Università di Pisa

Dipartimento di Informatica

Technical Report: TR-10-17

An inverse or negative auction

Lorenzo Cioni

lcioni@di.unipi.it

November 5, 2010

Address: Largo B. Pontecorvo 3, 56127 Pisa, Italy. Tel: +39 050 2212700 Fax: +39 050 2212726

An inverse or negative auction

Lorenzo Cioni
lcioni@di.unipi.it

November 5, 2010

Department of Computer Science, University of Pisa

Abstract

In this Technical Report (TR) we describe a type of auction mechanism where the auctioneer **A** wants to auction an item among a certain number of bidders $b_i \in B$ $(i = 1, \ldots, n)$ that submit bids in the auction with the aim of not getting that item ζ.
Owing to this feature we call this mechanism an **inverse** or **negative auction.**
The main motivation of this mechanism is that both the bidders and the auctioneer give a negative value to the auctioned item (and so they see it as a bad rather than a good).
The mechanism is presented in its basic simple version and with some possible extensions that account for the payment of a fee for not attending the auction, the interactions among the bidders and the presence of other supporting actors.

Contents

1 Introduction

In this TR we describe a type of auction mechanism[1] where the auctioneer **A** wants to auction an item among a certain number of bidders[2] $b_i \in B$ ($i = 1, \ldots, n$) that submit bids in the auction with the aim of not getting that item ζ.
Owing to this feature we call this mechanism an **inverse** or **negative auction.** The main motivation of this mechanism is twofold:

- both the bidders and the auctioneer give a negative value to the auctioned item (and so they see it as a bad rather than a good),
- the auctioneer has an imperfect knowledge of the bidders and so cannot contact any of them directly.

The mechanism[3], at least in its basic version, is simple and will be described in detail in the initial sections of the TR. It is based on the following steps:

- **A** selects the bidders b_i according to some private criteria that depend on the nature of ζ;
- the b_i submit their bids in a sealed bid auction;
- once they have been submitted the bids are revealed so that:
  - the bidder who made the lowest bid is the **losing bidder** and gets[4] ζ;
  - the other bidders are termed **winning bidders** and get the gain of having avoided the allocation of ζ;
- the losing bidder[5] b_1 gets ζ and, as a compensation, a sum equal to his bid x_1;
- each winning bidder b_i pays to the losing bidder a properly defined fraction of x_1.

This simple mechanism will be described in some detail in the following sections together with the possible strategies of the bidders and some possible extensions. Such extensions include a pre auction phase, where some of the bidders pay a fee for not attending the auction, and a post auction phase that can assume three forms and that aims at a reallocation of ζ depending on criteria that are different from those who drove the auction phase itself.

[1] In this TR we are going to use the term mechanism in a rather informal sense as a set of rules, strategies and procedures. For a more formal use of the term we refer, for instance, to [8, 10].

[2] In what follows we identify a bidder $b_j \in B$ also by the index $j \in N = \{1, \ldots, n\}$.

[3] The proposed mechanism is inspired by the Contract Net Protocol ([5, 15]).

[4] Possible ties among two or more losing bidders are resolved through a properly designed random device.

[5] We assume that after the bids have been revealed we renumber the bidders so that the losing bidder is the bidder b_1 whereas all the other bidder b_i (with $i \neq 1$) are the winning bidders.

2 Structure of the TR

This TR is structured as follows.
In the next sections we define a general framework for the proposed mechanism and make some analogies with both classical auctions and other mechanisms.
Then we describe in some detail the parameters that characterize both **A** and the members of B.
Successively we present the structure of the proposed mechanism, in its basic and simpler version, and the strategies for the bidders.
The following sections present some possible extensions to the basic mechanism that can form either the pre auction phase or the post auction phase.
The TR closes with a section devoted to some concluding remarks and to the description of future plans.

3 Pre auction and post auction phases

As a **pre auction** phase we examine the possibility to allow the bidders to pay to **A** a fee f (that **A** fixed and made common knowledge among the bidders) for not attending the auction. In this case, depending on the amount of the fee, we can have that:

- m bidders prefer to pay the fee in order to not attend the auction;
- $k = n - m$ bidders prefer to attend the auction.

In this case, at the end of the auction phase, **A** has collected an extra compensation equal to $e_c = mf$ that is awarded to the losing bidder.
For such value we may have two possibilities (see also section 13):

- it may be a public knowledge among the bidders that therefore know k and m before the auction phase;
- it may be a private knowledge of **A** to be revealed only after the execution of the auction phase.

As to the last point we note how this feature may be guaranteed or at least enforced through the design of the structure of the pre auction phase so to make the communication among the bidders either too difficult or too costly.
The easiest solution is to have the bidders, at least in this phase, to be unaware one of the others so to make any inter bidders communication impossible.
In the present TR we consider only the latter case so that the paid bids have no influence on the behavior of the remaining attending bidders that do not have such information when they submit their bids (see section 13).
We note indeed how even the m bidders who paid the fee can attend the possible post auction phase.
As a **post auction** phase we introduce some mechanisms that try to correct a simplifying assumption we have made in the basic mechanism.
The basic mechanism is, indeed, based on the assumption that the various b_i

are independent one from the others (in the sense that the allocation of ζ to one of the bidders has effect only on that bidder) and, similarly, do not influence any other actor[6].
The mechanisms of the post auction phase aim, indeed, at accounting for the following facts:

(pa_1) the bidders b_i are interdependent and so they may influence each other so for any pair of bidders (b_i, b_j) we can define as $d_{i,j}$ the damage caused to b_i from the allocation of ζ to b_j;

(pa_2) the bidders b_i may influence the actors of the set S so for any actor $s_i \in S$ we can define as $D_{i,j}$ the damage caused to s_i from the allocation of ζ to b_j

We may assume in general that $d_{i,j} \neq d_{j,i}$ so the cross damages between pairs of bidders are not symmetrically distributed.
In the (pa_1) case we assume that the bidders are interdependent but $S = \emptyset$. In this case the bidders can try to negotiate an allocation to another bidder that is more preferred by all the bidders depending on the values $d_{i,j}$ (for $i \neq j$) and not on the values $m_i = d_{i,i}$ that drive the auction phase. In this case we have a compensation for the newly chosen bidder.
On the other hand, in the (pa_2) case, we assume the bidders as independent but $S \neq \emptyset$. In this case the members of S try to obtain a reallocation depending on the values $D_{i,j}$ and through a compensation for the newly chosen bidder.
Last but not least the two cases (pa_1) and (pa_2) can be merged in a single case where we have both interdependent bidders and $S \neq \emptyset$.
In all the post auction cases the starting point is the allocation of ζ to one of the bidders on the basis of the outcome of the auction where we assume the bidders are independent and each is guided only by his self damage $m_i = d_{i,i}$.
At the end of the auction phase we can have two cases:

- the resulting allocation is satisfactory;
- the resulting allocation is unsatisfactory.

