Random termination auctions as allocation tools

Lorenzo Cioni
lcioni@di.unipi.it
Computer Science Department, University of Pisa
largo Pontecorvo 3, 56127, Pisa, Italy
tel.: (+39) 050 2212741 fax: (+39) 050 2212726

1 Introduction

Candle auctions have been used in the past as a variant of the English auction with a random termination time associated either to the going out of a candle or to the falling of a needle inserted in a random position in a burning candle ([9], [3], [2], [1]).
In this case such auctions are used by the auctioneer A for the allocation of a **good**[1] to one of the n bidders b_i of the set B.
Our basic motivation for the use of this type of auctions is the following.
We are planning to use such auctions for the allocation of a **chore**[2] ζ at one b_i from the set B whose members have been selected by A using a set of private criteria[3] that do not depend on the willingness to attend of the single bidders.
The to be selected bidder has to be chosen from the set B given that the available information about these bidders are imprecise or fuzzy. These features prevent the profitable and direct selection of a suitable bidder with the

[1]We say that an auctioned item is a **good** if:

- it has a non negative value for the auctioneer A;
- it has a non negative value for each bidder b_i;
- the b_i attend to the auction on a voluntary basis.

[2]With the term **chore** we denote an item that the auctioneer wants to give away but that the bidders are willing to accept only in exchange of a proper compensation.

[3]With this we mean that A can choose such criteria in any way he wishes without any need to justify to anybody his choices.

guarantee of choosing the best one.
For this reason we plan to adopt an auction mechanism[4] ([5]) where the bidders pay for not getting ζ but one of them has to get it though he also receives a compensation for being the[5] **wining bidder**.
The compensation to the winning bidder derives him from the other bidders, the so called **losing bidders**, and is accumulated during the various steps or rounds on which the auction is based.

2 The theoretical background

Auctions ([5], [6]) represent an allocation mechanism through which a resource (the so called auctioned item) is allocated to one actor from a set of actors called bidders. In classical cases the auction mechanism is characterized by the following features:

- the bidders attend the auction on a voluntary basis;
- the bidders attribute a positive value to the auctioned item so each of them is willing to bid for getting it;
- the rules of the auction are well known and are common knowledge among the bidders;
- the valuation that each bidder attributes to the auctioned item determines his strategy of bidding.

When the auction is over the winning bidder gets the auctioned item and pays a sum that depends on the structure of the auction. Ties among winning bidders are resolved with the use of a properly designed random device.
An auction is characterized by an **auctioneer** (who auctions an item) and a set of bidders b_i (who submit bids x_i) and are characterized by the evaluations m_i.
The bids may be ([5, 6]):

- **open cry** if they are publicly visible;
- **sealed** if they are made privately and are revealed all at the same time;

[4]In this paper we are going to use the term mechanism in a rather informal sense as a set of rules, strategies and procedures. For a more formal use of the term we refer, for instance, to [5], [6] and [7].

[5]This seemingly inconsistent naming scheme of winning bider and losing bidders (see further on) will be made clear in sections 3 and 4.

- **one shot** if they are submitted only once;

- **repeated** if they are repeatedly submitted until a termination condition is satisfied;

- **ascending** if they start low and then rise;

- **descending** if they start high and then decrease.

Classical auctions types include[6]:

- English auctions;

- Dutch auctions;

- First Price Sealed Bid ($FPSB$) auctions;

- Second Price Sealed Bid ($SPSB$) auctions.

In an English auction bids are open cry, repeated and ascending and the winner is the highest bidding bidder who pays the sum he bid that is coincident to the price at which the second last bidder dropped out.
In a Dutch auction bids are open cry, are offered by the auctioneer and are repeated and descending. The winner is the bidder who accepts the current value and that pays such a value.
In an $FPSB$ auction bids are sealed and one shot and the winner is the highest bidding bidder who pays the sum he bid.
In an $SPSB$ auction bids are sealed and one shot and the winner is the highest bidding bidder who pays the the second highest bid.
Each evaluation m_i represents the maximum sum that each bidder is willing to pay to accept the auctioned item. Such evaluations may be:

- **private** if they are independent one from the others so that a reciprocal knowledge would not change the individual values;

- **interdependent** if a reciprocal knowledge may change the individual values;

- **common** if the evaluations are ex-post the same among the bidders.

[6]Other possible types of auctions are: **all pay** auctions, where all the bidders bid and pay their own bids but only the highest bidding bidder wins the auction, and **third price** auctions that are similar to a $SPSB$ auction but for the fact that the paid price is the third highest bid.

On the basis of such definitions we note that[7]:

- Dutch auctions $\equiv$ $FPSB$ auctions;
- under private values, English auctions $\equiv$ $SPSB$ auctions.

Given such equivalences we note that, [5]:

- in a $SPSB$ auction it is a dominant strategy for a bidder to bid his own evaluation of an item so that we have $x_i = m_i$ for each bidder;
- if we assume a symmetric model (see further on) in a $FPSB$ auction it is a dominant strategy for a bidder to bid a little less $\delta > 0$ than his evaluation. Under the assumption that the evaluations of the bidders are independent and uniformly distributed over the same interval the value of δ tends to zero as the number of the bidders increases.

All the types of auctions we have seen so far are characterized by a fixed termination rule that depends either on the structure of the auction (as it is in sealed bid auctions) or on the actions of the bidders (as it is in open cry auctions).
On the other hand we may devise auction mechanisms that terminate independently from the actions of the bidders in the sense that they are implemented with an iterative multi step mechanism and at each step there is a non null probability that the auction ends without the bidder may perform any bid.
These types of auction have been used as variants of the English auctions and represent, at least in part, the subject of the present paper. In the literature ([1]) they are seen as a counterpart of the so called hard close auctions, those auctions that we formerly called classic auctions.

