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A purely probabilistic candle auction

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Abstract

Candle auctions have been used in the past as a variant of the English auction with a random termination time associated either to the going out of a candle or to the falling of a needle inserted in a random position in a burning candle.
In this case such auctions are used by the auctioneer A for the allocation of a **good** to one of the n bidders b_i of the set B.
Our basic motivation for the use of this type of auctions is the following. We are planning to use such auctions for the allocation of a **chore** ζ at one b_i from the set B whose members have been selected by A using a set of private criteria that do not depend on the willingness to attend of the single bidders.
The to be selected bidder has to be chosen from the set B given that the available information about these bidders are imprecise or fuzzy. These features prevent the profitable and direct selection of a suitable bidder with the guarantee of choosing the best one.
For this reason we plan to adopt an auction mechanism ([3]) where the bidders pay for not getting ζ but one of them has to get it though he also receives a compensation for being the **wining bidder**.
The compensation to the winning bidder derives him form the other bidders, the so called **losing bidders**, and is accumulated during the various steps or rounds on which the auction is based.

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1 The theoretical background

Auctions ([3], [4]) represent an allocation mechanism through which a resource (the so called auctioned item) is allocated to one actor from a set of actors called bidders. In classical cases the auction mechanism is characterized by the following features:

- the bidders attend the auction on a voluntary basis;
- the bidders attribute a positive value to the auctioned item so each of them is willing to bid for getting it;
- the rules of the auction are well known and are common knowledge among the bidders;
- the value that each bidder attributes to the auctioned item determines his strategy of bidding.

When the auction is over the winning bidder gets the auctioned item and pays a sum that depends on the structure of the auction. Ties among winning bidders are resolved with the use of a properly designed random device.
An auction is characterized by an **auctioneer** (who auctions an item) and a set of bidders (who submit bids) x_i and are characterized by the evaluations m_i.
The bids may be ([3, 4]):

- **open cry** if they are publicly visible;
- **sealed** if they are made privately and are revealed all at the same time;
- **one shot** if they are submitted only once;
- **repeated** if they are repeatedly submitted until a termination condition is satisfied;
- **ascending** if they start low and then rise;
- **descending** if they start high and then decrease.

Classical auctions types include[1]:

- English auctions;
- Dutch auctions;
- First Price Sealed Bid ($FPSB$) auctions;
- Second Price Sealed Bid ($SPSB$) auctions.

[1] Other possible types of auctions are: **all pay** auctions, where all the bidders bid and pay their own bids but only the highest bidding bidder wins the auction, and **third price** auctions that are similar to a $SPSB$ auction but for the fact that the paid price is the third highest bid.

In an English auction bids are open cry, repeated and ascending and the winner is the highest bidding bidder who pays the sum he bid that is coincident to the price at which the second last bidder dropped out.
In a Dutch auction bids are open cry and are offered by the auctioneer, repeated and descending and the winner is the bidder who accepts the current value and that pays such a value.
In an $FPSB$ auction bids are sealed and one shot and the winner is the highest bidding bidder who pays the sum he bid.
In an $SPSB$ auction bids are sealed and one shot and the winner is the highest bidding bidder who pays the the second highest bid.
The evaluations m_i are the maximum sums each bidder is willing to pay to get the auctioned item. Such evaluations may be:

- **private** if they are independent one from the others so that a reciprocal knowledge would not change the individual values;
- **interdependent** if a reciprocal knowledge may change the individual values;
- **common** if the evaluations are ex-post the same among the bidders.

On the basis of such definitions we note that[2]:

- Dutch auctions $\equiv FPSB$ auctions;
- under private values, English auctions $\equiv SPSB$ auctions.

Given such equivalences we note that, [3]:

- in a $SPSB$ auction it is a dominant strategy for a bidder to bid his own evaluation of an item so that we have $x_i = m_i$ for each bidder;
- if we assume a symmetric model (see further on) in a $FPSB$ auction it is a dominant strategy for a bidder to bid a little less $\delta > 0$ than his evaluation. Under the assumption that the evaluations of the bidders are independent and uniformly distributed over the same interval tis δ tends to zero as the number of the bidders increases.