In the former case no reallocation is required whereas in the latter case both the bidders of the set B and the supporters that form the set S may try to renegotiate it, within the different framework we have listed, so to identify a new bidder as the more preferred allocation.
We underline how such reallocation may require the raising of a further compensation for the new bidder in order to have him accept the allocation of ζ.

4 The theoretical framework

Auctions represent mechanisms for the allocation of one or more items to one or more bidders ([8, 9]). In the case of more items they can be either of

[6]With the term actor we denote a figure that is distinct from both **A** and the Bs but that wants to attend the auction since he thinks to be damaged from the allocation of ζ to one of the bidders. Such actors are termed **supporters** and form the set S.

homogeneous or of heterogeneous types.
The elements of an auction include the participants (i. e. the auctioneer and the bidders with their both private or common or interdependent values), the rules of participation, the rules through which the winning bidders are identified as well as the rules that define how much the single bidders have to pay.
In general we can have, indeed, that an action is used for the auctioning of a set of k either homogeneous or heterogeneous items among a set of bidders that compete for either at most one item or a subset of the items. For simplicity (and for an analogy with the proposed negative auction) in this section we examine only single item auctions in order to set up the framework for the understanding of the current proposal and of the analogies we present in section 5. The analogies will be used as an aid in the analysis of the proposed mechanism.
An auction is therefore characterized by an **auctioneer** (who auctions an item) and a set of bidders who submit bids x_i and are characterized by the evaluations m_i.
The bids may be ([8, 9]):

- **open cry** if they are publicly visible;
- **sealed** if they are made privately and are revealed all at the same time;
- **one shot** if they are submitted only once;
- **repeated** if they are repeatedly submitted until a termination condition is satisfied;
- **ascending** if they start low and then rise;
- **descending** if they start high and then decrease.

Classical types of auctions include[7]:

- English auctions;
- Dutch auctions;
- First Price Sealed Bid ($FPSB$) auctions;
- Second Price Sealed Bid ($SPSB$) auctions.

In an English auction bids are open cry, repeated and ascending and the winner is the highest bidding bidder who pays the sum he bid that is coincident with the price at which the second last bidder dropped out.
In a Dutch auction bids are open cry and are offered by the auctioneer, are repeated and descending and the winner is the bidder who accepts the current value and that pays such a value.

[7] Other possible types of auctions are ([8]): **all pay** auctions, where all the bidders bid and pay their own bids but only the highest bidding bidder wins the auction, and **third price** auctions that are similar to a $SPSB$ auction but for the fact that the paid price is the third highest bid.

In an $FPSB$ auction bids are sealed and one shot and the winner is the highest bidding bidder who pays the sum he bid.
In an $SPSB$ auction bids are sealed and one shot and the winner is the highest bidding bidder who pays the the second highest bid.
The evaluations m_i are the maximum sums each bidder is willing to pay to get the auctioned item. Such evaluations may be ([8, 9]):

- **private** if they are independent one from the others so that a reciprocal knowledge would not change the individual values;

- **interdependent** if a reciprocal knowledge may change the individual values;

- **common** if the evaluations are ex-post the same among the bidders.

On the basis of such definitions we note that[8]:

- Dutch auctions $\equiv FPSB$ auctions;

- under private values, English auctions $\equiv SPSB$ auctions.

Given such equivalences we note that, [8]:

- in a $SPSB$ auction (and so in an English auction) it is a dominant strategy for a bidder to bid his own evaluation of an item so that we have $x_i = m_i$ for each bidder;

- if we assume a symmetric model (see further on) in a $FPSB$ auction (and so in a Dutch auction) it is a dominant strategy for a bidder to bid a little less than his evaluation and so to bid $x_i = m_i - \delta$ with $\delta > 0$. Under the assumption that the evaluations of the bidders are independent and uniformly distributed over the same interval this δ tends to zero as the number of the bidders increases.

5 The analogies

Classical auctions (see [8, 9] and also section 4) are characterized by the following high level structure:

- **A** auctions one item ζ;

- the bidders of the set B bid, one of them (be it b_1) wins the auction, gets ζ and pays to **A** a certain sum s that depends on the rules of the auction;

- possibly the other bidders have to pay to **A** a certain sum.

[8]With $\equiv$ we denote a **strategic equivalence**. Two games are strategically equivalent if "they have the same normal form except for duplicate strategies. Roughly this means that for every strategy in one game a player has a strategy in the other game with the same outcomes", [8], note at page 4.

For the moment we disregard the last step (that characterizes for instance the **all pay** auctions).
In this case **A** is the seller, the Bs are the possible buyers and the b_1 who gets ζ (the winning bidder) is the effective buyer. In this classical mechanism we have a two way transfer:

- of the item ζ as a good[9] from **A** to the winning bidder b_1,
- of a sum s from b_1 to **A**.

In an **all pay** auction we can extend the analogy by saying that the winning bidder pays for getting ζ and all the others pay for having had the possibility to attend the auction and so all of them are buyers of something (either the item or that possibility).
In a **procurement auction** the auctioneer **A** pays the less requesting bidder b_1 a sum for acquiring from him either a good or a service[10]
Also in this case we have a two way transfer:

- of the item ζ as a good or a service from the winning bidder b_1 to **A**,
- of a sum s from **A** to b_1.

In both cases we have the transfer of an item with a positive or better a non negative value (for all the involved players[11]) from **A** to b_1 and of a corresponding positively valued item from b_1 to **A**. The difference is that in the former case **A** tries to maximize his gain whereas in the latter he tries to minimize his payment.
In our basic mechanism we have:

- the transfer of an bad ζ from **A** to b_1;
- the transfer of a total compensation equal to x_1 from the bidders b_i (for $i \neq 1$) to b_1.

In this case ζ has a negative value for **A** so, by giving it away to b_1, **A** is better off. It is, therefore, as if **A** received a sum of money from b_1 (in exchange of a fictitious good that represents the allocation of ζ) that, in his turn, receives a sum of money, subdivided in various percentages, from the other bidders b_i (for $i \neq 1$).
In this way it is as if we had, in sequence[12], the following two stages:

- a reverse $FPSB$ auction where the less offering bidder b_1 wins and gets ζ;

[9]With this we mean the fact that both **A** and the Bs assign to ζ a positive value or a worth that can be null.
[10]A service may be defined as the non-material equivalent of a good characterized by the fact of being intangible, insubstantial and of being represented as a set of singular and perishable benefits.
[11]We use the term player to denote both the auctioneer and the bidders.
[12]For a similar composite approach we refer to [6] and to [7].