3 Some preliminary remarks

In the present paper we propose an iterative mechanism that is characterized by a certain number L of rounds.
The rounds are numbered as $j = 0, 1, 2 \ldots, L-2, L-1, L$ but only L are useful rounds since at the $L+1-$th round the auction ends for sure without any bidder having the possibility of performing any action. This is the main reason why we speak in many cases of L ticks or times.

[7] With $\equiv$ we denote a **strategic equivalence**. Two games are strategically equivalent, [5], if for every strategy in a game a player has a strategy in the other game with the same outcome.

At each round j one of the n bidders $b_i \in B$ is randomly selected with a probability equal to π (see equation (1)) and can either **accept** or **refuse** (what this means will be explained in section 4). The presence of this random selection is enough to qualify the proposed mechanism as a purely probabilistic mechanism.
The auction goes on until a termination condition is verified and then it stops. At the end of the auction the last accepting bidder is the **winning bidder** whereas all the other bidders are the **losing bidders** (see section 4). In the proposed mechanism we have introduced the following termination conditions:

(a) the mechanism is executed a fixed number L of times[8] for $j = 0, 1, 2 \ldots, L-2, L-1$;

(b) the mechanism is executed L times but each time we have a non null probability of a premature termination.

We call the case (a) a **fixed termination** mechanism whereas the mechanism in the case (b) is termed a **variable termination** mechanism.
The fixed termination mechanism can be used to describe an allocation procedure where:

- the mechanism has a fixed duration,
- the bidders see it has having a random duration since each of them can choose an action only if he is selected,
- the bidders are equivalent with regard to the mechanism.

The variable termination mechanism drops the first condition so that we can use it to model a mechanism with a premature termination due to an exogenous random pressure. In this case a bidder faces a twofold uncertainty, on the termination and on the duration, so that when he is selected he can only make expected guesses.
If we denote as p_j the probability that the auction goes on at round j (and with $q_j = 1 - p_j$ the corresponding probability of termination at round j):

- in the case (a) we have $p_j = 1$ for $j \in [0, L-1]$ and $p_L = 0$;
- in the case (b) we have:
    - $p_0 = 1$,

[8]We recall that at $t = L$ the auction ends without none of the bidders performing any action so this last tick at L has a purely formal meaning.

· $p_L = 0$,

· for $j \in [1, L-1]$ p_j is monotonically non increasing.

The last condition allows us to define probabilities that are piecewise constant. A typical case is the following[9]:

- for $j \in [0, L_{min}]$ we have $p_j = 1$,
- for $j \in [L_{min}+1, L-1]$ we have that p_j is monotonically decreasing,
- $p_L = 0$.

In this case the auction has a minimum guaranteed duration that is common knowledge among the bidders.
We underline that at each step where the auction does not terminate each bidder is selected with a probability equal to:

$$\pi = \frac{1}{n} \tag{1}$$

whereas the complementary probability of not being selected is:

$$\bar{\pi} = 1 - \frac{1}{n} = 1 - \pi \tag{2}$$

Meaningful events in the case of a **fixed termination** are the following:

(ev_1) a bidder b_i is never selected;

(ev_2) a bidder b_i is selected at least once;

(ev_3) a bidder b_i is selected at round $h \in [0, L-1]$ and afterwards he is no more selected.

If we consider the various selections as independent events we can associate to the foregoing events, in that order, the following probability values:

$$P(ev_1) = (1-\pi)^L = \bar{\pi}^L \tag{3}$$

$$P(ev_2) = 1 - (1-\pi)^L = 1 - \bar{\pi}^L \tag{4}$$

$$P(ev_3) = \pi(1-\pi)^{L-h-1} = \pi\bar{\pi}^{L-h-1} \tag{5}$$

Such events may occur also in the case of **variable termination** but the corresponding probabilities must be modified to account for the presence of

[9] We note that $L_{min} < L-1$.

the values p_j.
We note, indeed, that the probability that the auction lasts for L rounds (from 0 to $L-1$) can be expressed as:

$$\Pi_L = \prod_{j=0}^{L-1} p_j \tag{6}$$

whereas the probability that it lasts for $0 < h < L-1$ rounds can be expressed as:

$$\Pi_h = \prod_{j=0}^{h-1} p_j \tag{7}$$

We underline how, from the definitions we have given for the values p_j, the probability that it lasts 1 round is equal to 1 and the probability that it lasts $L+1$ rounds is equal to 0 since $p_L = 0$.

4 The basic ingredients

The basic ingredients of the proposed mechanism are therefore:

- an auctioneer A and a set B of n bidders b_i, $i = 1, \ldots, n$;

- every bidder b_i has the following available individual strategies $S_i = \{a, r\}$ of either **acceptance** or **refusal**;

- every bidder b_i is characterized by the number k_i of his refusals and the number k_{-i} of the refusals of the other bidders, both to be initialized at 0 and one independent from the other[10];

- an integer $L > 0$ and a counter t that starts at 0 and stops not later than L;

- a fee f and a common pot P (initialized at $P = 0$) that is the compensation for the winning bidder;

- a set of values p_j for $j \in [0, L]$ that are common knowledge among all the bidders;

- a random number generator that generates (according to a uniform distribution) an integer in the interval $[1, n]$ at each tick of the counter;

[10]The independence derives from the fact that k_i depends on the behavior of b_i whereas k_{-i} depends on the behaviors of the other bidders.

- a private value v_i that represents the damage that each bidder receives from the allocation of the chore.

From the foregoing list it should be clear why we call the last accepting bidder as the winning bidder (so that the other bidders are termed losing bidders): because he is the one who gets the pot P that is formed, for what concerns his utility, by the payments of the others.
We note how both the value of L and the entity of the fee f play an important role in the mechanism.
The auctioneer A is free to select f at his will and to select L from an interval $[L_{min}, L_{max}]$. We note that f is common knowledge among the bidders whereas both L and the interval $[L_{min}, L_{max}]$ are a private information of A.
For what concerns f we note that:

- if it is fixed too low the bidders tend to refuse more often than they accept but the content of the pot may rise too slowly for effectively compensating the damage deriving from the allocation;
- if it is fixed too high the bidders tend to accept more often than they refuse so that the content of the pot may rise too slowly for effectively compensating the damage deriving from the allocation.