All the types of auctions we have seen so far are characterized by a fixed termination rule that depends either on the structure of the auction (as it is in sealed bid auctions) or on the actions of the bidders (as it is in open cry auctions).
On the other hand we may devise auction mechanisms that terminate independently from the actions of the bidder in the sense that they are implemented with an iterative multi step mechanism and at each step there is a non null probability that the auction ends without the bidder may perform any bid.
These types of auction have been used as variants of the English auctions and represent, at least in part, the subject of the present Technical Report (TR).
In the literature ([1]) they are seen as a counterpart of the so called hard close auctions, those auctions that we formerly called classic auctions.

[2]With $\equiv$ we denote a **strategic equivalence**. Two games are strategically equivalent, [3], if for every strategy in a game a player has a strategy in the other game with the same outcome.

2 Some preliminary remarks

In the present TR we propose an iterative mechanism that is characterized by a certain number L of rounds.
The rounds are numbered as $j = 0, 1, 2 \ldots, L-2, L-1, L$ but only L are useful rounds since at the $L+1-$th the auction ends for sure without any bidder having the possibility of performing any action. This is the main reason why we speak in many cases of L ticks or times.
At each round j one of the n bidders $b_i \in B$ is randomly selected with a probability equal to π (see equation (1)) and can either **accept** or **refuse** (what this means will be explained in section 3). The presence of this random selection is enough to qualify the proposed mechanism as a purely probabilistic mechanism.
The auction goes on until a termination condition is verified and then it stops.
At the end of the auction the last accepting bidder is the **winning bidder** whereas all the other bidders are the **losing bidders** (see section 3).
In the proposed mechanism we may introduce the following termination conditions:

(a) the mechanism is executed a fixed number L of times[3] for $j = 0, 1, 2 \ldots, L-2, L-1$;

(b) the mechanism is executed L times but each time we have a non null probability of a premature termination.

We call the case (a) a **fixed termination** mechanism whereas the mechanism in the case (b) is termed a **variable termination** mechanism.
If we denote as p_j the probability that the auction goes on at round j (and with $q_j = 1 - p_j$ the corresponding probability of termination at round j) we have:

- in the case (a) we have $p_j = 1$ for $j \in [0, L-1]$ and $p_L = 0$;
- in the case (b) we have:
    - $p_0 = 1$,
    - $p_L = 0$,
    - for $j \in [1, L-1]$ p_j is monotonically non increasing.

The last condition allows us to define probabilities that are piecewise constant. A typical case is the following[4]:

- for $j \in [0, L_{min}]$ we have $p_j = 1$,
- for $j \in [L_{min}, L-1]$ we have that p_j is monotonically decreasing,
- $p_L = 0$.

[3] We note that at $t = L$ the auction ends without none of the bidders performing any action so this last tick has a purely formal meaning.

[4] We note that $L_{min} < L - 1$.

In this case the auction has a minimum guaranteed duration that is common knowledge among the bidders.
We underline that at each step where the auction does not terminate each bidder is selected with a probability equal to:

$$\pi = \frac{1}{n} \tag{1}$$

whereas the complementary probability of not being selected is:

$$\bar{\pi} = 1 - \frac{1}{n} = 1 - \pi \tag{2}$$

Meaningful events in the case of **fixed termination** are the following:

(ev_1) a bidder b_i is never selected;

(ev_2) a bidder b_i is selected at least once;

(ev_3) a bidder b_i is selected at round $h \in [0, L-1]$ and afterwards he is no more selected.

If we consider the various selections as independent events we can associate to the foregoing events, in that order, the following probability values:

$$P(ev_1) = (1-\pi)^L = \bar{\pi}^L \tag{3}$$

$$P(ev_2) = 1 - (1-\pi)^L = 1 - \bar{\pi}^L \tag{4}$$

$$P(ev_3) = \pi(1-\pi)^{L-h-1} = \pi\bar{\pi}^{L-h-1} \tag{5}$$

Such events may occur also in the case of **variable termination** but the corresponding probabilities must be modified to account for the presence of the values p_j.
We note indeed that the probability that the auction lasts for L rounds (from 0 to $L-1$) can be expressed as:

$$\Pi_L = \prod_{j=0}^{L-1} p_j \tag{6}$$

whereas the probability that it lasts for $0 < h < L-1$ rounds can be expressed as:

$$\Pi_h = \prod_{j=0}^{h-1} p_j \tag{7}$$

We underline how, from the definitions we have given for the values p_j, the probability that it lasts 1 round is equal to 1 and the probability that it lasts $L+1$ rounds is equal to 0 since $p_L = 0$.