- an **all pay** auction where all the other bidders pay b_1 for having him to accept the bad ζ.

The analogy is, however, imperfect since the sums paid in the second stage are effectively defined in the first stage so that the leading analogy we can use in the analysis is with a $FPSB$ auction. In such type of auctions we know (see section 4) that, under some rather general assumptions, the best strategy for each bidder is to bid a little bit less than his own evaluation of the item and that such reduction tends to 0 as the number of the bidders increases. In our case we expect that each bidder bids a little bit more than his own evaluation of the item and that, under similar assumptions, such increase tends to 0 as the number of the bidders increases.
From the foregoing description of the two fictitious stages we have that in the first stage **A** is better off and the stage is efficient (see section 6) since ζ is allocated to the bidder who values it the less.
In the second stage the losing bidder b_1 is compensated and the winning bidders b_i are better off since each of them pays to b_1 a sum that is lower than b_i's evaluation of ζ. In this way every b_i has an utility that can be evaluated as the difference between the b_i's evaluation of the bad and the fraction of the compensation to b_1. Such utility can be expressed as $m_i - c_i$ where c_i depends on x_i, on x_1 and on the bids of all the other winning bidders. We note that the utility of b_1 can be similarly expressed as $x_1 - m_1$.
All these statements will be made clear in section 9.

6 The performance measuring criteria

In the literature ([15, 8, 9]) we can find a certain number of criteria that have been devised for the evaluation of the quality of the outcomes of a mechanism and that guide its design. Such criteria can be used also for the evaluation of the various types of auctions we have briefly examined in section 4 and are:

(c_1) guaranteed success,

(c_2) maximization of social welfare,

(c_3) [Pareto] efficiency,

(c_4) individual rationality,

(c_5) stability,

(c_6) simplicity.

We are going to use such criteria for the evaluation of the negative auction, without pre and post auction phases, that we propose in this TR. In this section we briefly recall the definition of each of such criteria.
We say that a mechanism (or a protocol) and so an auction[13] satisfies (c_1) if we

[13] We recall that an auction is a particular type of mechanism even if we use the proper and formal meaning of the term, see [8], so in the following criteria we refer directly to auctions.

are sure that the auction cannot be void so that the auctioned item is surely allocated to one of the bidders.
We say that an auction satisfies (c_2) if the outcome maximizes the total utility (as the sum of the utilities) of the participants and so, in our case, of both the auctioneer and the bidders. If we want to avoid any summation of utilities so to avoid both any form of compensations and any form of inter utilities comparison we can define a vector U of $n+1$ elements where $U(0)$ is the utility of **A** and each $U(i)$ is the utility of a b_i. We can then define such a vector before the auction (as U') and after the auction (as U''). In this way we say that U'' maximizes the social welfare if the following conditions hold:

- $U''(i) \geq U'(i)\ \forall i \in [0, n]$ with at least one strict inequality,

- none of the elements of U'' can attain a strictly higher value.

We say that an auction satisfies (c_3) if, given an allocation, there is not any other allocation where one bidder or the auctioneer is better off without none of the others being worse off.
We note that (c_2) implies (c_3) but the converse is not necessarily true.
We say that an auction satisfies (c_4) if it is in the best interest for the bidders to attend the auction or if by attending the auction they cannot derive a loss or a negative utility.
We say that an auction satisfies (c_5) if the bidders have a bidding strategy that defines an equilibrium so that none of them has any interest of performing an individual deviation. In this way we define a Nash Equilibrium (NE) of the auction ([11, 10, 1, 2]).
We say that an auction satisfies (c_6) if the foregoing strategy is easily understandable and implementable by even bidders with a bounded rationality ([3]).
We are going to use such criteria for the evaluation of the basic mechanism to see whether they are satisfied or not. In this way we can state if such proposal can be judged as rational or not ([3]).

7 The defining parameters

Both the auctioneer **A** and the bidders of the set B are characterized by some parameters that depend heavily on the nature of the item ζ but also on their individual characteristics.
For what concerns **A** we have only one parameter: the value m_A that **A** assigns to ζ as a measure of his utility since the only gain **A** receives from the auction is the allocation of ζ.
With m_A we denote:

- the damage or the negative utility that **A** will receive from ζ if the auction is void so the allocation fails;

- the benefit or the positive utility that **A** receives from the allocation of ζ to one of the $b_i \in B$.

In the former case m_A has a negative value whereas in the latter it has a positive value.
Every $b_i \in B$ is characterized by the following parameters (see also [8, 9]):

- a value m_i that he assigns to ζ;
- the amount x_i he is willing to bid;
- the random variables X_j that describe the bids of the other bidders;
- the interval of the values $[0, M_i]$ to which m_i belongs;
- the intervals of the values $[0, M_j]$ to which the X_j belong;
- the differentiable cumulative distributions F_j of the values X_j;
- the corresponding density functions $f_j = F'_j$ of such values.

We note that:

- the parameter m_i has a dual meaning in the sense that:
  - it represents the damage the b_i receives from the allocation of ζ;
  - it represents the benefit that b_i gets from the fact that ζ is allocated to some other bidder;
- the parameter x_i has a dual meaning in the sense that:
  - it represents the sum that b_i asks as a compensation for the allocation of ζ;
  - it defines the fraction c_i of the compensation that b_i has to pay to the losing bidder.

We can also define the following probabilities:

- the probability p_i for b_i of losing the auction;
- the dual probability $q_i = 1 - p_i$ for b_i of winning the auction.

We recall that the **losing bidder** is the bidder who gets ζ and is compensated for this fact by the other bidders, the so called **winning bidders**.

8 The basic assumptions

In this section we introduce the basic assumptions that we make on the parameters that characterize both the auctioneer and the bidders and that will be maintained through the rest of the TR. At the end of this section we comment a little on the possible relaxations of these assumptions.
The only assumption we can make on **A** is that his value m_A is a private information of the auctioneer so that it is not known to the bidders.