On the other hand the values L_{min} and L_{max} must be selected so that the value L, though randomly selected, is neither too low nor too high.
If L is too low the probability that all the bidders refuse for the whole duration of the auction is high. On the other hand it is meaningless to have L too high so that at each step from one value of the counter on all the bidders accept. In this case the pot is no more incremented and the auction is a mere waste of time.
We make some more comments in sections 5.2 and 6.2.

5 The *fixed termination* case

5.1 The basic steps

In the case where the auction has a fixed termination time the rules of the auctions are the following:

(1) we have an initialization phase where we put $P = 0$ and $t = 0$;

(2) at each tick t of the counter from 0 to $L - 1$ a random integer i is generated and a bidder b_i is selected;

(3) the bidder b_i can either accept or refuse;

(4) if he refuses he adds a fee f to the common pot so that $P = P + f$, $t = t + 1$;

(5) if he accepts he qualifies as the **current candle holder** or **cch**, $t = t + 1$;

(6) when the counter expires the possible **cch** wins the auction and gets both ζ and the content of the common pot P.

The counter is incremented of one unit at each acceptance or refusal and runs for $L + 1$ ticks (from 0 to L) and at $t = L$ it stops with no selection so that we have only L useful ticks.
At the end of the auction (so when the counter expires) we can have two cases:

(o_1) there is a last **cch** that is the winner of the auction,

(o_2) there is no last **cch** so the auction is void.

In the case (o_1) the last **cch** b_i gets ζ and P with a net utility of:

$$u_i = k_{-i} f - v_i \tag{8}$$

as the difference between the net gain that b_i receives from P and the damage he suffers from the allocation of ζ.
From relation (8) we can easily see how the last **cch** may have also a negative utility, depending on the value of the parameter k_{-i} in relation to the values f and v_i and so depending on the decisions of the other bidders.
The winning bidder may accept a negative utility if by refusing once more (and so by paying once more the fee) he would be worse off.
For what concerns the losing bidders $b_j \neq b_i$ we note that each of them gets a utility that can be expressed as:

$$u_j = v_j - k_j f \tag{9}$$

as the difference between the gain that b_j has from the missed allocation of ζ and the the sums that he has paid for refusing the allocation of ζ.
From relation (9) we can easily see how the losing bidders may have also a negative utility, depending on the values of the parameters k_j in relation to the values f and v_j. We note that k_j depends only on the decisions of the bidder b_j and on the chances of being selected at each step.
A losing bidder may accept a negative utility if by by accepting he would be worse off so that he is better off even if he has to pay once more the fee.
The case (o_2) can occur in the following two cases:

(1) if each bidder refuses at every tick from 0 to L;

(2) if each bidder refuses at every tick from h to L, including the last bidder who accepted at $h-1$.

In the case (1), at the end of the auction we have $P = Lf$ and the auctioneer can use this sum to allocate the chore to a different player not included in the set B by having the members of B pay him a compensation. In section 5.2 we are going to show how, at least in the current fixed termination case, this case can hardly ever occur in practice.
In the case (2) we have that one or more bidders, when selected, at the beginning accept but, from a certain round onwards, they all refuse so that we have no accepting bidder[11].
In this case at the end of the auction we have $P = kf$ with $k \in [1, L-1]$ and the auctioneer can use this sum to allocate the chore to a different player not included in the set B.

5.2 The possible collective and individual strategies

In the current fixed termination case every bidder knows how long the auction is going to last for sure and this feature is a common knowledge among the bidders.
What each bidder does not know for sure, before the end of the auction, is:

- if and when he can be selected,

- once selected, if and when he will be selected again.

We can express this fact by saying that the probability that the auction ends at step h for a given bidder has a value given by relation (5).
We recall indeed that the bidder b_i can play his individual strategies $S_i = \{a, r\}$ only if he is selected and this can occur, at every round, with a probability π.
From this policy we have that if a bidder is no more selected for him the auction has ended the last time he has been selected (though he is sure of this only when the auction actually ends).
When a bidder is selected at round h he can choose one of his available actions depending on:

- the value of f;

[11] In section 5.2 we are going to see under which conditions this collective behavior can occur.

- the value of h;
- the value of k_i;
- the value of k_{-i};
- the value of v_i.

In this section we start by examining some particular collective strategies and then we examine the various possibilities that a bidder has at a generic round $h \in [0, L-1]$.
For what concerns the collective strategies we want to verify if and under which conditions the following collective strategies are a Nash Equilibrium (NE, [8], [7], [4]):

(cs_1) each bidder, upon a selection, always accepts;

(cs_2) each bidder, upon a selection, always refuses.

In the case (cs_1) in order to verify that it is a NE we can proceed as follows. We assume to have $L-1$ consecutive acceptances (from 0 to $L-2$) and we see if a bidder selected at the $L-$th round is better off by accepting or by refusing. In the first case (cs_1) is a NE otherwise not.
So we suppose to have[12]:

$$a^0, a^1, \ldots, a^{L-2}, x^{L-1} \tag{10}$$

where x may be either a or r.
In order to make the desired verification we note that at round $L-2$ we have had only acceptances so that we have[13] $P = 0$ so when the currently selected bidder b_i has to choose an action he considers that, from relations (8) and (9):

- if he accepts he has a utility $u_i = -v_i$,
- if he refuses he has a utility $u_i = v_i - f$ since $k_i = 1$.

