders, the following succession of exchanges:

$$b_1 \to b_j \to b_h \to \cdots \to b_k \to b_1 \tag{34}$$

we have a cycle that could even give rise to a money pump for the initial b_1. To prevent this from occurring we impose a cut on the cycle so that the final accepting bidder must be b_k. This fact requires the recording of the various passages so to detect any cycle and to apply the correcting action.

7.3 The presence of the supporters

In this case we make the following assumptions:

- the bidders are independent so we have $d_{i,j} = 0\ \forall i \neq j$;
- we have s supporters $s_i \in S$ so that for each s_i we have the damages $D_{i,j}$ that he receives from the allocation of ζ to each bidder b_j.

Mechanism 7.2 *Also in this case (see section 7.2) the mechanism has the following structure:*

- *possible pre auction phase,*
- *auction phase,*
- *allocation and compensation phase,*
- *reallocation phase.*

The reallocation is driven, in this case, by the members of S with their values $D_{i,j}$.
We can consider S as partitioned[17]:

$$S = A \cup D \tag{35}$$

where:

- A is the set of the s_i that agree with the allocation of ζ to b_1 so that $s_i \in A$ if and only if $D_{i,1} < D_{i,j}$ for every $b_j \neq b_1$;
- D is the set of the s_i that disagree with the allocation of ζ to b_1 so that $s_i \in D$ if and only if[18] exists at least a bidder $j_i \neq 1$ such that $D_{i,j_i} < D_{i,1}$.

We can have the following cases:

(1) $A = S$ and $D = \emptyset$ so no reallocation is required;

(2) $A = \emptyset$ and $D = S$ so every s_i has at least a more preferred allocation;

(3) $A \neq \emptyset$ and $D \neq \emptyset$.

[17] In a classic way we have $S = A \cup D$ and $A \cap D = \emptyset$.
[18] We note that every $s_i \in D$ may have his own j_i.

In the case (1) the procedure is obviously over.
In the case (2) for every $s_i \in D$ we can partition N as $N = L_i \cup \{b_1\} \cup U_i$ where:

- L_i identifies the bidders that cause to s_i a lower damage than b_1 or the more preferred bidders;
- U_i identifies the bidders that cause to s_i a greater damage than b_1 or the less preferred bidders.

We can have two cases:

- $\cap_{s_i} L_i = \emptyset$,
- $\cap_{s_i} L_i \neq \emptyset$

In the former case no compromise is possible among the members of D so the allocation of ζ at the current b_1 is unchanged.
In the latter case we can have two sub cases.
In the former sub case we have $\cap_{s_i} L_i = b_j$ so the members of D offer to b_j both Σ_j(see section 7.2) and $\gamma_j = x_j - \Sigma_j$ to be shared proportionally among the members of D as:

$$\gamma_{i,j} = \gamma_j \frac{D_{i,1} - D_{i,j}}{\sum_{s_i} (D_{i,1} - D_{i,j})} \tag{36}$$

We note that a proposal to b_j is feasible only if, for each supporter s_i, the following feasibility condition holds:

$$\gamma_{i,j} \leq D_{i,1} - D_{i,j} \tag{37}$$

If condition (37) is violated for at least one supporter then no proposal can be made so the Ss must consider another of the available bidders, if they have one, otherwise the procedure ends with a failure.
If b_j accepts we have a new allocation otherwise the procedure ends with a failure and the allocation is unchanged. For the conditions of acceptance for b_j we refer to section 7.2. In this case b_j accepts if the offered total compensation is enough to cover the damage m_j from the allocation of ζ since the bidders are assumed to be independent.
In the latter sub case we have $L = \cap_{s_i} L_i \subset N$ so we identify a set of $l = |L|$ elements. In this case the members of D can use the Borda method[19]

[19]Given n alternatives the method is based on the fact that each voter assigns $n-1$ points to the top ranked alternative, $n-2$ to the second top ranked alternative up to 0 point to the lowest ranked alternative. The points are added together and the alternatives ordered in a weakly descending order (ties are indeed possible) so that the alternative that receives the highest number of points, in absence of ties, is the Borda winner. If we have ties on the top ranked alternatives we can choose one of them at random as the Borda winner.

([12, 13]) on such elements so to define the Borda winner (be it b_j) and apply to it what we have seen for the single outcome sub case. In the case of a tie on the Borda winners one of such winners can be selected at random since they can be seen as equivalent alternatives.
If the new allocation is feasible and the Borda winner accepts the procedure is over otherwise the members of D discard him and repeat the procedure on the reduced set $L \setminus \{b_j\}$ until one of the bidders accepts (so the procedure ends with success) or there is no more Borda winners to be contacted so that the procedure ends with a failure.
In the case (3) we have:

- $\forall s_i \in A$ b_1 is the best choice;

- $\forall s_i \in D$ there are preferred choices to b_1.