If we relax this assumption so that m_A becomes a common knowledge of the bidders nothing changes since that knowledge has no effect on the bidding strategy of the bidders.
On the other hand, some other basic assumptions involve the characteristic parameters of the bidders and may be summarized as follows[14]:

- the bidders are assumed to be **risk neutral** so that their utility is linearly separable ([8]) and can be expressed as the difference between a benefit and a damage and so as $x_i - m_i$ if the bidder loses the auction or as $m_i - c_i$ if he wins it;

- the random variables X_j of the bidders distinct form b_i are assumed to belong to a common interval $[0, M]$ for a suitable $M > 0$;

- the random variables X_j of such bidders are assumed to be independent random variables;

- the valuations are assumed to be **private values** of the single bidders;

- the bidders b_j are assumed to be **symmetric** so they are characterized by the same F and by the same f;

- the random variables X_j are assumed to be uniformly distributed on the interval $[0, M]$ so that we have, for $x \in [0.M]$:

$$P(X_j \leq x) = F(x) = \frac{x}{M} \tag{1}$$

  and, correspondingly:

$$f(x) = \frac{1}{M} \tag{2}$$

From the foregoing assumptions we derive that the probability for each bidder of losing the auction p_i is the same for all the bidders so we can denote it as p and use $q = 1 - p$ to denote the dual probability of winning the auction.
Possible relaxations of the foregoing assumptions involve:

- the possibility that the bidders are risk adverse[15] so that his utility is no more linearly separable but it is a convex function of x_i;

- the possibility that the evaluations are either common or interdependent among the bidders;

- the possibility that the bidders are asymmetric so that we can have:

[14] See also sections 4 and 7 and [8, 9]

[15] We recall that, in classical terms, a player is **risk neutral** ([4]) if he is indifferent between attending a lottery and receiving a sum equal to its expected monetary value whereas he is **risk averse** if he prefers the expected value to attending the lottery. We can also say that a player is risk neutral if his utility function is linearly separable in gain and loss whereas, if he is risk averse, it can be seen as a concave function. In our context we have to consider the opposite perspective and so we consider the utility function of risk averse bidders as a convex function of its parameters.

- · different intervals $[0, M_j]$ for each bidder b_j,
- · different functions F_j and f_j for each bidder b_j;

- the possibility to have different distributions such as a Gaussian or a triangular distribution also under the symmetry assumption.

Such relaxations can be introduced either one at a time or in combinations. Their treatment, that makes the analysis more complex, is out of the scope of the present TR and is the subject of further research efforts (see section 13 for further details).

9 The basic mechanism and its strategies

The **basic mechanism** is composed only by the auction phase among independent bidders. We can describe it as follows[16]:

(ph_1) **A** auctions ζ;

(ph_2) the b_i make their bids x_i in a sealed bid one shot auction;

(ph_3) the bids are revealed;

(ph_4) the lowest bidding bidder b_1 gets ζ and x_1 as a compensation for this allocation;

(ph_5) each of the other bidders b_i pays a fraction c_i such that:

$$\sum_{i \neq 1} c_i = x_1 \tag{3}$$

For what concerns the values c_i we assume a **proportional repartition** among the bidders so we have:

$$c_i = x_1 \frac{x_i}{X} \tag{4}$$

where $X = \sum_{j \neq 1} x_j$. In this way we account for the fact that the bidders who receive a bigger advantage from the allocation of ζ to b_1 pay the higher fractions of the compensation.
At this point we state and prove the following proposition.

Proposition 9.1 (Weakly dominant strategy) *From the assumptions made so far it is a weakly dominant strategy for each bidder to submit a bid equal to his evaluation of the auctioned item.*
Proof and some remarks
From what we have stated in sections 7 and 8 we derive easily that the expected

[16] In this section we assume that, when the phase (ph_3) is over we can renumber the bidders so that b_1 is the losing bidder whereas the b_i (with $i \neq 1$) are the winning bidders. Possible ties are resolved with the random selection of one of the tied bidders.

utility from the auction for every bidder when he faces phase (ph_2) *can be expressed as:*

$$E(b_i) = p(x_i - m_i) + (1-p)(m_i - x_1 \frac{x_i}{X}) \tag{5}$$

as the sum of the utility if he loses the auction multiplied with the probability of losing and the utility if he wins it multiplied with the probability of winning.
It is obvious that at phase (ph_3) *each* b_i *knows if he is the loser or one of the winners.*
In the former case he has a utility:

$$x_1 - m_1 \tag{6}$$

whereas in the latter he has a utility:

$$m_i - x_1 \frac{x_i}{X} \tag{7}$$

Relation (5) can be rewritten as:

$$E(b_i) = (1 - \frac{x_i}{M})^{n-1}(x_i - m_i) + (1 - (1 - \frac{x_i}{M})^{n-1})(m_i - x_1 \frac{x_i}{X}) \tag{8}$$

by using the following relations:

$$p = (1 - \frac{x_i}{M})^{n-1} \tag{9}$$

$$q = 1 - p = 1 - (1 - \frac{x_i}{M})^{n-1} \tag{10}$$

that have been derived by using the hypotheses of independence and identical and uniform distribution of the X_j *and by imposing that the* x_i *is lower than the* X_j *for* $j \neq i$*.*
Since in relations (5) and (8) we want to impose that in any case each bidder b_i *has a non negative utility we get the following constraints:*

- $y_1 = x_i - m_i \geq 0$
- $y_2 = m_i - x_1 \frac{x_i}{X} = m_i - x_1 \frac{x_i}{x_i + X'} \geq 0$

where[17] y_1 *is the utility for* b_i *if he loses and* y_2 *is his utility if he wins.*
From the former constraint we derive:

$$x_i \geq m_i \tag{11}$$

For what concerns the latter constraint, from the definition of y_2 *and by performing the derivations with respect to* x_i*, we easily derive that:*

- $y_2' < 0$

[17]We note how we can write $X = x_i + X'$ where X' accounts for the bids of the bidders distinct from b_1 and b_i.