In this case b_i refuses if $v_i - f > -v_i$ or if $f < 2v_i$ (and therefore the collective strategy of all acceptances is not a NE) but accepts if $f \geq 2v_i$ so that that the collective strategy of all acceptances would be a NE. We are going to make some more comments shortly.

[12]We use the notation a^h to denote an acceptance from any of the bidders at step h and x^h to denote a generic action at step h. We are not interested in putting in evidence repeated acceptances from the same bidder.

[13]We note that $P = (k_i + k_{-i})f$ so if $P = 0$ we have $k_i = 0$ and $k_{-i} = 0$ and vice versa.

In the case (cs_2) we have a succession of refusals[14] and we want to verify if the bidder b_i selected at round $L-1$ is better off by accepting or by refusing. We therefore want to verify if, in the succession (11), the x must be an a or an r:

$$r^0, r^1, \ldots, r^{L-2}, x^{L-1} \tag{11}$$

In order to verify this we note that at round $L-2$ we have, owing to $L-1$ consecutive refusals, $P = (L-1)f = (k_i + k_{-i})f$ so when the currently selected bidder b_i has to choose an action at round $L-1$ he considers that, from relations (8) and (9):

- if he accepts he has a utility $u_i = k_{-i}f - v_i$,
- if he refuses he has a utility $u_i = v_i - (k_i + 1)f$ (since by refusing he has to pay once more the fee).

In this case b_i at round $L-1$ refuses if:

$$v_i - (k_i + 1)f \geq k_{-i}f - v_i \tag{12}$$

or if:

$$2v_i \geq k_{-i}f + (k_i + 1)f = (k_{-i} + k_i)f + f = (L-1)f + f = Lf \tag{13}$$

So if:

$$f \leq \frac{2v_i}{L} \tag{14}$$

then b_i refuses and we have that the collective strategy of all refusals is a NE otherwise he accepts and that collective strategy is not a NE.
We have therefore derived that if:

$$\frac{2v_i}{L} < f < 2v_i \tag{15}$$

the foregoing collective strategies are not NE so that we are sure that at the end of the auction:

- there will be a winning bidder,
- there will be an effective content of the pot $P \neq 0$ to compensate him.

[14] We use the notation r^h to denote a refusal from any of the bidders at step h and x^h to denote a generic action at step h. We are not interested in putting in evidence repeated refusals from the same bidder.

Such conditions depend, however, on the value v_i of the last selected bidder and on the random value L. In order to make them operational we can choose a suitable pair $\underline{v}, \bar{v}$ such that:

$$v_i \in [\underline{v}, \bar{v}] \tag{16}$$

or:

$$\underline{v} < v_i < \bar{v} \tag{17}$$

for every b_i. In this way we can fix f such that:

$$\frac{2v_i}{L_{min}} < \frac{2\bar{v}}{L_{min}} < f < 2\underline{v} < 2v_i \tag{18}$$

Before examining other strategies that are individually available to the bidders we make some comments and introduce some definitions.
At each step $t \in [0, L-1]$ b_i, if he is selected, can either accept or refuse and is characterized by the following parameters that influence his decision at that step:

(1) $k_{-i}(h-1)$ or the number of the refusals from the bidders b_j with $j \neq i$ up to the step $h-1$ with the conventional initialization, for $h = 0$, as $k_{-i}(-1) = 0$;

(2) $k_i(h-1)$ or the number of the b_i's own refusals up to the step $h-1$ with the conventional initialization, for $h = 0$, as $k_i(-1) = 0$.

In this way we can state that:

(1) if b_i is not selected at a step h we have:

$$k_i(h) = k_i(h-1) \tag{19}$$

so that k_i is constant with h whereas we have:

$$k_{-i}(h) \geq k_{-i}(h-1) \tag{20}$$

so that k_{-i} is not decreasing with h;

(2) if b_i is selected at step h we have:

$$k_{-i}(h) = k_{-i}(h-1) \tag{21}$$

so that k_{-i} is constant with h whereas we have:

$$k_i(h) = k_i(h-1) + \delta_h \tag{22}$$

In relation (22) we have $\delta_h = 1$ if, at step h, b_i refuses and $\delta_h = 0$ if at step h b_i accepts so that k_i is not decreasing with h.

In order to state the criteria according to which b_i chooses an action at step h we define the maximum number of refusals from the other bidders from step $h+1$ to step $L-1$ as:

$$R(h) = L - h - 1 \tag{23}$$

and the following parameters that define the decision of b_i at step h according to their signs:

$\alpha_1(h) = k_{-i}(h-1)f - v_i$

$\alpha_2(h) = (k_{-i}(h-1)+R(h))f - v_i = \alpha_1(h)+R(h)f$ so that $\alpha_2(h) \geq \alpha_1(h)$,

$\alpha_3(h) = v_i - k_i(h)f = v_i - (k_i(h-1)+\delta_h)f$ from (22).

In this way $\alpha_1(h)$ represents what b_i gains if he accepts at step h in the prospect of being the last **cch** whereas $\alpha_2(h)$ represents the highest gain b_i can obtain if he accepts at step h in the prospect of being the last **cch**.
It is easy to understand how the signs of $\alpha_1(h)$ and $\alpha_2(h)$ are related, in some way to be specified, whereas that of $\alpha_3(h)$ is determined independently. From the foregoing definitions it is easy to derive the following conditions:

(1) if $\alpha_1(h) \geq 0$ then $\alpha_2(h) \geq 0$,

(2) if $\alpha_2(h) < 0$ then $\alpha_1(h) < 0$,

(3) if $\alpha_1(h) < 0$ then $\alpha_2(h)$ can be of any sign depending on the value of $R(h)$ (so it is unconstrained),

(4) if $\alpha_2(h) \geq 0$ then $\alpha_1(h)$ can be of any sign depending on the value of $R(h)$ (so it is unconstrained),

(5) $\alpha_3(h)$ is positive up to a value $\bar{k}_i$ and from that value on it is negative. With this we mean that we can define:

$$\bar{k}_i = \lfloor \frac{v_i}{f} \rfloor \tag{24}$$

such that:

- for $k_i < \bar{k}_i$ if b_i refuses at h we have $\alpha_3(h) \geq 0$,
- for $k_i \geq \bar{k}_i$ if b_i refuses at h we have $\alpha_3(h) < 0$.