3 The basic ingredients

The basic ingredients of the proposed mechanism are therefore:

- an auctioneer A and a set B of n bidders b_i, $i = 1, \ldots, n$;
- every bidder b_i has the following available individual strategies $S_i = \{a, r\}$ of either acceptance or refusal;
- every bidder b_i is characterized by the number k_i of his refusals and the number k_{-i} of the refusals of the other bidders, both to be initialized at 0 and one independent from the other[5];
- an integer $L > 0$ and a counter t that starts at 0 and stops not later than L;
- a fee f and a common pot P (initialized at $P = 0$) that is the compensation for the winning bidder;
- a set of values p_j for $j \in [0, L]$ that are common knowledge among all the bidders;
- a random number generator that generates (according to an uniform distribution) an integer in the interval $[1, n]$ at each tick of the counter;
- a private value v_i that represents the damage that each bidder receives from the allocation of the chore.

From the foregoing list it should be clear why we call the last accepting bidder as the winning bidder (so that the other bidders are termed losing bidders): because he is the one who gets the pot P that is formed, for what concerns his utility, by the payments of the others.
We note how both the value of L and the entity of the fee f play an important role in the mechanism.
The auctioneer A is free to select f at his will and to select L from an interval $[L_{min}, L_{max}]$.
For what concerns f we note that:

- if it is fixed too low the bidders tend to refuse more often than they accept but the content of the pot may rise too slowly to effectively compensating the damage deriving from the allocation;
- if it is fixed too high the bidders tend to accept more often than they refuse so that the content of the pot may rise too slowly to effectively compensate the damage deriving from the allocation.

[5]The independence derives from the fact that k_i depends on the behavior of b_i whereas k_{-i} depends on the behaviors of the other bidders.

On the other hand the values L_{min} and L_{max} must be selected so that the value L is neither too low nor too high.
If L is too low the probability that all the bidders refuse for the whole duration of the auction is high. On the other hand it is meaningless to have L too high so that at each step from one value of the counter on all the bidders accept. In this case the pot is no more incremented and the auction is a mere waste of time.
We make some more comments in sections 4.2 and 5.2.

4 The *fixed termination* case

4.1 The basic steps

In the case where the auction has a fixed termination time the rules of the auctions are the following:

- we have an initialization phase where we put $P = 0$ and $t = 0$;
- at each tick t of the counter from 0 to $L-1$ a random integer i is generated and a bidder b_i is selected;
- the bidder b_i can either accept or refuse;
- if he refuses he adds a fee f to the common pot so that $P = P + f$, $t = t + 1$;
- if he accepts he qualifies as the **current candle holder** or **cch**, $t = t + 1$;
- when the counter expires the **cch** wins the auction and gets both ζ and the content of the common pot P.

The counter is incremented of one unit at each acceptance or refusal and runs for $L + 1$ ticks (from 0 to L) and at $t = L$ it stops with no selection so that we have only L useful ticks.
At the end of the auction (so when the counter expires) we can have two cases:

(o_1) there is a **cch** that is the winner of the auction,

(o_2) there is no **cch** so the auction is void.

In the case (o_1) the **cch** gets ζ and P with a net utility of:

$$u_i = k_{-i} f - v_i \tag{8}$$

as the difference between the net gain that b_i receives form P and the damage he suffers from the allocation of ζ.
From relation (8) we can easily understand how the last **cch** may have also a negative utility, depending on the value of the parameter k_{-i} in relation to the values f and v_i and so depending on the decisions of the other bidders.