If, for each $s_i \in D$, we define the set $L_i = \{j \in N \,|\, D_{i,j} < D_{i,1}\}$ we can define the set $L = \cap_{s_i \in D} L_i$ so that we have three cases:

(a) $|L| = 0$,

(b) $|L| = 1$,

(c) $|L| > 1$.

In the case (a) no reallocation is possible since there is no possible compromise among the members of D that are not able to agree on a feasible alternative to b_1.
In the case (b) we have a b_j (with $j \in N$) that is better than b_1 for the members of D. The members of D can proceed as follows:

- each $s_i \in D$ evaluates his individual gain $D_{i,1} - D_{i,j}$;

- they evaluate the collective gain $\Gamma_i = \sum_{s_i \in D}(D_{i,1} - D_{i,j})$;

- they ask to the member of A how much they (as a whole) want to be paid to switch from b_1 to b_j, be it $\rho_{1,j}$.

If the total of $\rho_{1,j}$ and the sum that the D have to pay to b_j (that accounts also of the payments of the other bidders but b_1) to have him to accept ζ is lower than Γ_i the reallocation is feasible and the procedure may end with success otherwise it surely ends with a failure.
We note that:

- the reallocation actually succeeds if b_j accepts so if the proposed total compensation cannot be lower than m_j;

- the sum $\rho_{1,j}$ is defined by the members of the set A through a negotiation and is proportionally shared among the members of A so that each can compensate the major damages deriving from the new allocation.

In the case (c) we have $L \subset N$ such that b_j is a better choice than b_1 for any $j \in L$. In this case the members of D can use the Borda method to select the best choice from the set $L$and use it as in the case (b) . If they succeed the procedure is over otherwise they discard that bidder from the set L, choose another bidder from the reduced L (if there is at least one bidder available) and repeat the procedure. If all the attempts fail the procedure of reallocation ends with a failure.

7.4 Interaction and support

In this section we sketch a possible algorithm that can be used in the case where:

- the bidders are interdependent so that we have, in general, $d_{i,j} \geq 0$ for any $i \neq j \in N$;

- $S \neq \emptyset$ so that we have, in general, $D_{i,j} \neq 0$ for any $s_i \in S$ and $j \in N$.

Mechanism 7.3 *Also in this case (see section 7.2) the mechanism has the following structure:*

- *possible pre auction phase,*

- *auction phase,*

- *allocation and compensation phase,*

- *reallocation phase.*

The reallocation depends on both the values $d_{i,j}$ (where i and j identify the bidders) and the values $D_{i,j}$ (where i identify the supporters and j identify the bidders). In the current version of the proposed algorithm we assume that the sets B and S can act independently one from the other.

Algorithm 7.2 *In this case we can adopt a procedure based on the following steps:*

(1) the Bs define the set J_B of suitable bidders as we have seen in section 7.2;

(2) *the Ss define the set J_S of suitable bidders as we have seen in section 7.3;*

(3) *they evaluate the set $J = J_B \cap J_S$;*

(4) *if $J = \emptyset$ go to 9;*

(5) *if $J \neq \emptyset$ order J;*

(6) *select the best b_j from J, $J = J \setminus \{b_j\}$;*

(7) *b_j is contacted and he is offered a compensation;*

(8) *b_j can:*

 (8a) *accept so he gets ζ and the compensation; go to 9;*

 (8b) *refuse so that if $J \neq \emptyset$ go to 6 else go to 9;*

(9) *end;*

Observation 7.2 *The steps* (1) *and* (2) *are simultaneous moves in the sense of Game Theory ([9, 10, 11]).*
The steps (4) *and* (8*b*) *define the termination conditions with failure.*
At the step (8*b*) *the contacted bidder has refused so that, if $J \neq \emptyset$, the members of B and S have another bidder to contact otherwise the procedure must end with a failure. On the other hand, at step* (4)*, if $J = \emptyset$ the procedure neither effectively starts since the two sets B and S have no common bidder to whom propose the allocation.*
At the step (5) *the bidders of the set J are ordered*[20] *from the best to the worst by applying the Borda method to the following preference profiles:*

- *the one produced by the members of B over the set J that derives from the ordering on the set J_B;*

- *the one produced by the members of S over the set J that derives from the ordering on the set J_S.*

The use of the Borda method avoids the carrying out of direct comparisons between the evaluations of the bidders through the use of scores that account for the position of each bidder in the corresponding ordering.
If the resulting profile contains tied alternatives they can be contacted in any order since they are seen as equivalent from both the members of B and the members of S.

[20]If $|J| = 1$ the proposed ordering operation proves obviously useless since there is only one bidder to be contacted.