From the foregoing definitions and considerations we have that if b_i is selected at step h he can choose his best action from the set[15] $S_i = \{a, r\}$ according to the following algorithm[16,17]:

if with $\delta_h = 1$ $\alpha_3(h) \geq \alpha_2(h)$ then b_i selects r

else [here b_i, by refusing, is not sure to have a greater gain]

if $\alpha_1(h) \geq 0$ or $\alpha_2(h) \geq 0$ then b_i selects a
else [we are in the case $\alpha_1(h) < 0$ and $\alpha_2(h) < 0$]
if with $\delta_h = 1$ $\alpha_3(h) \geq 0$ then b_i selects r
else [we are in the case $\alpha_1(h) < 0$, $\alpha_2(h) < 0$ and $\alpha_3(h) < 0$]
if $\alpha_3(h) > \alpha_2(h)$ then b_i selects r
else b_i selects a

The condition $\alpha_3(h) \geq \alpha_2(h)$ with $\delta_h = 1$ (and so with a refusal from b_i at step h) can be expressed as:

$$v_i - k_i(h-1)f - f > k_{-i}(h-1)f - v_i + R(h)f \tag{25}$$

or:

$$2v_i > k_{-i}(h-1)f + k_i(h-1)f + f + R(h)f = P(h) + f + R(h)f \tag{26}$$

From relations 18 between f and the values v_i we have that such condition is rarely, if ever, verified so that we are going to discard completely the corresponding case in the considerations that follow.
We recall that we have[18] $\alpha_2(h) = \alpha_1(h) + R(h)f$ where $R(h)$ is the number of steps from $h+1$ inclusive to $L-1$ inclusive. In this way at the step h the best that b_i can expect is $R(h)$ refusals from the other bidders. Moreover we can make the following considerations.

- If at step h we have $\alpha_2(h) < 0$ then $\forall h' > h$ we have $\alpha_2(h') < 0$. Such feature derives from the fact that we have $\alpha_2(h') \leq \alpha_2(h)$ since we have $R(h') \leq R(h)$ owing to the fact that some players b_j with $j \neq i$ can accept and that b_i can be selected again so that not all the steps can turn into useful refusals.

[15] We recall that in the set $S_i = \{a, r\}$ a stands for acceptance and r for refusal.

[16] We recall that the negative of the proposition $a\, or\, b$ is $\neg a\, and\, \neg b$ and that the negative of $x \geq 0$ is $x < 0$.

[17] In the pseudo code we represent comments as text within square brackets.

[18] We note that $\alpha_2(h)$ attains its maximum value at $h = 0$ where $R(h) = L - 1$.

- If at step h we have $\alpha_2(h) \geq 0$ with a given $R(h)$ at $h' > h$, since we have $R(h') < R(h)$, we can have $\alpha_2(h') < 0$.

- From such considerations we have that for $\alpha_2(h)$ we can have either a permanence of the sign $-$ or $+$ or a switch of sign from $+$ to $-$.

- If at step h we have $\alpha_1(h) \geq 0$ then at every step $h' > h$ we have $\alpha_1(h') \geq 0$ since we have $k_{-i}(h') \geq k_{-i}(h)$.

- If at step h we have $\alpha_1(h) < 0$ (and this is true at step 0) then at the step $h' > h$ we can have $\alpha_1(h') \geq 0$ since we have $k_{-i}(h') \geq k_{-i}(h)$.

- From such considerations we have that for $\alpha_1(h)$ we can have either a permanence of the sign $-$ or a switch of sign from $-$ to $+$.

We can therefore devise the following strategies[19] for a bidder b_i:

- if we have $\alpha_2(0) < 0$ then $\alpha_2(h) < 0$ at every step $h > 0$ so that, by property (2) and since $\alpha_2(h) \geq \alpha_1(h)$ for every h, we have $\alpha_1(0) < 0$ and $\alpha_1(h) < 0$ and therefore b_i is better off if he refuses at every selection at least until $k_i < \bar{k}_i$ since from that point on he may be better off by accepting according to the algorithm we provided;

- if we have $\alpha_1(0) < 0$ and $\alpha_2(0) \geq 0$ then at the beginning, upon being selected, b_i is better off by accepting until at some h we have $\alpha_2(h) < 0$ so that b_i is better off if he refuses at every selection at least until $k_i < \bar{k}_i$ since from that point on he may be better off by accepting according to the algorithm we provided;

- if we have $\alpha_1(0) < 0$ and $\alpha_2(0) \geq 0$ then at the beginning, upon being selected, b_i is better off by accepting and he is better off if he keeps on accepting if there is a step h form where we have also $\alpha_1(0) \geq 0$ (see also the algorithm we provided).

Such strategies can be summarized by Table 1 that is used by each bidder b_i if he is selected at step h. With *yes* we denote the fact that the sign of $\alpha_3(h)$ plays a role in the decision that may be either a refusal (r) or an acceptance (a) whereas if an entry contains *no* this means that the sign of $\alpha_3(h)$ plays no a role in the decision. The other entries have an easy interpretation.
If a bidder b_i accepts at step h we have:

---

[19]We note how the cases:

$\alpha_1(0) \geq 0$ and $\alpha_2(h) < 0$,

$\alpha_1(0) \geq 0$ and $\alpha_2(h) \geq 0$,

are impossible cases from the very definitions of $\alpha_i(h)$ and $\alpha_i(h)$.