For what concerns the losing bidders $b_j \neq b_i$ we note that each of them gets an utility that can be expressed as:

$$u_j = v_j - k_j f \qquad (9)$$

as the difference between the gain that b_i has from the missed allocation of ζ and the the sums he has paid for refusing the allocation of ζ.
From relation (9) we can easily understand how the losing bidders may have also a negative utility, depending on the values of the parameters k_j in relation to the values f and v_j. We note that k_i depends only on the decisions of the bidder b_i and on the chances of being selected at each step.
The case (o_2) can occur if all the bidders refuse at every tick from 0 to L.
In this case at the end of the auction we have $P = Lf$ and the auctioneer can use this sum to allocate the chore to a further player not included in the set B.
In section 4.2 we are going to show how, at least in the current fixed termination case, this case can hardly ever occur in practice.

4.2 The possible collective and individual strategies

In the current case every bidder knows how long the auction is going to last for sure and this feature is a common knowledge among the bidders.
What each bidder does not know for sure, before the end of the auction, is:

- if and when he can be selected,
- once selected, if and when he will be selected again.

We can express this fact by saying that the probability that the auction ends at step h for a given bidder has a probability given by relation (5).
We recall indeed that the bidder b_i can play his individual strategies S_i only if he is selected and this can occur, at every round, with a probability π. So if a bidder is no more selected his auction has ended the last time he has been selected (though he is sure of this only when the auction actually ends).
When a bidder is selected he can choose one of his available actions depending on:

- the value of h;
- the value of k_i;
- the value of k_{-i};
- the value of v_i.

At this point we start by examining some particular collective strategies and then we examine the various possibilities that a bidder has at a generic round $h \in [0, L-1]$.
For what concerns the collective strategies we want to verify if and under which conditions the following collective strategies may be a Nash Equilibrium (NE, [6], [5], [2]):

(cs_1) each bidder, upon a selection, always accepts;

(cs_2) each bidder, upon a selection, always refuses.

In the case (cs_1) to verify it is a NE we can proceed as follows. We assume to have $L-1$ consecutive acceptances (from 0 to $L-2$) and we see if a bidder selected at the $L-$th round is better off by accepting or by refusing. In the first case (cs_1) is a NE otherwise not.
So we suppose to have[6]:

$$a^0, a^1, \ldots, a^{L-2}, x^{L-1} \tag{10}$$

where x may be either a or r.
In order to make the desired verification we note that at round $L-2$ we have[7] $P=0$ so when the currently selected bidder b_i has to choose an action he considers that, from relations (8) and (9):

- if he accepts he has an utility $u_i = -v_i$,
- if he refuses he has an utility $u_i = v_i - f$ since $k_i = 1$.

In this case b_i refuses if $v_i - f > -v_i$ or if $f < 2v_i$ (and therefore the collective strategy of all acceptances is not a NE) but accepts if $f > 2v_i$ so that that the collective strategy of all acceptances would be a NE. We are going to make some more comments shortly.
In the case (cs_2) we have a succession of refusals[8] and we want to verify if the bidder b_i selected at round $L-1$ is better off by accepting or by refusing. We want to verify if, in the succession (11) the x must be an a or an r:

$$r^0, r^1, \ldots, r^{L-2}, x^{L-1} \tag{11}$$

In order to verify this we note that at round $L-2$ we have, owing to $L-1$ consecutive refusals, $P = (L-1)f = (k_i + k_{-i})f$ so when the currently selected bidder b_i has to choose an action at round $L-1$ he considers that, from relations (8) and (9):

- if he accepts he has an utility $u_i = k_{-i}f - v_i$,
- if he refuses he has an utility $u_i = v_i - (k_i + 1)f$ (since by refusing he has to pay once more the fee).

In this case b_i refuses if:

$$v_i - (k_i + 1)f > k_{-i}f - v_i \tag{12}$$

[6]We use the notation a^h to denote an acceptance from any of the bidders at step h and x^h to denote a generic action at step h. We are not interested in putting in evidence repeated acceptances from the same bidder.

[7]We note that $P = (k_i + k_{-i})f$ so if $P = 0$ we have $k_i = 0$ and $k_{-i} = 0$ and vice versa.