- $y_2'' > 0$

*so y_2 is **concave decreasing** with:*

- *a maximum value equal to m_i for $x_i = 0$,*
- *a minimum value for $x_i = M$ equal to :*

$$m_i - x_1 \frac{M}{M + X'} \tag{12}$$

From the foregoing observations we derive that:

- *if we impose $y_1 = y_2$ we derive a value $\hat{x}$;*
- *for $x_i < \hat{x}$ we have $y_1 < y_2$ so by winning b_i is better off than by losing;*
- *for $x_i > \hat{x}$ we have $y_1 > y_2$ so by losing b_i is better off than by wining.*

On the other hand, from relations (9) and (10) we can easily see how:

- *p has a maximum value of 1 for $x_i = 0$, decreases for x_i increasing and attains a null value for $x_i = M$;*
- *q has dual behavior since it has a minimum value of 0 for $x_i = 0$, increases for x_i increasing and attains the maximum value of 1 for $x_i = M$;*

We note that the rates of both decrease and increase are higher the higher is the number n of the bidders.
At this point we want to find the value $\bar{x}_i$ where we have

$$p = q \tag{13}$$

so that for $x_i < \bar{x}_i$ we have that p dominates q whereas we have the opposite for $x_i > \bar{x}_i$. From relation (13) and relations (9) and (10) we get:

$$(1 - \frac{x_i}{M})^{n-1} = 1 - (1 - \frac{x_i}{M})^{n-1} \tag{14}$$

From relation (14), with some easy algebra, we derive:

$$\bar{x}_i = (1 - (\frac{1}{2})^{\frac{1}{n-1}})M \tag{15}$$

We note that $\bar{x}_i \to 0$ as $n \to \infty$ so that q tends to dominate p for any x_i. According to all this we have that b_i should maximize y_2 so to bid no less than m_i (given the constraint we have imposed on y_1) and so he should bid a sum equal to m_i.

Remark 9.1 *We have in this way verified how the truthful bidding is a weakly dominant strategy for each bidder in the basic mechanism of the negative auction.*

In practical terms and owing the approximations and simplifications we have made (and that are really verified only for high values of n) b_i should tend to bid a little more than his evaluation of ζ (with this quantity tending to 0 as the number of the bidders increases) so confirming what we have derived from the analogy with a $FPSB$ auction (see section 5). This argumentation is enforced also by the considerations we have made about the behaviors of both y_1 and y_2 as well by those we made about the behaviors of both p and q.

10 The basic mechanism and the performance measuring criteria

We now want to verify if the proposed mechanism satisfies the criteria we have introduced in section 6. In this case we can assess what follows.

(c_1) Guaranteed success is satisfied since the auction cannot be void so ζ is allocated to one of the b_i.

(c_2) Maximization of social welfare, according to our vector based definition, is not satisfied since b_1 would be better off from not being the losing bidder. If we exclude b_1 (that is anyway compensated according to his claim) the criterion is satisfied since **A** is better off and the other bidders b_i cannot attain a higher utility since b_1 is the less offering bidder.

(c_3) [Pareto] efficiency is satisfied since ζ is allocated to the less evaluating/offering bidder (who is compensated) and all the winning bidders have a non negative maximum utility.

(c_4) Individual rationality is satisfied since any bidder has a non negative utility both if he wins and if he loses.

(c_5) Stability is satisfied since all the bidders have an equilibrium strategy that they can follow and such a strategy is simple both to understand and to implement so that also (c_6) (or simplicity) is satisfied.

As to (c_4) we remark how U' defines the status quo ante where the auctioneer, if we consider only the allocation of ζ, has a negative utility whereas the bidders have a null utility so that U'' represents an improvement for both the auctioneer and the bidders.

11 The use of the fee

In this section we present the pre auction phase where:

- m bidders pay the fee f in order to not attend the auction;

- $k = n - m$ bidders prefer to attend the auction.

We make the hypothesis that the sum mf is a private information of **A** so it is unknown to the other k bidders that neither know n. For the attending bidders (those who do not pay the fee) we can repeat what we have said in sections 9 and 10.
In this case the losing bidder, at the end of the auction phase, gets the following final compensation f_c:

$$f_c = x_1 + mf \tag{16}$$

If the mechanism has a post auction phase all the n bidders can attend to it, as we will show in the following sections.
At this point we define the following profiles:

(ne_1) all the n bidders pay the fee f,

(ne_2) none of the n bidders pays the fee f.

We want to see if such profiles are NE or not.
In the case (ne_1) we have that if the bidders collude among themselves and decide that they all pay the fee f they collect $e_c = nf$. In this case, every bidder would have a utility equal to[18] $m_i - f$. If a bidder b_j individually violates the collusive agreement he gets a utility equal to:

$$(n-1)f - m_j \tag{17}$$

since no further compensation from the auction phase is possible. The individual deviation is profitable (so that (ne_1) is not a NE) if we have:

$$(n-1)f - m_j > m_j - f \tag{18}$$

or if:

$$m_j < f\frac{n}{2} \tag{19}$$

So if the fee f is such that the constraint (19) is satisfied for at least one b_j the collusive agreement is not a NE and the auction cannot be void since **A** is able to find a bidder to which to allocate ζ with a compensation paid by the other bidders.
We note that if **A** fixes f such that we have:

$$f > \frac{2M}{n} \tag{20}$$

we have:

$$\frac{n}{2}f > M \geq m_i \, \forall b_i \tag{21}$$

and so relation (19) is surely verified.
In the case (ne_2) the individual deviation depends on the possible policies of the single bidders since we have that $e_c = 0$ so from this condition we cannot derive any incentive for the bidders to deviate.

[18]This requires $f < m_i$ for every b_i. We comment on this assumption shortly.

In order to understand under which conditions the case (ne_2) can occur we therefore examine a more general case and so under which conditions a bidder is better off if he pays the fee than if he attends the auction.
A bidder b_i has indeed the following possibilities[19]:

(1) he pays the fee f and has an utility[20] $u_i^p = m_i - f$;

(2) he does not pay and attend the auction and so:

  (2a) he has an utility $u_i^l = x_i - m_i$ if he loses the auction,

  (2b) he has an utility $u_i^w = m_i - x_1 \frac{x_i}{x_i + X'}$ if he wins the auction.

From the case (1) we derive the first constraint since we have that if $u_i^p < 0$ then b_i does not pay the fee and attends the auction. This requires that:

$$u_i^p = m_i - f \geq 0 \tag{22}$$

or:

$$f \leq m_i \tag{23}$$

If condition (23) is violated for every b_i so that we have:

$$f > m_i \tag{24}$$

for every b_i we have that no bidder pays the fee. In this way we have that if $f > max\{m_i\}$ or if f is very high no bidder pays the fee and so they all attend the auction phase.
Once we have established that relation (22) is satisfied we want to make a comparison with the cases $(2a)$ and $(2b)$ so that we can make the following comparisons:

$$m_i - f \geq x_i - m_i \tag{25}$$

and:

$$m_i - f \geq m_i - x_1 \frac{x_i}{x_i + X'} \tag{26}$$

If such relations are satisfied then b_i is better off by paying the fee and so by not attending the auction.
From relation (25) we derive:

$$f \leq 2m_i - x_i \leq m_i \tag{27}$$

(since we have assumed $x_i \geq m_i$) and so not really a new constraint since it is the same as relation (23).
On the other hand from relation (26) we get:

$$f \leq x_1 \frac{x_i}{x_i + X'} \leq x_1 \frac{x_i}{(n-1)x_1} \leq \frac{M}{n-1} \tag{28}$$

[19] We use the decorations p, l and w as exponents to denote, in the order, a payment, a loss and a win.