Observation 7.3 *At the step* (7) *it is necessary to collect a sum equal to* Σ *so that the members of* B *must collect a sum* c_B *and the members of* S *must collect a sum* c_S *such that:*

- *the offer* Σ *to* b_j *is enough to compensate him for the allocation of* ζ *and so together with what the bidders already committed to pay to* b_1 *is not lower than* x_j *or* $\Sigma \geq x_j - \Sigma_j$*;*

- *the sum* Σ *is proportionally subdivided between the two sets* B *and* S *as, respectively:*

$$c_B = \frac{|B|}{|B| + |S|}\Sigma \tag{38}$$

and:

$$c_S = \frac{|S|}{|B| + |S|}\Sigma \tag{39}$$

- *the sum* c_B *is to be shared among the members of* B *proportionally according to ratios:*

$$\frac{d_{i,1} - d_{i,j}}{\sum_{i \neq j}(d_{i,1} - d_{i,j})} \tag{40}$$

- *the sum* c_S *is to be shared among the members of* S *proportionally according to ratios:*

$$\frac{D_{i,1} - D_{i,j}}{\sum_{i \neq j}(D_{i,1} - D_{i,j})} \tag{41}$$

8 Concluding remarks and future plans

In this paper we presented the structure of a negative auction mechanism under the form of a basic mechanism together with some possible extensions. The extensions include both a pre auction phase and a post auction phase: the first aims at reinforcing the requirement of individual rationality[21] whereas the latter aims at describing possible interactions among the bidders and the supporters.

The proposed extensions are still under development so that the full formal characterization is under way. One of the refinement we are planning to introduce, in the case of the interactions among the bidders without supporters (see section 7.2), is the use of **pull** operations (in addition to the push operations) through which a set of bidders distinct from the current losing bidder

[21] A mechanism satisfies the property of individual rationality ([3], [7], [9]) if the involved players do not have a negative utility from attending to it and so have some incentives from attending the mechanism.

can try to pull the allocation of ζ towards other more preferred bidders by sharing among themselves the cost of this switching between bidders.
A **push** operation can, indeed, be executed only by the currently losing bidder so that, if he is satisfied with the allocation, no reallocation is possible though some other bidders may wish to pay him to have the item to be pulled to another and more preferred bidder.
Other future plans include the relaxations we have listed in section 4 so that we plan to examine what happens if we assume that:

- the bidders are risk adverse so that they prefer either to pay the fee or to pay a fixed amount for not getting ζ for sure than attending the auction with the risk of getting ζ though together with a compensatory sum;

- the evaluations are either common or interdependent among the bidders and in any way may vary either after the pre auction phase (if the associated values are common knowledge) or after the auction phase itself if a post auction phase is present;

- the bidders are asymmetric so we can have different intervals $[0, M_i]$ and different functions F_i and f_i for each bidder b_i.

Last but not least we are planning to see what changes we may have in the auction phase if the sum $e_c$is a common knowledge among the bidders before they attend the auction phase.
As a first approximation we can expect that if the k attending bidders know the value of m (and so the number of the bidders who paid the fee) they may be willing to bid less than m_i since each of them may consider to have a fixed compensation equal to mf, in case of loss, and so he may wish to increase the probability of losing the auction and such an increase may be obtained by simply bidding less than m_i.

References

[1] Ken Binmore. *Playing fair, Game Theory and the Social Contract.* Massachusetts Institute of Technology, 1998.

[2] Ken Binmore. *Just playing, Game Theory and the Social Contract.* Massachusetts Institute of Technology, 1998.

[3] Ken Binmore. *Rational Decisions.* Princeton University Press, 2009.

[4] Lorenzo Cioni. An inverse or negative auction. Technical Report TR-10-17, Computer Science Department, November 2010.

[5] Simon French. *Decision Theory. An introduction to the mathematics of rationality.* John Wiley and Sons, 1986.

[6] Michael N. Huhns and Larry M. Stephens. Distributed methods for decision support. *Multiagent Systems and Societies of Agents*, pages 78–120, 2001. editor Gerhard Weiss.

[7] Vijay Krishna. *Auction Theory.* Academic Press, 2002.

[8] Paul Milgrom. *Putting Auction Theory to Work.* Cambridge University Press, 2004.

[9] Roger B. Myerson. *Game Theory. Analysis of conflict.* Harvard University Press, 1991.

[10] Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory.* MIT Press, 1994.

[11] Fioravate Patrone. *Decisori (razionali) interagenti. Una introduzione alla teoria dei giochi.* Edizioni plus, 2006.

[12] Donald G. Saari. *Decisions and Elections, Explaining the Unexpected.* Cambridge University Press, 2001.

[13] Donald G. Saari. *Disposing Dictators, Demistifying Voting Paradoxes. Social Choice Analysis.* Cambridge University Press, 2009.

[14] Michael Wooldridge. *An Introduction to MultiAgent Systems.* John Wiley and Sons, 2002.