[8]We use the notation r^h to denote a refusal from any of the bidders at step h and x^h to denote a generic action at step h. We are not interested in putting in evidence repeated refusals from the same bidder.

or if:

$$2v_i > k_{-i}f + (k_i + 1)f = (k_{-i} + k_i)f + f = (L - 1)f + f = Lf \tag{13}$$

So if:

$$f < \frac{2v_i}{L} \tag{14}$$

then b_i refuses and we have that the collective strategy of all refusals is a NE otherwise he accepts and that collective strategy is not a NE.
We have therefore derived that if $2v_i/L < f < 2v_i$ the foregoing collective strategies are not NE so that we are sure that at the end of the auction:

- there will be a winning bidder,
- there will be a pot P to compensate him.

Such conditions depend, however, on the value v_i of the last selected bidders. In order to make it operational we can choose $2\underline{v}/L < f < 2\underline{v}$ where $\underline{v}$ is such that $v_i \in [\underline{v}, \bar{v}]$ for every b_i and for a suitable pair $\underline{v}, \bar{v}$.
Before going on we want to examine:

(is_1) when it is optimal for a selected bidder b_i to choose a at $t = 0$;

(is_2) when it is optimal for a selected bidder b_i to choose r at $t = 0$.

In the case (is_1) after the acceptance we can have the following meaningful cases:

- $L - 1$ consecutive refusals of the other bidders so that b_1 is the final **cch**;
- at least one acceptance from one of the other bidders so that b_1 is not the final **cch**.

In the former case b_i is surely better off by accepting if $(L - 1)f \geq v_i$ and this is true also if the same bidder is selected again at step h if, between step 0 and step h, we had only refusals.
On the other hand if $(L - 1)f < v_i$ then b_i is better off by refusing and so reducing his utility from v_i to $v_i - f$ under the hypothesis that $v_i \geq f$ (we are in the case (is_2)).
In other words b_i is better off by accepting at $t = 0$ if he is sure either to have a gain if he will be the final **cch** or if he is sure not to be the final **cch** so he avoids paying once more than it is necessary the fee[9] f.
We now focus the attention on the behavior of the single bidder and assume:

- to be at round $h \in [1, L - 1]$
- that the bidder b_i is selected;

[9] We recall that if at step $h - 1$ the utility of b_i is $u_i(h - 1)$ and if at step h he refuses his utility becomes $u_i(h) = u_i(h - 1) - f$. On the other hand if b_i accepts at step h his utility remains unchanged.

- that he has already accumulated k_i refusals;
- that the net content[10] of the pot for him is $k_{-i}f$

We want to see which strategy b_i should select in the various possible cases.
We start by considering the case where b_i has never been selected before the current step h. In this case we have $k_i = 0$ and possibly $k_{-i} > 0$. We can define for b_i:

- $a(h-1) = u_i(h-1|w)$ as the utility at step $h-1$ if he will be the final **cch**;
- $b(h-1) = u_i(h-1|l)$ as the utility at step $h-1$ if he will not be the final **cch**.

It is easy to see how we have:

$$a(h-1) = u_i(h-1|w) = k_{-i}f - v_i \tag{15}$$

and:

$$b(h-1) = u_i(h-1|l) = v_i \tag{16}$$

At step h we have that b_i accepts if he has already a gain or expects to have a gain by winning the auction otherwise he refuses.
More formally we have the following cases:

- if $a(h-1) \geq 0$ then b_i accepts since he can only become better off if in subsequent rounds other bidders refuse;
- if $a(h-1) < 0$ then b_i considers that if $a(h-1)(L-h-1)f \geq 0$ he accepts otherwise he refuses.