[20] In this case we evaluate the utility as the difference between the benefit, as represented by the missed allocation of ζ, and the payment as represented by the fee f.

since, by the definition of x_1, we get $X = x_i + X' \geq (n-1)x_1$ and, by definition, $x_1 \leq x_i \leq M$ for every b_i. From relation (28) we derive that if f is small enough then the bidders have incentive to pay it otherwise they have incentives to attend the auction. From this we may derive that if **A** fixes f high enough (for instance $f = M/2$) he can be sure to have a non void auction even if some bidders may prefer to pay the fee f.

12 The post auction phase

12.1 Introductory remarks

In the simplest mechanism when the auction phase is over the allocation is performed by the bidders on the basis of the values $m_i = d_{i,i}$ only. This way of proceeding is based on the assumption that the bidders are independent so that the allocation damages only the individual bidders and neither other bidders nor other actors that form the set S of the supporters.
In section 12.2 we see how we can account for the interdependence of the bidders and so for the damages among the bidders. We therefore present an algorithm based on a succession of **push** operations by which a bidder can push ζ towards another more preferred bidder (according to the values attributed to the cross damages $d_{i,j}$). In this case we have no supporters so that $S = \emptyset$.
In section 12.3 we assume that the bidders are independent but $S \neq \emptyset$ and we examine if the supporters can push ζ towards another more preferred bidder (according to the values attributed to the cross damages $D_{i,j}$ of the $s_i \in S$).
Last but not least in section 12.4 we present an attempt to merge the two approaches and so we assume to have both interdependent bidders and $S \neq \emptyset$.

12.2 The interaction among the bidders

In addition to the parameters we have seen in section 7 and the assumptions we have made in section 8 we introduce the following parameters, for every bidder b_i:

- $d_{i,j} \geq 0$ is the damage that b_i receives if ζ is allocated to b_j;
- $c_{i,j} \geq 0$ is the contribution that b_i is willing to pay to b_j to have him accept the allocation of ζ.

It is obvious that $m_i = d_{i,i}$ and $c_{i,i} = 0$.
Before going on we recall that the auction phase ends with the allocation of ζ to b_1 who receives a compensation equal to x_1. For every bidder $b_i \neq b_1$ we can write the due payment as:

$$\sigma_{i,1} = x_1 \frac{x_i}{X} \tag{29}$$

(with $X = \sum_{j \neq 1} x_j$) so that we have:

$$\Sigma_1 = \sum_{i \neq 1} \sigma_{i,1} = x_1 \tag{30}$$

We can also define:

$$\Sigma_j = \Sigma_1 - \sigma_{j,1} \tag{31}$$

to be used shortly.
In this case the mechanism has the following structure:

- possible pre auction phase,
- auction phase,
- allocation and compensation phase,
- reallocation phase.

In the allocation phase b_1 gets, from the members of $N_{-1} = N \setminus \{1\}$, the commitments of payment $\sigma_{i,1}$ that form the compensatory sum Σ_1 whereas the **reallocation** phase depends on the values $d_{i,j}$.
When the allocation phase is over, b_1 orders the $d_{1,j}$ $\forall j \neq 1$ with regard to $d_{1,1} = m_1$. We can have two cases:

- $d_{1,1} < d_{1,j}$ $\forall j \neq 1$ so b_1 is satisfied and no reallocation is required;
- $\exists J_1 \subset N_{-1}$ such that $\forall j \in J_1$ $d_{1,j} < d_{1,1}$.

In the former case the mechanism **ends** and b_1 receives the commitments at payment as effective compensations from the other bidders.
In the latter case b_1 may negotiate a reallocation with the members of J_1 that he orders in increasing order of received damage. We note that for any b_j with $j \in J_1$ we define as $\bar{c}_{1,j} = d_{1,1} - d_{1,j}$ the maximum contribution that b_1 is willing to pay, in a way to be specified, to b_j to have him accept ζ, whereas with $c_{1,j} < \bar{c}_{1,j}$ we denote the current value of the contribution.
The attempt of reallocation may proceed along the following steps:

(1) b_1 defines J_1;

(2) we have two cases:

  (2a) $J_1 = \emptyset$ so go to (5);

  (2b) $J_1 \neq \emptyset$ so go to (3);

(3) b_1 contacts (in the order) a b_j with $j \in J_1$ and offers him a further compensation $c_{1,j} < \bar{c}_{1,j}$ so that b_j would get $\Sigma = \Sigma_j + c_{1,j}$;

(4) at this point we have two cases:

  (4a) b_j accepts and so becomes the new b_1 with $\Sigma_1 = \Sigma_j$; go to (1);

  (4b) b_j refuses so we have two cases:

    (4b1) there is one more b_j that can be contacted so b_1 chooses him; go to (3);

(4b2) there is no b_j to contact so the procedure ends with a failure; go to (5);

(5) end;

The operation at step (3) is a **push** operation through which the current b_1 tries to allocate ζ to some other bidder b_j by having a gain. Such procedure may either succeed or fail. For it to succeed the current b_j must accept the proposal of b_1. It is easy to see that b_j accepts if the following conditions are verified:

(ac_1) $\Sigma \geq m_j$

(ac_2) $d_{j,1} \geq d_{j,j}$

If condition (ac_1) is violated b_j surely refuses the push proposal whereas if the condition (ac_2) is violated b_j can accept, with a risky decision, ζ if he is sure he can push it to some other bidder b_h such that $d_{j,h} < d_{j,1} < d_{j,j}$.
The procedure has the following termination conditions:

- when no bider accepts a push proposal from the current b_1;
- when for a bidder b_1 we have $J_1 = \emptyset$ so the current item holder is satisfied of the allocation;
- when there would be a cycle.

The last case deserves some more comments. If we have, avoiding to rename the successive losing bidders:

$$b_1 \to b_j \to b_h \to \cdots \to b_k \to b_1 \tag{32}$$

we have a cycle that could even give rise to a money pump for the initial b_1. To prevent this from occurring we impose a cut on the cycle so that the final accepting bidder must be b_k. This fact requires the recording of the various passages so to detect any cycle and to apply the halt condition.