$\alpha_1(h)$	-	-	+
$\alpha_2(h)$	-	+	+
$\alpha_3(h)$	yes	no	no
action	r/a	a	a

Table 1: Signs and actions

- he is no more selected for $h' > h$ and there are only refusals from the other bidders then he has a gain otherwise, if there is at least one acceptance, he is better off since he did not pay the fee f at step h;

- he is selected again at step $h' > h$ he can revise his decision and in any case he is better off since he did not pay the fee f at step h.

We recall that b_i at step h, if he is selected, takes a decision according to the signs of $\alpha_1(h)$, $\alpha_2(h)$ and $\alpha_3(h)$ but he has no influence, unless he is selected again, at step $h+1$ where a bidder b_j (with $j \neq i$) can replace him as **cch**.
As a last issue we examine the cases where we can have no last **cch** so that we can define the auction as void. From what we have seen up to this point this eventuality can happen in the following cases:

(1) every bidder refuses at every step $h \in [0, L-1]$,

(2) we may have some acceptances from 0 to $h-1$ but from h on to $L-1$ we have only refusals even from the last accepting bidder.

For what concerns the case (1) we refer to what we have said about it at the beginning of this section.
For what concerns the case (2) we note what follows.
Since we are in a case different from case (1) we must have at least one acceptance and since we are in the case (2) we must have a last acceptance followed by only refusals.
We assume such last acceptance at step h from bidder b_k. We have the following two cases:

- b_k accepts for $\alpha_1(h) \geq 0$;

- b_k accepts for $\alpha_2(h) \geq 0$ but with $\alpha_1(h) < 0$.

In the former case this last acceptance can never be turned into a refusal since the characterizing condition holds for any $h' > h$ so b_k is the last **cch** since all the distinct bidders that are selected afterwards refuse and b_k cannot be selected again since otherwise he should refuse notwithstanding

the acceptance condition is verified (so we have a contradiction).
In the latter case for all $h' \in [h+1, L-1]$ we can have only refusals from the bidders b_i with $i \neq k$ since we are assuming to have the last acceptance at step h. From this we have that the acceptance condition for b_k is possibly backed up (since we can have also $\alpha_1(h) \geq 0$) so also in this case b_k is the last **cch**.
In this way we assume implicitly that b_k is no more selected for all $h' \in [h+1, L-1]$. Since we are assuming a last acceptance at step h we have that if the condition of acceptance for b_k is verified for any $h' \in [h+1, L-1]$ then b_k cannot be selected again otherwise he should refuse notwithstanding the acceptance condition is verified.
We have to prove the last claim we made about the validity of the acceptance condition for b_k if he is selected again at $h' > h$.
Upon a new selection at $h' > h$ if we have $\alpha_1(h') \geq 0$ then b_k should accept against the fact that the acceptance at h is the last one so that new selection of b_k is impossible.
If we have $\alpha_2(h') \geq 0$ and $\alpha_1(h') < 0$ we can make similar considerations.
The hard case is if $\alpha_2(h') < 0$ (and so $\alpha_1(h') < 0$) though this case can occur only if we have:

$$\alpha_2(h') = \alpha_2(h) - f < 0 \tag{27}$$

whereas in all the other cases it is not verified and we are back to the preceding cases.
We have therefore proved that case (2) can hardly ever occur. If anyway it occurs we have that the auctioneer can use the content of the pot:

$$P \geq (L - h - 1)f \tag{28}$$

(since there can have been some refusals in the steps from 0 to $h-1$) to allocate the auctioned item to one actor distinct from the bidders b_i.

6 The *variable termination* case

6.1 The basic steps

In this case at every step $j \in [0, L-1]$ we have a probability p_j that the auction ends at that step (see section 3). The proposed procedure is therefore based on the following steps:

(1) starts at $j = 0$ with all the variables properly initialized among which we have $P = 0$;

(2) at step j we see if the auction can go on (with a probability p_j) or must stop (with a probability $1 - p_j$);

(3) if it must stop go to (7);

(4) if it can go on a bidder b_i is randomly selected;

(5) if b_i accepts then b_i is the **cch**; $j = j + 1$; go to (2);

(6) if b_i refuses then $P = P + f$; $j = j + 1$; go to (2);

(7) the final **cch** gets P and ζ;

(8) end;

The auction can end either at $t = L$ or at any step $h < L$.
The former case occurs with a probability:

$$\bar{p} = \prod_{j=0}^{L-1} p_j \tag{29}$$

and for its analysis we refer also to section 5.
The latter case occurs with a probability:

$$1 - \bar{p} \tag{30}$$

In both cases the final **cch** at step (7) (if it exists) is the current **cch** when the auction ends otherwise the auction is void and the auctioneer can use the content P of the pot to allocate the auctioned item to an actor distinct from the bidders b_i.
The termination of the auction at every step is determined with the use of a properly defined random device that uses a predefined distribution of probability values (so p_j and $1 - p_j$ at every step j) that are assumed to be common knowledge among the bidders.
From the structure of the auction and from what we have seen in section 3 we easily derive that at step h:

(1) every bidder b_i knows his current situation as represented by the values $k_i(h-1)$ and $k_{-i}(h-1)$;

(2) every bidder b_i can evaluate the probability that the auction goes on for k more rounds up to round $L - 1$;

(3) every bidder b_i can evaluate the probability of being selected once again before the end of the auction.