In the acceptance cases we have:

$$a(h) = a(h-1) \tag{17}$$

and:

$$b(h) = b(h-1) \tag{18}$$

whereas in the refusal case we have:

$$a(h) = a(h-1) \tag{19}$$

and, under the assumption that we have $b(h-1) - f > 0$:

$$b(h) = b(h-1) - f \tag{20}$$

We now consider a more general case where up to step $h-1$ we had k_i refusals from b_i and possibly $k_{-i} > 0$ refusals from the other bidders. In this case we define the following quantities:

$$a(h-1) = u_i(h-1|w) = k_{-i}f - v_i \tag{21}$$

[10] Each bidder b_i sees the content of the pot as $P = (k_i + k_{-i})f$ so that the net content for bidder i is represented by the payments made by the others and so by $k_{-i}f$.

$$b(h-1) = u_i(h-1|l) = v_i - k_i f \tag{22}$$

$$c(h-1) = k_{-i} f + (L-h-1)f - v_i \tag{23}$$

Also in this case we have that b_i accepts (so that both relations (17) and (18) are satisfied) if either $a(h-1) \geq 0$ or $a(h-1) < 0$ and $c(h-1) \geq 0$.
On th other hand if we have:

- $c(h-1) < 0$
- $b(1) = u_i(h|l) = v_i - (k_i + 1)f \geq 0$

then b_i refuses so that both relations (19 and 20 are satisfied.
The problematic case occurs whenever we have:

- $c(h-1) < 0$
- $b(1) < 0$

so that b_i has to take a decision by choosing the current lower loss. In this case we have that if $a(h-1) \geq b(h)$ then b_i accepts otherwise he refuses.
We recall that at any refusal the utility of b_i is worsened by f whereas at any acceptance it remains unchanged

5 The *variable termination* case

5.1 The basic steps

In this case at every step $j \in [0, L-1]$ we have a probability p_j that the auction ends at that step (see section 2). In this case the proposed procedure is based on the following steps:

(1) starts at $j = 0$ with all the variables properly initialized;

(2) at step j we see if the auction can go on (with a probability p_j) or must stop (with a probability $1 - p_j$);

(3) if it must stop go to (7);

(4) if it can go on a bidder b_i is randomly selected;

(5) if b_i accepts then b_i is the **cch**; $j = j + 1$; go to (2);

(6) if b_i refuses then $P = P + f$; $j = j + 1$; go to (2);

(7) the final **cch** gets P and ζ;

(8) end;

The final **cch** at step (7) is the current **cch** when the auction ends. The termination of the auction at every step is determined with the use of a properly defined random device that uses a predefined distribution of probability values that are assumed to be common knowledge among the bidders.
From this structure and from what we have seen in section 2 we easily derive that at step h:

- every bidder b_i knows his current situation as represented by the values k_i and k_{-i};

- every bidder b_i can evaluate the probability that the auction goes on for k more rounds and also until round $L-1$;

- every bidder b_i can evaluate the probability of being selected once again before the end of the auction.

With this we mean that every bidder b_i knows if the past is profitable or not and if the future is promising or not. The past is profitable if for b_i at step h we have (see relations (21), (22)) we have:

- $a(h-1) \geq 0$

- $b(1) = u_i(h|l) = v_i - (k_i+1)f \geq 0$

On the other hand the future is promising if the expected gain considering also it is positive or if (see relation (24)):

$$c(h-1) \geq 0 \tag{24}$$

where:

$$c(h-1) = k_{-i}f + (L-h-1)P(L-h-1)|h)f - v_i \tag{25}$$

In relation (24) we define as $P(L-h-1|h)$ the probability that the auction lasts until round $L-1$ having reached round h.
We recall that p_h is the probability that at round h the auction goes on so that a bidder can be selected so that we may define the probability that the auction lasts for k more rounds having lasted until round h as:

$$P(k|h) = \prod_{j=h+1}^{k} p_j \tag{26}$$

where $k \geq h+1$.
From relation (26) we derive:

- $P(L-h-1|h) = \prod_{j=h+1}^{L-1} p_j$

- $P(1|h) = \prod_{j=h+1}^{h+1} p_j = p_{h+1}$

We recall indeed that between $h+1$ and $L-1$ we have $L-1-h-1+1 = L-h-1$ rounds.