12.3 The presence of the supporters

In this case we make the following assumptions:

- the bidders are independent so we have $d_{i,j} = 0 \ \forall i \neq j$;
- we have s supporters $s_i \in S$ so that for every supporter s_i we have the damages $D_{i,j}$ that he receives from the allocation of ζ to each bidder b_j.

Also in this case the mechanism has the following structure:

- possible pre auction phase,
- auction phase,
- allocation and compensation phase,

- reallocation phase.

The reallocation is driven, in this case, by the members of S.
We can consider S as partitioned[21]:

$$S = A \cup D \tag{33}$$

where:

- A is the set of the s_i that agree with the allocation of ζ to b_1 so that $s_i \in A$ if and only if $D_{i,1} < D_{i,j}$ for every $b_j \neq b_1$;
- D is the set of the s_i that disagree with the allocation of ζ to b_1 so that $s_i \in D$ if and only if[22] exists at least a $j_i \neq 1$ such that $D_{i,j_i} < D_{i,1}$.

We can have the following cases:

(1) $A = S$ and $D = \emptyset$ so no reallocation is required;

(2) $A = \emptyset$ and $D = S$ so every s_i has at least a preferred allocation;

(3) $A \neq \emptyset$ and $D \neq \emptyset$.

In the case (1) the procedure is over.
In the case (2) for every $s_i \in D$ we can partition N as $N = L_i \cup \{b_1\} \cup U_i$ where:

- L_i identifies the bidders that cause to s_i a lower damage than b_1 or the more preferred bidders;
- U_i identifies the bidders that cause to s_i a greater damage than b_1 or the less preferred bidders.

We can have two cases:

- $\cap_{s_i} L_i = \emptyset$,
- $\cap_{s_i} L_i \neq \emptyset$

In the former case no compromise is possible among the members of D so the allocation at b_1 of ζ is unchanged.
In the latter case we can have two sub cases.
In the former sub case we have $\cap_{s_i} L_i = b_j$ so the members of D offer to b_j both Σ_j(see section 12.2) and $\gamma_j = x_j - \Sigma_j$ to be shared proportionally among the members of D as:

$$\gamma_j \frac{D_{i,1} - D_{i,j}}{\sum_{s_i}(D_{i,1} - D_{i,j})} \tag{34}$$

If b_j accepts we have a new allocation otherwise the procedure ends with a failure and the allocation is unchanged. For the conditions of acceptance for b_j we refer to section 12.2. In this case b_j accepts if the offered compensation is

[21]In a classic way we have $S = A \cup D$ and $A \cap D = \emptyset$.
[22]We note that every $s_i \in D$ may have his own j_i.

enough to cover the damage m_j from the allocation of ζ since the bidders are assumed to be independent.
In the latter sub case we have $\cap_{s_i} L_i \subset N$ so we identify a set of k elements. In this case the members of D can use the Borda method[23] ([13, 14]) on such elements so to define the Borda winner and apply to it what we have seen for the previous sub case. In the case of a tie on the Borda winners one of such winners can be selected at random since they can be seen as equivalent alternatives.
If the Borda winner accepts the procedure is over otherwise the members of D discard him and repeat the procedure until one of the bidders accepts (so the procedure ends with success) or there is no more Borda winners to be contacted so that the procedure ends with a failure.
In the case (3) we have:

- $\forall s_i \in A$ b_1 is the best choice;

- $\forall s_i \in D$ there are preferred choices to b_1.

If, for each $s_i \in D$, we define the set $L_i = \{j \in N \mid D_{i,j} < D_{i,1}\}$ we can define the set $L = \cap_{s_i \in D} L_i$ so that we have three cases:

(a) $|L| = 0$,

(b) $|L| = 1$,

(c) $|L| > 1$.

In the case (a) no reallocation is possible since there is no possible compromise among the members of D that are not able to agree on a feasible alternative to b_1.
In the case (b) we have a b_j (with $j \in N$) that is better than b_1. The members of D can proceed as follows:

- each $s_i \in D$ evaluates the individual gain $D_{i,1} - D_{i,j}$;

- they evaluate the collective gain $\Gamma_i = \sum_{s_i \in D}(D_{i,1} - D_{i,j})$;

- they ask to the member of A how much they want to be paid to switch from b_1 to b_j, be it $\rho_{1,j}$.

If the total of $\rho_{1,j}$ and the sum that the D have to pay to b_j (that accounts also of the payments of the other bidders but b_1) to have him to accept ζ is lower than Γ_i the reallocation is feasible and the procedure may end with success otherwise it surely ends with a failure.
We note that:

[23] Given n alternatives the method is based on the fact that each voter assigns $n-1$ points to the top ranked alternative, $n-2$ to the second top ranked alternative up to 0 point to the lowest ranked alternative. The points are added together and the alternatives ordered in a weakly descending order (ties are therefore possible) so that the alternative that receives the highest number of points, in absence of ties, is the Borda winner. If we have ties on the top ranked alternatives we can choose one of them at random as the Borda winner.

- the reallocation actually succeeds if b_j accepts so if the proposed compensation cannot be lower than m_j;

- the sum $\rho_{1,j}$ is defined by the members of the set A through a negotiation and is shared among the members of A so that each can compensate the major damages deriving from the new allocation.

In the case (c) we have that $\exists L \subset N$ such that b_j is a better choice than b_1 for $j \in L$. In this case the members of D can use the Borda method to select the best choice and use it as in the case (b) . If they succeed the procedure is over otherwise they discard that bidder, choose another bidder from L (if there is at least one bidder available) and repeat the procedure. If all the attempts fail the procedure of reallocation ends with a failure.

12.4 Interaction and support

In this section we sketch a possible algorithm that can be used in the case where:

- the bidders are interdependent so that we have, in general, $d_{i,j} \geq 0$ for any $i \neq j \in N$;

- $S \neq \emptyset$ so that we have, in general, $D_{i,j} \neq 0$ for any $s_i \in S$ and $j \in N$.

Also in this case (see section 12.2) the mechanism has the following structure:

- possible pre auction phase,

- auction phase,

- allocation and compensation phase,

- reallocation phase.