The values at point (1) define the past for each bidder whereas those at points (2) and (3) define his future for what concerns the length of the auction and the probability of a further involvement in the auction. With this we mean that every bidder b_i knows if his past is profitable or not and can guess if his future is promising or not.
The past is profitable for b_i if at step h we have (see section 5.2):

$$\alpha_1(h) = k_{-i}(h-1)f - v_i \geq 0 \tag{31}$$

(upon an acceptation) or:

$$\alpha_3(h) = v_i - k_i(h)f \geq 0 \tag{32}$$

where:

$$k_i(h) = k_i(h-1) + 1 \tag{33}$$

upon a refusal so we have $\delta_h = 1$.
On the other hand the future is promising if the expected gain (including the gain from the past) is positive or if:

$$\tilde{\alpha}_2(h) \geq 0 \tag{34}$$

with:

$$\tilde{\alpha}_2(h) = k_{-i}(h-1)f + \tilde{k}f - v_i = \alpha_1(h) + \tilde{k}f \tag{35}$$

where $\tilde{k}$ assumes a value $k \in [1, R(h)]$ with a probability $P(k|h)$ (to be defined shortly) and $R(h) = L - h - 1$ is the number of rounds after round h up to the maximum duration of the auction at $L - 1$.
With $P(k|h)$ we define the probability that the auction lasts k more rounds having reached round h.
We recall that p_h is the probability that at round h the auction goes on so that we can define the probability that the auction lasts for k more rounds having reached round h as:

$$p_{hk} = P(k|h) = \prod_{j=h+1}^{h+k} p_j \tag{36}$$

From definition (36) we derive:

$P(L-h-1|h) = \prod_{j=h+1}^{L-1} p_j$

$P(1|h) = \prod_{j=h+1}^{h+1} p_j = p_{h+1}$

We recall indeed that between $h+1$ and $L-1$ we have:

$$L - 1 - h - 1 + 1 = L - h - 1 \tag{37}$$

rounds.
In relation (35) we have a random contribution $P(k|h)kf$ that is composed of two parts:

- one that increases with k or kf;

- one that decreases (since $p_j < 1$ for each $j \neq 0$) with k or

$$P(k|h) = \prod_{j=h+1}^{h+k} p_j \tag{38}$$

Since the values p_j form a non increasing succession of values with j we have:

$$p_{h+k}^k \leq P(k|h) \leq p_{h+1}^k \tag{39}$$

From the definition (36) and from the properties of the values p_j we can easily derive the following properties:

(a) $p_{hk} > p_{hk'}$ for every $k' > k$;

(b) $p_{hk} > p_{h'k}$ for every $h' > h$;

(c) $p_{hk} > p_{h'k'}$ for every $h' > h$ and $k' > k$.

We moreover have, since $p_j < 1$ for every $j \neq 0$ that:

$p_{h+1}^k \to 0$ as k increases,

$p_{h+k}^k \to 0$ as k increases,

so, from relation (39), we have that:

$p_{hk} \to 0$ as k increases.

In this case we have assumed (see section 3):

(1) $p_0 = 1$,

(2) $p_j < 1$ and decreasing for any $j > 0$,

(3) $p_L = 0$.

If, on the other hand, we assume (see section 3) for instance:

(4) $p_j = 1$ for any $j \in [0, L_{min}]$,

(5) $p_j < 1$ and decreasing for any $j \in [L_{min} + 1, L - 1]$,

(6) $p_L = 0$,

we can make the following considerations:

- if $h > L_{min}$ the bidders behave as in the previous case,

- if $h \leq L_{min}$ the bidders must consider that the auction has a minimum guaranteed duration so that they define:

$$\tilde{\alpha}_2(h) = \alpha_i(h) + R(h)f + \tilde{k}f \tag{40}$$

where $R(h) = L_{min} - h$ is the maximum number of refusals from the other bidders in the guaranteed rounds of the auction and $\tilde{k}$ assumes a value $k \in [1, L - L_{min} - 1]$ with a probability $P(k|h)$. In this case the bidders can use a variant of the algorithm we are going to present in section 6.2 to account for the presence of the term $R(h)f$.

In what follows, owing to space constraints, we focus only on the case where assumptions (1), (2) and (3) hold.

6.2 The possible collective and individual strategies

At this point we can consider what we have seen in section 5.2 and extend it to the new situation where the bidders know the probabilities p_j (for $j \in [0, L-1]$ since $p_L = 0$) and can guess the probabilities $p_{hk} = P(k|h)$ for any $h \in [1, L-1]$ and $k \in [1, L-h]$.
It is easy to see how, for $h = L - 1$, we have (by similar arguments to those we used in section 5.2):

- a succession of L acceptances is not a NE for the same reasons we saw in section 5.2;

- a succession of L refusals may not be a NE for the same reasons we saw in section 5.2.

We underline how, from property (a) of section 6.1, the value $P(L-1|0)$ (with $p_0 = 1$) represents the minimum value of p_{hk} so that such successions are events with a low probability of occurring.
Within the current framework where the auction can end at step h independently from the actions of the bidders we must consider the following successions:

(1) of h refusals from 0 to $h-1$,

(2) of h acceptances from 0 to $h-1$.

Both successions occur with the following probability:

$$(1-p_h)\prod_{j=0}^{h-1} p_j \tag{41}$$

In the case (1) the auction is void but the auctioneer can use the final content of the pot $P = hf$ to allocate the auctioned item to an actor that is distinct from the bidders.
A succession of h refusals is motivated from the fact that no bidder finds the content of the pot as worth enough and, at the same time, none of them thinks the auction is going to last long enough to get a gain from it.
In the case (2) the auction is not void (since the last accepting bidder is the final **cch**) but the bidder who gets the item ζ has no compensation since the pot is empty (no bidder ever paid the fee f).
A succession of h acceptances can depend on the fact that either refusing is too costly or that each bidder thinks the auction is going to last enough so to have a gain from accepting or that both conditions hold.
In order to understand if such successions can occur we must examine the individual strategies of the bidders to be formalized with an algorithm, according to the same approach we have used in section 5.2.
In the current case the behavior of each bidder depends on the sign of the following parameters:

(1) $\alpha_1(h) = k_{-i}(h-1)f - v_i$,

(2) $\tilde{\alpha}_2(h) = k_{-i}(h-1)f - v_i + \tilde{k}f = \alpha_1(h) + \tilde{k}f$,

(3) $\alpha_3(h)v_i - k_i(h)f$ with $k_i(h) = k_i(h-1) + \delta_h$.