5.2 The possible collective and individual strategies

At this point we have to consider what we have seen in section 4.2 and extend it to the new situation where the bidders know the probabilities p_j (for $j \in [0, L-1]$) and can guess the probabilities $P(k|h)$ for any $h \in [1, L-2]$. and $k > h$.
It is easy to see how, for $h = L - 1$, we have (see also section 4.2):

- a succession of L acceptances is not a NE for the same reasons we saw in section 4.2;
- a succession of L refusals may not be a NE for the same reasons we saw in section 4.2.

Similar considerations hold for a succession of h acceptances (for $h \in [1, L-2]$) if a bidder b_i thinks that, being selected at the turn $h - 1$, he evaluates that $P(h + 1|h)$ is a very low value.
At step $h = 0$ (see section 4.2) we have that it is usually better for a bidder b_i to accept than to refuse.
At that step we have $k_i = 0$ and $k_{-i} = 0$ so that we have the following utilities for the bidder b_i:

- if accepts $u_i(0) = -v_i$;
- if refuses $u_i(0) = v_i - f$.

It would seem that accepting is dominated by refusing unless we have $f > 2v_i$ but if b_i thinks that the auction can last at least h more rounds he can evaluate the following probability:

$$P(h|0) \tag{27}$$

If he thinks that such a value is high enough he may be tempted to accept since he can imagine the following scenarios:

- h consecutive refusals so that his expected utility is $u_i(h|w) = hfP(h|0) - v_i$,
- at least one acceptance from one of the other bidders.

In the former case of only refusals from the other bidders he will be the final **cch**. In the latter case where there is at least one acceptance from one of the other bidders b_i will not be the final **cch** and, at the same time, he saved to pay one more time the fee f. In this case he is surely better off since $v_i > v_i - f$ for every value of $f > 0$.
If b_i refuses at $t = 0$ his utility becomes $v_i - f$ instead of v_i and he is not the **cch** (and under this condition he cannot be the final[11] **cch**).
At this point we can say that b_i is better off by accepting if the following constraint is satisfied:

$$hfP(h|0) - v_i > v_i - f \tag{28}$$

[11] We note that a bidder becomes firstly the **cch** and when the auction ends and he has that title he becomes the final **cch**.

or if the utility from an acceptance (in the best case where he is the final **cch**) is higher than the utility from a refusal. Such relation can be rewritten as:

$$hfP(h|0) > 2v_i - f \tag{29}$$

from where we derive that if:

$$P(h|0) > \frac{2v_i - f}{hf} \tag{30}$$

the b_i is better off by accepting otherwise he is better off by refusing.
At a generic step h (for $h \in [1, L-2]$) we may argue that every bidder b_i knows:

- his k_i,
- his k_{-i},

so we can repeat the considerations we have made in section 4.2.
Informally we have that:

- b_i accepts if the past is enough rewarding;
- b_i accepts if the foregoing condition is false and he may expect to gain from the expected future rounds;
- b_i refuses if he has not already refused too much and the foregoing conditions are not satisfied.

We say that the past is rewarding if we have:

$$k_{-i}f - v_i > 0 \tag{31}$$

whereas we say that the past and the foreseen future are enough rewarding if we have:

$$k_{-i}f + kfP(k|h) - v_i > 0 \tag{32}$$

Last but not least we say that b_i has not refused too much if we have:

$$v_i - (k_i + 1)f > 0 \tag{33}$$

since we have to account for a further refusal and so a further payment of the fee.

6 Concluding remarks

The present TR introduces two repeated or multi shot auction mechanisms. In both the mechanisms the bidders are selected at each step according to a uniform distribution so that each bidder can perform a choice (accept or refuse) only if he is selected.
In this way each bidder has a random termination time for his participation to

the auction as the last time he is selected. On the other hand every bidder is influenced by the decisions of the others.
If b_i at step h is selected and accepts he becomes the **cch**. He keeps this title upon successive refusals from the other bidders (that make him better off) and upon his own successive acceptances (upon being selected) and loses it upon any acceptance of one of the other bidders.
In the former mechanism this random selection is the only probabilistic element we introduced in it whereas in the latter we introduced a further probabilistic device since we allowed each step to be assigned a probability of termination at that step.
Both mechanisms have been presented and described together with some strategies for the bidders.
Their formal treatment must, however, still be completed and will be the subject of further research efforts.

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