The reallocation depends on both the values $d_{i,j}$ (where i and j identify the bidders) and the values $D_{i,j}$ (where i identify the supporters and j identify the bidders).
In the current version of the proposed algorithm we assume that the sets B and S can act independently from each other.
In this case they can adopt a procedure based on the following steps:

(0) if there is any suitable bidder then go to (1) else go to (6);

(1) the Bs can define a new b_j as we have seen in section 12.2;

(2) the Ss can define a new b_h as we have seen in section 12.3;

(3) we can have two cases:

   (3a) $b_j = b_h$ so there is an agreement on the bidder to be contacted; we call it b_j, go to (5);

(3b) $b_j \neq b_h$ so there is a selection between b_j and b_h; let us suppose that the selection is b_h go to (4);

(4) b_h is contacted and he is offered a compensation; b_h can either accept and go to (6) or can refuse and go to (5);

(5) b_j is contacted and he is offered a compensation; b_j can either accept and go to (6) or can refuse and so go to (0);

(6) end;

The steps (1) and (2) are simultaneous moves in the sense of Game Theory ([11, 10, 12]). The step (0) defines a termination condition with failure when none of the contacted bidders has accepted and there is no more a suitable bidder to be contacted.
In the cases (4) and (5) it is necessary to collect a sum equal to Σ (to be defined shortly) so that the members of B must collect a sum c_B and the members of S must collect a sum c_S such that:

- the offer Σ to b_j is such to compensate b_j for the allocation of ζ and so together with what the bidders already committed to pay to b_1 is not lower than x_j or $\Sigma \geq x_j \Sigma_j$;
- the sum Σ is subdivided between the two sets B and S as, respectively:

$$c_B = \frac{|B|}{|B| + |S|} \Sigma \tag{35}$$

  and:

$$c_S = \frac{|S|}{|B| + |S|} \Sigma \tag{36}$$

- the sum c_B is to be shared among the members of B proportionally according to ratios:

$$\frac{d_{i,1} - d_{i,j}}{\sum_{i \neq j} (d_{i,1} - d_{i,j})} \tag{37}$$

- the sum c_S is to be shared among the members of S proportionally according to ratios:

$$\frac{D_{i,1} - D_{i,j}}{\sum_{i \neq j} (D_{i,1} - D_{i,j})} \tag{38}$$

We note that the preliminary selection between b_j (proposed by the Bs) and b_h (proposed by the Ss) is not neutral since it may involve different payments from both the members of B and the members of S
Such a selection may be performed through a sealed bid one shot auction where the Bs and the Ss submit respectively two bids γ_B (as the sum that the B ask for accepting b_h) and γ_S (as the sum that the S ask for accepting b_j).
We have the following cases:

- $\gamma_S = \gamma_B$ so there is a random selection between the two bidders b_h and $b_j 2$;

- $\gamma_S < \gamma_B$ so the starting bidder is b_j;

- $\gamma_B > \gamma_S$ so the starting bidder is b_h.

In this way the less offering party loses since the selected bidder is the one proposed by the other party.

13 Concluding remarks and future plans

In this TR we presented the structure of a negative auction mechanism under the form of a basic mechanism together with some possible extensions.
The extensions include both a pre auction phase and a post auction phase: the first aims at reinforcing the requirement of individual rationality whereas the latter aims at introducing possible interactions among the bidders and some other actors (the supporters).
The proposed extensions are still under development so that the full formal characterization is under way. One of the refinement we are planning to introduce, in the post auction phase, in the case of the interactions among the bidders without supporters (see section 12.2) is the use of **pull** operations (in addition to the push operations) through which a set of bidders distinct from the current losing bidder can try to pull the allocation of ζ towards other more preferred bidders by sharing among themselves the cost of this switching between bidders. A **push** operation can, indeed, be executed only by the currently losing bidder so that, if he is satisfied with the allocation, no reallocation is possible though some other bidders may wish to pay him to have the item to be pulled to another and more preferred bidder.
Other future plans include the relaxations we have listed in section 8 so that we plan to argue what happens if we assume that:

- the bidders are risk adverse so that they prefer either to pay the fee or to pay a fixed amount for not getting ζ for sure than attending the auction with the risk of getting ζ though together with a compensatory sum;

- the evaluations are either common or interdependent among the bidders and in any way may vary either after the pre auction phase (if the associated values are common knowledge, see further on) or after the auction phase itself if a post auction phase is present;

- the bidders are asymmetric so we can have different intervals $[0, M_i]$ and different functions F_i and f_i for each bidder b_i.

Last but not least we are planning to see what changes we may have in the auction phase if the fees that are paid in the pre auction phase are a common knowledge among the bidders.
As a first approximation we can say that if the k attending bidders know the

value of m (and so the number of bidders who paid the fee) they may be willing to bid less than m_i since each of them may consider to have a fixed compensation equal to mf, in case of loss, and so he may wish to increase the probability of losing the auction and such an increase may be obtained by simply bidding less than m_i.
Beyond all this, after the proposed extensions have been fully formalized, we have to apply them the performance measuring criteria to verify whether they are satisfied or not and so whether the proposed extensions may be classified as rational or not.

References

[1] Ken Binmore. *Playing fair, Game Theory and the Social Contract*. Massachusetts Institute of Technology, 1998.

[2] Ken Binmore. *Just playing, Game Theory and the Social Contract*. Massachusetts Institute of Technology, 1998.

[3] Ken Binmore. *Rational Decisions*. Princeton University Press, 2009.

[4] Simon French. *Decision Theory. An introduction to the mathematics of rationality*. John Wiley and Sons, 1986.

[5] Michael N. Huhns and Larry M. Stephens. Distributed methods for decision support. *Multiagent Systems and Societies of Agents*, pages 78–120, 2001. editor Gerhard Weiss.

[6] Paul Klemperer. A survey on auction theory. *originally on the Journal of Economic Surveys*, 1999. Internet version.

[7] Paul Klemperer. Why every economist should learn some auction theory. *http://www.paulklemperer.org*, 2001. Internet version.

[8] Vijay Krishna. *Auction Theory*. Academic Press, 2002.

[9] Paul Milgrom. *Putting Auction Theory to Work*. Cambridge University Press, 2004.

[10] Roger B. Myerson. *Game Theory. Analysis of conflict*. Harvard University Press, 1991.

[11] Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory*. MIT Press, 1994.

[12] Fioravate Patrone. *Decisori (razionali) interagenti. Una introduzione alla teoria dei giochi*. Edizioni plus, 2006.

[13] Donald G. Saari. *Decisions and Elections, Explaining the Unexpected*. Cambridge University Press, 2001.

[14] Donald G. Saari. *Disposing Dictators, Demistifying Voting Paradoxes. Social Choice Analysis*. Cambridge University Press, 2009.

[15] Michael Wooldridge. *An Introduction to MultiAgent Systems*. John Wiley and Sons, 2002.