We recall that $\tilde{k}$ is a random variable that represents the maximum number of refusals from the bidders b_j with $j \neq i$ from the round $h+1$ to the end of the auction ans is associated with the probability $p_{hk} = P(k|h)$ so that we have $\tilde{k} = k$ with that probability.
The algorithm we are going to present shortly is based on the following strategy. At the step h b_i is selected so that he can evaluate the maximum value of $\tilde{k}$ such that, with $\delta_h = 1$ (and so by refusing at h) he has:

$$\alpha_3(h) > \tilde{\alpha}_2(h) \tag{42}$$

If such a value is associated with a low probability then he is better off by refusing. Similar considerations hold in all the cases where b_i has to deal with $\tilde{\alpha}_2(h)$ and so with $\tilde{k}$. If the needed value is associated with a sufficiently high probability then b_i can bet on it otherwise he is better off by non betting and by choosing the alternative course of auction (if any).
At the beginning of the auction (and so at $h = 0$) we conventionally have $k_{-i}(-1) = 0$ and $k_i(-1) = 0$ so we have:

(1') $\alpha_1(0) = -v_i$,

(2') $\tilde{\alpha}_2(0) = -v_i + \tilde{k}f$ where $\tilde{k}$ assumes a value in $[1, L-1]$ with a probability $P(k|0)$,

(3') $\alpha_3(0)v_i - k_i(0)f$ with $k_i(0) = \delta_h$.

From the above definitions (1), (2) and (3) we derive the following properties (fully analogous to those we have seen in section 5.2):

(1) if $\alpha_1(h) \geq 0$ then $\tilde{\alpha}_2(h) \geq 0$,

(2) if $\tilde{\alpha}_2(h) < 0$ then $\alpha_1(h) < 0$,

(3) if $\alpha_1(h) < 0$ then $\tilde{\alpha}_2(h)$ is unconstrained,

(4) if $\tilde{\alpha}_2(h) \geq 0$ then $\alpha_1(h)$ is unconstrained,

(5) if $\alpha_1(h) \geq 0$ then for any $h' > h$ we have $\alpha_1(h') \geq 0$,

(6) $\tilde{\alpha}_2(h) \geq \alpha_1(h)$.

From these premises we can devise the following algorithm that each bidder b_i can use to determine his choice of an action from the set $S_i = \{a, r\}$ at the generic step h (after the termination test has been executed, see the steps (2), (4) and (5) of the procedure at the beginning of section 6.1)[20]:

```
if with δ_h = 1 α_3(h) ≥ α̃_2(h) then b_i selects r

else [here b_i, by refusing, is not sure to have a greater gain]

    if α_1(h) ≥ 0 or α̃_2(h) ≥ 0 then b_i selects a
    else [we are in the case α_1(h) < 0 and α̃_2(h) < 0]
        if with δ_h = 1 α_3(h) ≥ 0 then b_i selects r
        else [we are in the case α_1(h) < 0, α̃_2(h) < 0 and α_3(h) < 0]
```

[20] In the pseudo code we represent comments as text within square brackets.

if $\alpha_3(h) > \tilde{\alpha}_2(h)$ then b_i selects r
else b_i selects a

The condition $\alpha_3(h) \geq \tilde{\alpha}_2(h)$ means that b_i thinks it is unlikely that the auction is going to last a number of rounds that is sufficient to have him collect the minimum number of refusals from the other bidders so he is better off by refusing at the current step.
If this condition is falsified $b : i$ tries to see if he can have a gain by accepting. If also this attempt fails he then tries to minimize his damage by choosing the best worst action. The difference with the algorithm in the fixed termination case is due to the fact that b_i may guess the needed value of $\tilde{k}$ with its associated probability and then he behaves in the proper way depending on such value of probability.

7 Concluding remarks

The present paper introduces two repeated or multi shot auction mechanisms. In both mechanisms the bidders are selected at each step according to a uniform distribution and each bidder can perform a choice (accept or refuse) only if he is selected.
In this way each bidder has a random termination time for his participation to the auction as the last time he is selected. On the other hand every bidder is influenced by the decisions of the others.
If b_i at step h is selected and accepts he becomes the **cch**. He keeps this title upon successive refusals from the other bidders (that make him better off) and upon his own successive acceptances (upon being selected) and loses it upon any acceptance of one of the other bidders or upon his successive refusal.
In the former mechanism this random selection is the only probabilistic element we introduced in the procedure whereas in the latter we introduced a further probabilistic device since we allowed each step to be assigned a probability of premature termination at that step.
Both mechanisms have been presented and described together with some strategies for the bidders.
Their formal treatment must, however, still be completed and will be the subject of further research efforts.

References

[1] Sascha Füllbrun. A Comparison of Candle Auctions and Hard Close Auctions with Common Values. *FEMM Working Paper*, No 19, 2009. Internet version.

[2] Sascha Füllbrun and Tim Hoppe. Stochastic Deadlines: A Comparison of Parallel Multiple Auctions Design. *FEMM Working Paper*, No 15, 2009. Internet version.

[3] Sascha Füllbrun and Abdlolkarim Sadrieh. Sudden Termination Auctions - An Experimental Study. *FEMM Working Paper*, No 24, 2006. Internet version.

[4] Robert Gibbons. *Game Theory for Applied Economists*. Princeton University Press, 1992.

[5] Vijay Krishna. *Auction Theory*. Academic Press, 2002.

[6] Paul Milgrom. *Putting Auction Theory to Work*. Cambridge University Press, 2004.

[7] Roger B. Myerson. *Game Theory. Analysis of conflict*. Harvard University Press, 1991.

[8] Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory*. MIT Press, 1994.

[9] E. Carleton Williams. The Aldermaston Candle Auction. 1935? Internet version.